

A split symmetric pair that is p -stable, not stable, and has unramified multiplicity one

24 October 2026

Abstract

We give an explicit symmetric pair over $\mathbf{Q}_3$ whose ambient group is split and semisimple. The pair is p -stable but not stable in the sense of Carmeli. For the specified split integral model and its hyperspecial subgroup $K = \mathcal{G}(\mathbf{Z}_3)$, every irreducible K -spherical representation has at most one invariant linear form. The proof includes an explicit closed double coset witnessing failure of stability and treats the exceptional self-twist case in the multiplicity argument. The fixed choice of K is part of the assertion; the involution does not preserve it.

1 Statement and conventions

Throughout,

$$k = \mathbf{Q}_3, \quad \mathfrak{o} = \mathbf{Z}_3, \quad v = v_3, \quad v(3) = 1.$$

Representations are smooth and complex, and tensor products are algebraic tensor products. The smooth contragredient is denoted by $\tau^\vee$. Irreducible smooth representations of the reductive groups considered here are admissible. The space $\mathrm{Hom}_H(\pi, \mathbf{C})$ consists of all H -invariant linear forms on the smooth representation space.

An *unramified representation* here means an irreducible representation with a nonzero vector fixed by the particular subgroup K below. This includes every irreducible K -spherical representation, including those arising at reducible or singular principal-series parameters.

Define group schemes over $\mathfrak{o}$ by

$$\tilde{\mathcal{G}} = \{(A, B) \in \mathrm{GL}_4 \times \mathrm{GL}_4 : \det A \det B = 1\}, \quad (1.1)$$

$$\mathcal{G} = \tilde{\mathcal{G}} / \{(tI_4, t^{-1}I_4) : t \in \mathbf{G}_m\}. \quad (1.2)$$

The quotient is by the displayed central split torus. Put

$$\mathbf{G} = \mathcal{G}_k, \quad G = \mathbf{G}(k), \quad K = \mathcal{G}(\mathfrak{o}).$$

Write $[A, B]$ for the class of a matrix pair. Set $D = \mathrm{diag}(3, 1, 1, 1)$ and define

$$\theta[A, B] = [DBD^{-1}, D^{-1}AD], \quad \mathbf{H} = \mathbf{G}^\theta, \quad H = \mathbf{H}(k) = G^\theta. \quad (1.3)$$

In particular, H is the *full* fixed subgroup.

Theorem 1.1. *The construction has the following properties.*

- (1) $\mathcal{G}$ is split semisimple of type $A_3 \times A_3$, and $K = \mathcal{G}(\mathfrak{o})$ is a hyperspecial subgroup of G .
- (2) The pair is p -stable: H has exactly one open orbit on G/P for every k -parabolic subgroup P .

(3) *The pair is not stable: some double coset HgH is closed but $\theta(g^{-1}) \notin HgH$.*

(4) *For every irreducible smooth representation π of G ,*

$$\pi^K \neq 0 \implies \dim_{\mathbf{C}} \operatorname{Hom}_H(\pi, \mathbf{C}) \leq 1.$$

We use stability and p -stability as in [1, Definitions 2.1 and 5.1]. Openness and closedness refer to the local-field topology on rational points. With

$$\sigma(g) = \theta(g^{-1}), \quad s(g) = g \sigma(g),$$

stability requires every closed H -double coset to be preserved by σ . We prove the four assertions in the following sections.

