TRANSCENDENTAL VERSION OF THE THEORY OF SPHERICAL HARMONICS

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1. Introduction

This is a study note for Howe's proof of the transcendental version of the theory of spherical Harmonics, which is the decomposition of the oscillator representation for the dual pair $(\widetilde{SL}_2(\mathbb{R}), O(p))$.

2. Oscillator representations

We write $\widetilde{SL}_2(\mathbb{R})$ the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$, it exists because the fundamental group of $\mathrm{SL}_2(\mathbb{R})$ is $\mathbb{Z}$.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$, we can define a representation ω of sl_2 on $\mathcal{S}(\mathbb{R})$ via

$$\omega(h) = x\frac{d}{dx} + \frac{1}{2}$$
$$\omega(e^+) = \frac{i}{2}x^2$$
$$\omega(e^-) = \frac{i}{2}\frac{d^2}{dx^2}$$

here $\{h,\ e^+,\ e^-\}$ is the standard basis of $\mathfrak{sl}_2$. The operator $x\frac{d}{dx}$ is called the *Euler operator* over $\mathbb{R}$.

Theorem 2.1. *(Shale-Weil) The $\mathfrak{sl}_2$ module $\mathcal{S}(\mathbb{R})$ exponentiates to a unitary representation $\widetilde{SL}_2(\mathbb{R})$ on $L^2(\mathbb{R})$.*

We will call this representation the *oscillator representation* of $\widetilde{SL}_2(\mathbb{R})$.

The one-parameter subgroups generated by $h,\ e^+,\ e^-$ can be also described as

$$\omega(\exp(th))f(x) = e^{t/2}f(e^t x)$$
$$\omega(\exp(te^+))f(x) = e^{itx^2}f(x)$$
$$\omega(\exp(te^-)) = \text{convolution with}\frac{i+1}{2}(\pi t)^{-\frac{1}{2}}e^{-ix^2/2t}$$

The operator

$$2\omega(\mathfrak{k}) = 2i(\omega(e^-) - \omega(e^+)) = x^2 - \frac{d^2}{dx^2}$$

is known as the *Hermitian operator*.

We consider the following operators on $\mathcal{S}(\mathbb{R})$

$$a = x + \frac{d}{dx}$$
$$a^+ = x - \frac{d}{dx}$$

here $+$ stands for the adjoint of an operator under the usual inner product on $\mathcal{S}(\mathbb{R})$. For $v_0 = e^{-x^2/2}$, we can set $v_j = (a^+)^j v_0$.

Let P_j be the j-th Hermitian polynomials, which is a polynomial of degree j, then we have $v_j = P_j(x)e^{-\frac{x^2}{2}}$, then for the Hermitian functions $\{v_j\}$ we have

Lemma 2.2. *The Hermitian functions $\{v_j\}$ form an orthogonal basis of $L^2(\mathbb{R})$.*

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From the relations

$$x = \frac{1}{2}(a + a^+)$$
$$\frac{d}{dx} = \frac{1}{2}(a - a^+)$$
$$\omega(\mathfrak{k}) = \frac{1}{4}(aa^+ + a^+a)$$

then we can compute $\omega(\mathfrak{k})v_j = (j + \frac{1}{2})v_j$. This shows that $\{v_j\}$ is an orthogonal $\mathfrak{k}$-basis. $\{v_{2j}\}$ spans the lowest weight module $V_{1/2}$ of $\mathfrak{sl}_2$ with lowest weight vector v_0, $\{v_{2j+1}\}$ spans the lowest weight module $V_{3/2}$ of $\mathfrak{sl}_2$ with lowest weight vector $v_{3/2}$. The oscillator representation decomposes as

$$\mathcal{S}(\mathbb{R}) = V_{\frac{1}{2}} \oplus V_{\frac{3}{2}}$$

We can also define the n-th tensor power of the oscillator representation, it can be identified with $\mathcal{S}(\mathbb{R}^n)$ and we have the representation $(\omega^n, \mathbb{R}^n)$ is given by

$$\omega^n(h) = \sum_{i=1}^{n} x_j \frac{\partial}{\partial x_j} + \frac{n}{2}$$
$$\omega^n(e^+) = \frac{i}{2}\sum_{j=1}^{n} x_j^2$$
$$\omega^n(e^-) = \frac{i}{2}\sum_{j=1}^{n} \frac{\partial}{\partial x_j^2} = \frac{i}{2}\Delta_n$$

Let $O(p)$ be the compact orthogonal group then it acts on $\mathcal{S}(\mathbb{R}^p)$ via

$$g \cdot f(x) = f(g^{-1}x)$$

it turns out that the action of $O(p)$ and $\widetilde{SL}_2(\mathbb{R})$ are intimately related and the spectral decomposition of one of them completely determines the spectral decomposition of the other.

We have the following remarkable connection between the Casimir operators of $O(p)$ and $\mathfrak{sl}_2$

$$\mathcal{C}_{\mathfrak{o}(p)} = \omega^p(\mathcal{C}_{\mathfrak{sl}_2}) - (\frac{p}{2} - 1)^2 + 1$$

this can be proven using the explicit description of the basis of Lie algebra in terms of the differential operators on $\mathcal{S}(\mathbb{R}^p)$. Relations of this sort are predicted by the theory of dual pairs.