2 The split group and its integral and fixed points

2.1 The integral model

Lemma 2.1. *There is an isomorphism of split semisimple group schemes over $\mathfrak{o}$*

$$\mathcal{G} \simeq \frac{\operatorname{SL}_4 \times \operatorname{SL}_4}{\{(zI_4, z^{-1}I_4) : z \in \mu_4\}}.$$

Moreover,

$$\begin{aligned} G &= \{(A, B) \in \operatorname{GL}_4(k)^2 : \det A \det B = 1\} / \{(tI_4, t^{-1}I_4) : t \in k^\times\}, \\ K &= \{(A, B) \in \operatorname{GL}_4(\mathfrak{o})^2 : \det A \det B = 1\} / \{(tI_4, t^{-1}I_4) : t \in \mathfrak{o}^\times\}. \end{aligned}$$

Proof. The inclusion of $\operatorname{SL}_4 \times \operatorname{SL}_4$ into $\tilde{\mathcal{G}}$ induces a map to $\mathcal{G}$ with kernel the displayed anti-diagonal copy of μ_4 . Given $(A, B) \in \tilde{\mathcal{G}}$, after an fppf extension choose t with $t^4 = (\det A)^{-1}$. The equivalent representative $(tA, t^{-1}B)$ has both determinants equal to 1. This proves surjectivity as fppf sheaves and the asserted central quotient. A central quotient of a split semisimple group scheme is split semisimple. Thus $\mathcal{G}$ is a split reductive integral model, and $K = \mathcal{G}(\mathfrak{o})$ is hyperspecial.

For the formulas on points, use

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{G} \longrightarrow 1.$$

The obstruction to lifting an R -point of $\mathcal{G}$ lies in $H_{\text{fppf}}^1(R, \mathbf{G}_m) = \operatorname{Pic}(R)$. This vanishes for both $R = k$ and $R = \mathfrak{o}$. Hence every point lifts, and two lifts differ by the stated central torus. $\square$

Remark 2.2. One must not identify G with $(\operatorname{SL}_4(k) \times \operatorname{SL}_4(k)) / \mu_4(k)$: taking rational points of that finite central quotient need not be surjective. Lemma 2.1 uses a split central torus and therefore gives the actual rational and integral points.

Let

$$\theta_0[A, B] = [B, A], \quad \alpha_D[A, B] = [DAD^{-1}, B].$$

These are well-defined k -automorphisms where appropriate, and

$$\theta = \alpha_D \theta_0 \alpha_D^{-1}. \tag{2.1}$$

Thus θ is an involution. The group $\mathbf{G}$ is connected, so $\mathbf{G}/\mathbf{H}$ is connected; the construction is a symmetric pair with the conventions of [1].

Lemma 2.3. For $H_0 = G^{\theta_0}$, one has

$$H_0 = \{[C, tC] : C \in \mathrm{GL}_4(k), t \in k^\times, (\det C)^2 t^4 = 1\}, \quad (2.2)$$

$$H = \{[DCD^{-1}, tC] : C \in \mathrm{GL}_4(k), t \in k^\times, (\det C)^2 t^4 = 1\}. \quad (2.3)$$

Every C occurring in these formulas has $v(\det C)$ even.

Proof. The equality $[B, A] = [A, B]$ means $B = tA$ and $A = t^{-1}B$ for some $t \in k^\times$. Taking $C = A$, the determinant condition is $(\det C)^2 t^4 = 1$. This proves (2.2); applying α_D gives (2.3). The determinant equation is equivalent to $\det C = \pm t^{-2}$, proving the parity assertion. $\square$

2.2 A finite-index description

Set

$$A = \mathrm{PGL}_4(k), \quad K_A = \mathrm{PGL}_4(\mathfrak{o}), \quad \Delta = k^\times / (k^\times)^4.$$

For a matrix lift $\tilde{a}$ of $a \in A$, define

$$\delta(a) = \det(\tilde{a}) \bmod (k^\times)^4.$$

This is well-defined because scalar changes of lift multiply the determinant by a fourth power. Put

$$\begin{aligned} N &= A \times A, & M &= \{(a, b) \in N : \delta(a)\delta(b) = 1\}, \\ J &= M \cap (K_A \times K_A), \\ \epsilon(a) &= (-1)^{v(\det \tilde{a})}, & L &= \ker \epsilon. \end{aligned} \quad (2.4)$$

The character ϵ is likewise independent of the lift.

Lemma 2.4. The map $q : G \rightarrow N$, $[A, B] \mapsto (\overline{A}, \overline{B})$, has image M and central kernel

$$Z_0 = \{[I_4, I_4], [-I_4, I_4]\} \subset K \cap H.$$

Its images on K and H are

$$q(K) = J, \quad q(H) = H' = \{(dhd^{-1}, h) : h \in L\}, \quad d = \overline{D}. \quad (2.5)$$

Furthermore,

$$N/M \simeq \Delta \simeq C_4 \times C_2, \quad (K_A \times K_A)/J \simeq C_2.$$

Proof. The image of q is contained in M . Conversely, if $(a, b) \in M$, choose matrix lifts A, B . Then $\det A \det B = c^4$ for some $c \in k^\times$. Replacing A by $c^{-1}A$ yields a pair in $\tilde{\mathcal{G}}(k)$. Thus $q(G) = M$.

A kernel element is represented by (aI_4, bI_4) with $(ab)^4 = 1$, and its class equals $[(ab)I_4, I_4]$. Since $\mu_4(\mathbf{Q}_3) = \{1, -1\}$, the kernel is exactly Z_0 . Its elements are integral and fixed by θ .

Clearly $q(K) \subset J$. Given an element of J , choose lifts $A, B \in \mathrm{GL}_4(\mathfrak{o})$; these exist because $\mathrm{Pic}(\mathfrak{o}) = 0$. Again $\det A \det B = c^4$. Since the left-hand side is a unit, $c \in \mathfrak{o}^\times$. The same determinant correction is therefore integral, proving $q(K) = J$.

The decomposition

$$k^\times = 3^{\mathbf{Z}} \times \{1, -1\} \times (1 + 3\mathbf{Z}_3)$$

and the bijectivity of fourth powers on $1 + 3\mathbf{Z}_3$ show that $\Delta \simeq (\mathbf{Z}/4\mathbf{Z}) \times \{1, -1\}$. The elements of Δ whose squares equal 1 are exactly those with even valuation. Also, each element of $k^\times$ of even valuation is either a square or the negative of a square.

Lemma 2.3 now gives $q(H) \subset H'$. Conversely, for $h \in L$, choose a lift C . Its determinant has even valuation, so $\det C = \pm t^{-2}$ for some $t \in k^\times$. Then $[DCD^{-1}, tC] \in H$ maps to (dhd^{-1}, h) . This proves $q(H) = H'$.

Finally, the product-determinant map $N \rightarrow \Delta$ is surjective with kernel M . Its image on $K_A \times K_A$ is $\mathfrak{o}^\times / (\mathfrak{o}^\times)^4 \simeq C_2$, giving the last assertions. $\square$

3 p -stability

Proposition 3.1. *The pair $(\mathbf{G}, \mathbf{H}, \theta)$ is p -stable.*

Proof. Let Fl be the complete flag variety of k^4 . The full flag variety of $\mathbf{G}$ is $\text{Fl} \times \text{Fl}$, since central isogenies do not change flag varieties. Its rational points equal G/B_G for a minimal parabolic B_G . This is also clear from the transitivity of each factor $\text{SL}_4(k)$ on complete flags.

The subgroup

$$H_1 = \{[DSD^{-1}, S] : S \in \text{SL}_4(k)\}$$

lies in H . Consider

$$\Omega = \{(F_1, F_2) : D^{-1}F_1 \text{ and } F_2 \text{ are opposite complete flags}\}.$$

Opposition means

$$(D^{-1}F_1)^r \oplus F_2^{4-r} = k^4 \quad (r = 1, 2, 3).$$

These are nonvanishing determinant conditions, so Ω is nonempty and open.

Choosing bases adapted to opposite flags proves that $\text{GL}_4(k)$ acts transitively on ordered opposite pairs. The stabilizer of a standard opposite pair contains every invertible diagonal matrix, and its determinant map is therefore surjective. Multiplying a transporting matrix by a stabilizer element makes its determinant 1. Thus $\text{SL}_4(k)$ also acts transitively. Apply this to $(D^{-1}F_1, F_2)$ to conclude that H_1 is transitive on Ω .