3. Theory of spherical harmonics

Theorem 3.1. *(Transcendental version of the theory of spherical harmonics)*

- *We have a decomposition*

$$\mathcal{S}(\mathbb{R}^p) = \sum_{m=0}^{\infty}(\tilde{\mathcal{H}}_m^p \otimes V_{\frac{p}{2}+m})^-$$

of $\mathcal{S}(\mathbb{R}^p)$ into $O(p) \times \widetilde{SL}_2(\mathbb{R})$ modules, here "-" indicates closure in $\mathcal{S}(\mathbb{R}^p)$.
- *The space $\tilde{\mathcal{H}}_m^p$ is an irreducible $O(p)$ module of dimension $\beta(p,m)$, in particular the $\tilde{\mathcal{H}}_m^p$ are all distinct as $O(p)$ modules, and the decomposition in* (3.1) *is a decomposition into irreducible $O(p) \times \widetilde{SL}_2(\mathbb{R})$ modules.*

Decomposition of this kind is typical in the theory of dual pairs 3.1.

Remark 3.2. There is also a L^2 version of the decomposition (3.1).

Proof. We consider $\mathcal{S}(\mathbb{R}^p)$ the p-th fold tensor power of the oscillator representation on $\mathcal{S}(\mathbb{R}^p)$, from the decomposition of the tensor product of representations of $\mathfrak{sl}_2$, we have

$$\mathcal{S}(\mathbb{R}^p)|_{\mathfrak{sl}_2} \cong V_{\frac{p}{2}} \oplus \sum_{m\geq 1} \beta(p,m) V_{\frac{p}{2}+m}$$

this is a decomposition of the $\mathfrak{sl}_2$ modules with explicitly known multiplicities. We know that in addition to the $\mathfrak{sl}_2$-action, there is also a commuting action of $O(p)$. Thus $\mathcal{S}(\mathbb{R}^p)$ is a module for $\widetilde{SL}_2(\mathbb{R}) \times O(p)$.

We now describe the structure of this module: for each possible $\mathfrak{k}$ eigenvalue $\frac{p}{2} + m$, denote by $\tilde{\mathcal{H}}^p_m$ the space of n^--null vectors of that $\mathfrak{k}$-eigenvalue. Each function $\phi \in \tilde{\mathcal{H}}^p_m$ generates an $\mathfrak{sl}_2$ module isomorphic to $V_{\frac{p}{2}+m}$, hence defines an embedding $\varphi_\phi : V_{\frac{p}{2}+m} \to \mathcal{S}(\mathbb{R}^p)$, let ϕ vary over $\tilde{\mathcal{H}}^p_m$, we get a mapping

$$\begin{aligned} \mathrm{Har}_m :\ \tilde{\mathcal{H}}^p_m \otimes V_{\frac{p}{2}+m} &\to \mathcal{S}(\mathbb{R}^p) \\ \phi \otimes v &\mapsto \varphi_\phi(v) \end{aligned}$$

the map Har_m is an isomorphism between the sum of $\mathfrak{sl}_2$ submodules of $\mathcal{S}(\mathbb{R}^p)$ that is isomorphic to $V_{\frac{p}{2}+m}$.

Since $\tilde{\mathcal{H}}^p_m$ is defined by the $\mathfrak{sl}_2$ action, it will be preserved by $O(p)$, and in fact Har_m is an isomorphism of $O(p) \times \widetilde{SL}_2(\mathbb{R})$ module.

We now describe the $O(p)$ module structure further. From the formula for $\omega^p(n^-)$

$$\omega^p(n^-) = \frac{1}{2} e^{-\frac{r^2}{2}} \Delta_p e^{\frac{r^2}{2}}$$

we can show that $\ker\omega^p(n^-)$ are precisely the functions of the form $fe^{-r^2/2}$ for f harmonic, and for $fe^{-r^2/2}$ to be a $\mathfrak{k}$-eigenvector of eigenvalue $\frac{p}{2} + m$, it is necessary for f to be a harmonic polynomial of degree m, hence $\tilde{\mathcal{H}}^p_m = \mathcal{H}^p_m e^{-r^2/2}$ where

$$\mathcal{H}^p_m = \ker\, \Delta_p : \mathcal{P}^m(\mathbb{R}^p) \longrightarrow \mathcal{P}^{m-2}(\mathbb{R}^p)$$

Considering the unit sphere $S^{p-1} \subseteq \mathbb{R}^p$, we may identify the stabilizer of e_P as $O(p-1)$. We can use $f \mapsto f(e_P)$ to define an $O(p-1)$-invariant functional on $\mathcal{H}^p_m$. In this way we get

$$e_Y = \sum_{j=0}^{[\frac{m}{2}]} c_j x_p^{m-2j} (r^2_{p-1})^j$$

e_Y is harmonic and it is the only nonzero $O(p)$ submodule of $\mathcal{H}^p_m$, namely $\mathcal{H}^p_m$ itself. Since $\mathcal{H}^p_m$ is irreducible, we can calculate its dimension from the fact that

$$\Delta_p :\ \mathcal{P}^m(\mathbb{R}^p) \longrightarrow \mathcal{P}^{m-2}(\mathbb{R}^p)$$

is a surjective map. $\square$