By (2.5), every element of H acts on this transformed pair by simultaneous projective change of basis. Hence H preserves Ω , so Ω is a single H -orbit. Its complement is a proper algebraic closed subset of the flag product and has empty interior in the local-field topology; equivalently, the other Bruhat relative-position strata have smaller dimension. No orbit in that complement can be open. Thus there is exactly one open H -orbit on the minimal parabolic quotient.

All minimal parabolic subgroups are G -conjugate. By [1, Definition 5.1 and the following sentence], checking these quotients suffices for p -stability, proving the assertion for every parabolic quotient. $\square$

4 An explicit closed double coset that is not stable

4.1 An even-valuation centralizer

Let

$$E = k(i, u), \quad i^2 = -1, \quad u^2 = 3.$$

The extension $k(i)/k$ is unramified quadratic, since $T^2 + 1$ is irreducible over $\mathbf{F}_3$. The polynomial $T^2 - 3$ is Eisenstein over $k(i)$. Thus E/k is biquadratic, of degree 4, ramification index 2, and residue degree 2.

Set

$$x = u \frac{1 + u + i}{1 - u + i} = -\frac{6 + 5u + 30i + 12ui}{13}. \quad (4.1)$$

Its multiplication matrix in the basis $(1, u, i, ui)$ is

$$X = \frac{1}{13} \begin{pmatrix} -6 & -15 & 30 & 36 \\ -5 & -6 & 12 & 30 \\ -30 & -36 & -6 & -15 \\ -12 & -30 & -5 & -6 \end{pmatrix}. \quad (4.2)$$

Lemma 4.1. *The matrix X is semisimple, $k[X] \simeq E$, and*

$$\det X = 9, \quad \operatorname{tr} X = -\frac{24}{13}.$$

If $C \in \operatorname{GL}_4(k)$ and $CXC^{-1} = aX$ for $a \in k^\times$, then $a = 1$ and $v(\det C)$ is even. Every projective centralizer of X^{-1} also has even determinant valuation.

Proof. Expansion gives (4.1). Each coefficient of u, i, ui is nonzero. Every nonidentity element of the biquadratic Galois group changes the signs of at least two of those basis elements, so none fixes x . Therefore $k[x] = E$, and multiplication by x has a separable degree-four minimal polynomial. This proves semisimplicity.

Let $c \in \operatorname{Gal}(E/k(i))$ send u to $-u$. It sends the fraction in (4.1) to its inverse, so that fraction has norm 1 to $k(i)$, and hence to k . Since

$$N_{E/k}(u) = N_{k(i)/k}(-3) = 9,$$

we get $\det X = N_{E/k}(x) = 9$. The trace is four times the constant coefficient of x , namely $-24/13$.

If $CXC^{-1} = aX$, taking traces forces $a = 1$. Because $k[X] = E$, every endomorphism commuting with X is E -linear on the one-dimensional E -space E . Hence

$$Z_{\operatorname{GL}_4(k)}(X) = E^\times$$

under the multiplication embedding. Its determinants are norms. If v_E is normalized to have value group $\mathbf{Z}$, the local norm valuation formula gives

$$v(N_{E/k}(e)) = f(E/k)v_E(e) = 2v_E(e),$$

which is even. Finally, $CX^{-1}C^{-1} = aX^{-1}$ implies $CXC^{-1} = a^{-1}X$, so the same conclusion applies to X^{-1} . $\square$

4.2 The double-coset obstruction

Proposition 4.2. *For*

$$g = [DX^{-1}, D],$$

the double coset HgH is closed, but $\theta(g^{-1}) \notin HgH$. Thus the pair is not stable.

Proof. Begin with the swap pair (G, H_0, θ_0) and put

$$g_0 = [X^{-1}D, D].$$

This is in G , since $\det(X^{-1}D) \det D = 9^{-1} \cdot 3 \cdot 3 = 1$. For $\sigma_0(z) = \theta_0(z^{-1})$, direct multiplication yields

$$\sigma_0(g_0) = [D^{-1}, D^{-1}X], \quad g_0\sigma_0(g_0) = [X^{-1}, X]. \quad (4.3)$$

The second element is semisimple. By [1, Theorem 4.13, p. 11], a double coset for a symmetric pair is closed precisely when its symmetrization is semisimple. Thus $H_0g_0H_0$ is closed.

Use the map

$$r : G \longrightarrow \operatorname{PGL}_4(k), \quad r([A, B]) = \overline{AB^{-1}}.$$

This is well-defined: changing the representative by (t, t^{-1}) multiplies AB^{-1} by a scalar. For $h_1 = [C, tC]$ and $h_2 = [F, sF]$ in H_0 , multiplication gives

$$r(h_1zh_2) = \overline{C}r(z)\overline{C}^{-1}. \quad (4.4)$$

Before projectivizing, the relative matrix is $(ts)^{-1}C(AB^{-1})C^{-1}$. By Lemma 2.3, $v(\det C)$ is even.

Now

$$r(g_0) = \overline{X^{-1}}, \quad r(\sigma_0(g_0)) = \overline{D^{-1}X^{-1}D}.$$

If $\sigma_0(g_0) \in H_0 g_0 H_0$, equation (4.4) gives a C of even determinant valuation with

$$\overline{CX^{-1}C^{-1}} = \overline{D^{-1}X^{-1}D}.$$

Thus DC projectively centralizes X^{-1} . But $v(\det DC) = 1 + v(\det C)$ is odd, contradicting Lemma 4.1. This proves nonstability of that closed double coset. Only the displayed equivariance of r is used; no identification of a rational homogeneous quotient with all of $\mathrm{PGL}_4(k)$ is required.

Finally, α_D is a homeomorphism on rational points, sends H_0 to H , and intertwines the associated anti-involutions. It sends g_0 to

$$\alpha_D(g_0) = [DX^{-1}, D] = g.$$

It therefore carries the closed, nonstable double coset to the one asserted. Directly, its symmetrization is

$$g\theta(g^{-1}) = [DX^{-1}D^{-1}, X].$$

□

5 Multiplicity one for every irreducible unramified representation

Retain the notation of (2.4). The following identities will be decisive:

$$\epsilon(d) = -1, \quad \epsilon(a)\epsilon(b) = 1 \quad ((a, b) \in M). \quad (5.1)$$

5.1 Descent and a spherical ambient representation

Lemma 5.1. *Every irreducible K -spherical representation π of G descends to a representation π_M of M , with*

$$\pi^K = \pi_M^J, \quad \mathrm{Hom}_H(\pi, \mathbf{C}) = \mathrm{Hom}_{H'}(\pi_M, \mathbf{C}).$$

Proof. The kernel Z_0 of q is central and lies in K . Schur's lemma makes each of its elements a scalar on π ; that scalar is 1 because it fixes a nonzero K -fixed vector. Thus π descends. The equalities follow from $q(K) = J$ and $q(H) = H'$. □

Define the quadratic character $\lambda : k^\times \rightarrow \{1, -1\}$ by

$$\lambda(3^n u) = \begin{cases} 1, & u \equiv 1 \pmod{3}, \\ -1, & u \equiv -1 \pmod{3}, \end{cases} \quad u \in \mathfrak{o}^\times.$$

It is trivial on fourth powers, so $\kappa(a) = \lambda(\det \tilde{a})$ defines a character of A . Its restriction to K_A is nontrivial.

Lemma 5.2. *If τ is irreducible and K_A -spherical, then*

$$\dim \tau^{K_A} = 1, \quad \mathrm{Hom}_{K_A}(\kappa, \tau) = 0.$$

Proof. Fix an Iwahori subgroup inside K_A . The representation τ has Iwahori-fixed vectors, so [2, Proposition 2.6, p. 396] embeds it in an unramified principal series $I(\chi)$. Here χ is trivial on the integral diagonal torus, and induction from the upper triangular Borel B_A is normalized. Iwasawa decomposition gives

$$I(\chi)|_{K_A} \simeq \text{Ind}_{B_A \cap K_A}^{K_A} \mathbf{1},$$

because both χ and the modulus are trivial on $B_A \cap K_A$. The fixed vectors of this compact induced model are the constant functions, a one-dimensional space. The embedding and $\tau^{K_A} \neq 0$ prove the first assertion.

Compact Frobenius reciprocity gives

$$\text{Hom}_{K_A}(\kappa, \text{Ind}_{B_A \cap K_A}^{K_A} \mathbf{1}) = \text{Hom}_{B_A \cap K_A}(\kappa, \mathbf{1}) = 0.$$

Indeed, κ is nontrivial on the class of $\text{diag}(-1, 1, 1, 1)$ in $B_A \cap K_A$. The same vanishing holds in the subrepresentation τ . $\square$

Lemma 5.3. *For every irreducible admissible J -spherical representation π of M , there exist irreducible K_A -spherical representations τ_1, τ_2 of A such that*

$$V = \tau_1 \boxtimes \tau_2$$

restricts semisimply to M and contains π as its unique J -spherical irreducible summand, with multiplicity one. In particular, $\dim V^J = 1$, and π is generated by the usual spherical tensor.

Proof. We first supply the finite-index argument. Form $W = \text{c-Ind}_M^N \pi$. Since M is open of finite index, its restriction to M is a finite direct sum of conjugates of π , and W is admissible. It is cyclic as an N -module: any nonzero vector of the irreducible π generates that copy of π , and its N -translates generate all of W . A cyclic module has a maximal proper submodule, hence an irreducible quotient V_0 . This quotient is smooth and admissible: taking invariants under a compact open subgroup is exact over $\mathbf{C}$, by averaging.

Frobenius reciprocity applied to the nonzero quotient map $W \rightarrow V_0$ gives a nonzero map $\pi \rightarrow V_0|_M$, which is injective by irreducibility. The finitely many translates of this copy of π , one for each coset of M in N , span V_0 because V_0 is irreducible. Each translate is a simple M -submodule since M is normal. A finite sum of simple submodules is semisimple. Thus $V_0|_M$ is a finite direct sum and contains π as a direct summand. These are also the standard finite-index conclusions recorded in [4, Lemmas 2.1(a) and 2.3].

Now $V_0^J \neq 0$. The finite group $(K_A \times K_A)/J \simeq C_2$ acts on this space, and has the two characters 1 and $\kappa \boxtimes \kappa$. The latter character extends to N and is trivial on M , because

$$\kappa(a)\kappa(b) = \lambda(\det \tilde{a} \det \tilde{b}) = 1 \quad ((a, b) \in M).$$

Choose an eigenspace in V_0^J , and twist V_0 by either 1 or $\kappa \boxtimes \kappa$ to make that eigenspace fixed by $K_A \times K_A$. Denote the twisted representation by V . Its restriction to M is unchanged.

Using the tensor-product description of irreducible admissible representations of a direct product (see [3]), write $V = \tau_1 \boxtimes \tau_2$. Both factors are K_A -spherical. Since the compact quotient has order two,

$$V^J = V^{K_A \times K_A} \oplus V[\kappa \boxtimes \kappa],$$

where the second term is its indicated character-isotypic space. It is zero by Lemma 5.2. Consequently

$$V^J = \tau_1^{K_A} \otimes \tau_2^{K_A}$$

is a line.

The semisimple restriction $V|_M$ therefore has exactly one irreducible summand with J -fixed vectors, occurring once. It is the prescribed π . Its nonzero spherical tensor generates it by irreducibility. $\square$

5.2 The ambient period space

Lemma 5.4. *For $V = \tau_1 \boxtimes \tau_2$ as above,*

$$\mathrm{Hom}_{H'}(V, \mathbf{C}) \simeq \mathrm{Hom}_L(\tau_1, \tau_2^\vee),$$

and

$$\dim \mathrm{Hom}_{H'}(V, \mathbf{C}) = \mathbf{1}_{\{\tau_1 \simeq \tau_2^\vee\}} + \mathbf{1}_{\{\tau_1 \simeq \tau_2^\vee \otimes \epsilon\}} \leq 2. \quad (5.2)$$

Proof. A linear form on the tensor product is a bilinear form. Replacing its first argument v by $\tau_1(d)v$ converts invariance under (dhd^{-1}, h) to invariance under (h, h) . The latter bilinear forms are L -intertwiners $\tau_1 \rightarrow \tau_2^\vee$.

There is no smooth-dual issue: if v is fixed by an open subgroup U of L , invariance makes the functional in the second argument U -fixed. It is therefore smooth for L , and hence for A , since L is open in A .

Since A/L has order two, finite-index Frobenius reciprocity, or the decomposition $\mathrm{Ind}_L^A \mathbf{1} = \mathbf{1} \oplus \epsilon$, yields

$$\mathrm{Hom}_L(\tau_1, \tau_2^\vee) \simeq \mathrm{Hom}_A(\tau_1, \tau_2^\vee) \oplus \mathrm{Hom}_A(\tau_1, \tau_2^\vee \otimes \epsilon).$$

Explicitly, A/L acts on the left side by

$$a \cdot T = \tau_2^\vee(a) T \tau_1(a)^{-1}.$$

Its trivial and sign eigenspaces give the two displayed spaces. Schur's lemma gives (5.2). $\square$

Lemma 5.5. *Every H' -invariant linear form on the summand π of Lemma 5.3 extends to an H' -invariant linear form on V .*

Proof. Semisimplicity gives an M -equivariant projection $p : V \rightarrow \pi$ that is the identity on π . Since $H' \subset M$, the form $\ell \circ p$ extends any H' -period ℓ on π . $\square$

5.3 The exceptional self-twist case

Proposition 5.6. *If the dimension in (5.2) is 2, then the J -spherical summand satisfies $\mathrm{Hom}_{H'}(\pi, \mathbf{C}) = 0$.*

Proof. Both indicators in (5.2) are 1. Identify

$$\tau_1 = \tau, \quad \tau_2 = \tau^\vee, \quad \tau \simeq \tau \otimes \epsilon.$$

Choose an invertible intertwiner S such that

$$S\tau(a) = \epsilon(a)\tau(a)S \quad (a \in A). \quad (5.3)$$

It preserves the spherical line because $\epsilon|_{K_A} = 1$. Normalize S to fix a nonzero spherical vector v_0 . Then S^2 commutes with $\tau(A)$, so Schur's lemma makes it scalar; evaluating at v_0 gives $S^2 = 1$.

On the smooth dual set $S^\vee(w) = w \circ S$. This fixes the dual spherical line. Indeed, averaging over K_A shows that evaluation identifies $(\tau^\vee)^{K_A}$ with the dual of τ^{K_A} . Choose w_0 in that line with $w_0(v_0) = 1$. The operator $S^\vee$ preserves the line and $(S^\vee w_0)(v_0) = w_0(Sv_0) = 1$, so $S^\vee w_0 = w_0$.

Put $U = S \otimes S^\vee$ on $V = \tau \boxtimes \tau^\vee$. Equation (5.3) and its dual give

$$U(\tau(a) \otimes \tau^\vee(b)) = \epsilon(a)\epsilon(b)(\tau(a) \otimes \tau^\vee(b))U.$$

By (5.1), U commutes with M . It fixes $v_0 \otimes w_0$, which generates π as an M -representation. Hence

$$U|_{\pi} = 1. \quad (5.4)$$

The period space on V is identified with $\text{End}_L(\tau)$, which is spanned by 1 and S . Its corresponding basis is

$$\ell_C(v \otimes w) = w(C\tau(d)^{-1}v), \quad C = 1, S. \quad (5.5)$$

To check invariance directly, C commutes with $\tau(L)$, and for $h \in L$,

$$\begin{aligned} \ell_C(\tau(dhd^{-1})v \otimes \tau^\vee(h)w) &= w(\tau(h)^{-1}C\tau(h)\tau(d)^{-1}v) \\ &= \ell_C(v \otimes w). \end{aligned}$$

The sign $\epsilon(d) = -1$ gives $S\tau(d)^{-1} = -\tau(d)^{-1}S$. Since C commutes with S and $S^2 = 1$,

$$\begin{aligned} \ell_C(U(v \otimes w)) &= w(SC\tau(d)^{-1}Sv) \\ &= -w(C\tau(d)^{-1}v) = -\ell_C(v \otimes w). \end{aligned}$$

Thus every H' -period on V is U -odd. By (5.4), every such form vanishes on π . Lemma 5.5 extends every period on π to one on V , so the period space on π is zero. $\square$

Proof of the multiplicity assertion in Theorem 1.1. Let π be any irreducible K -spherical representation of G . Descend it to a J -spherical representation of M by Lemma 5.1, and choose V as in Lemma 5.3.

If $\dim \text{Hom}_{H'}(V, \mathbf{C}) \leq 1$, Lemma 5.5 gives $\dim \text{Hom}_{H'}(\pi, \mathbf{C}) \leq 1$. If the ambient dimension is 2, Proposition 5.6 gives the stronger conclusion that the period space on π is zero. These exhaust the possibilities by (5.2). The descent identifies this space with $\text{Hom}_H(\pi, \mathbf{C})$, proving the claim.

Together with Lemma 2.1, Proposition 3.1, and Proposition 4.2, this proves all of Theorem 1.1. $\square$

6 The role of the specified hyperspecial subgroup

The conclusion is relative to $K = \mathcal{G}(\mathbf{Z}_3)$ for the split model (1.2). Twisting by D changes the position of the fixed subgroup relative to K , while stability and p -stability are unchanged under α_D .

The involution does not preserve K . For example, with E_{12} the standard matrix unit,

$$[I_4 + E_{12}, I_4] \in K, \quad \theta[I_4 + E_{12}, I_4] = [I_4, I_4 + 3^{-1}E_{12}] \notin K.$$

Indeed, another representative of the latter class is $(tI_4, t^{-1}(I_4 + 3^{-1}E_{12}))$. Its first component belongs to $\text{GL}_4(\mathfrak{o})$ only if t is a unit; then the second still has a nonintegral entry.

The sign $\epsilon(d) = -1$ makes the two potentially competing periods vanish on the fixed- K spherical constituent. The theorem does not assert a bound uniformly over arbitrary choices of hyperspecial subgroup, and does not impose $\theta(K) = K$. It proves the requested bound for every irreducible representation with $\pi^K \neq 0$, without genericity, temperedness, or unitarity restrictions. The bound is attained by the trivial representation.

References

- [1] Shachar Carmeli, *On the Stability and Gelfand Property of Symmetric Pairs*, arXiv:1511.01381v3 (2019). [arXiv version 3](#). Definitions 2.1 and 5.1; Theorem 4.13. Page numbers refer to version 3.
- [2] W. Casselman, *The unramified principal series of p -adic groups. I. The spherical function*, Compositio Mathematica **40** (1980), no. 3, 387–406. [Full text](#). Proposition 2.6, p. 396.
- [3] D. Flath, *Decomposition of representations into tensor products*, in *Automorphic Forms, Representations and L -Functions*, Proceedings of Symposia in Pure Mathematics **33**, Part 1, American Mathematical Society, 1979, pp. 179–183.
- [4] S. S. Gelbart and A. W. Knap, *L -indistinguishability and R groups for the special linear group*, Advances in Mathematics **43** (1982), 101–121. [Full text](#). Lemmas 2.1(a) and 2.3, pp. 105–108. Only the unconditional finite-index results of Section 2 are used.