CLASSIFICATION OF SMOOTH AFFINE SPHERICAL VARIETIES

RUI CHEN

1. Introduction

This is a study note for the classification of smooth affine spherical varieties based on the papers [Kno98], [KVS06].

2. Multiplicity free representations

This is a summary for Knop's paper [Kno98].

We study multiplicity free representations of connected reductive groups, we first give a simple criterion to decide the multiplicity freeness of a representation. Then we determine all invariant differential operators in terms of a finite reflection group, the little Weyl group, and give a characterization of the spectrum of the Capelli operators. At the end, we reproduce the classification of multiplicity free representations.

A finite-dimensional representation V of a connected reductive group G is called multiplicity free if its coordinate ring contains every simple G-module at most once. Multiplicity free representations form a very restricted class of representations, nevertheless they are very important due to Roger Howe's philosophy that every "nice" result in the invariant theory of particular representations can be traced back to a multiplicity free representation. Also all of Weyl's first and second fundamental theorems can be explained by some multiplicity freeness result.

The multiplicity freeness criterion is a corollary of the local structure theorem of Brion-Luna-Vust. The determination of invariant differential operators is a very special result of a much general result valid for any G-variety by Knop, the calculation for the Weyl group is very interesting.

2.1. **The local structure theorem.**

We present the local structure theorem in the form of [Kno94].

Let G be a connected reductive group and X any affine G-variety, for a function f on X, we denote X_f the points of where f is non-zero, the Lie algebra acts on functions by derivations, hence we get a morphism

$$\psi_f:\ X_f \to \mathfrak{g}^*,\ :\ x \mapsto [\xi \mapsto \frac{\xi f(x)}{f(x)}]$$

let $B \subseteq G$ be a Borel subgroup and $f \in \mathbb{C}[X]$ be a highest weight vector with respect to B. Then $P_f := \{g \in G \mid gf \in \mathbb{C}^* f\}$ is a parabolic subgroup of containing B having a character χ_f with $gf = \chi_f(g)f$ for all $g \in P_f$. Let P_f^u be the unipotent radical and $\mathfrak{p}_f,\ \mathfrak{p}_f^u$ the Lie algebra of $P_f,\ P_f^u$.

Proposition 2.1. *The roots of P_f are exactly those α for which $\langle \chi_f,\ \alpha^\vee \rangle \geq 0$.*

We will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using a G-invariant scalar product $(\cdot,\cdot)$. Let $T \subseteq B$ be a maximal torus with Lie algebra $\mathfrak{t}$, for a character $\chi \in \mathfrak{t}^*$, we let χ' be the corresponding element in $\mathfrak{t}$.

Theorem 2.2. *Let $f \in \mathbb{C}[X]$ be a highest weight vector, then the morphism $\psi_f :\ X_f \to \mathfrak{g}^*$ is P_f equivariant, its image is a single P_f-orbit $\chi'_f + \mathfrak{p}_f^u$, and every isotropy group of this orbit is a Levi component of P_f.*

Corollary 2.3. *For $x \in X_f$ and let L be the isotropy group of $\psi_f(x)$ in P_f, then L is a Levi subgroup of P_f and $\Sigma := \psi^{-1}(\psi_f(x))$ is an affine L-stable subvariety of X_f such that $P_f \times^L \Sigma \to X_f$ is a P_f-equivariant isomorphism.*

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2.2. **Multiplicity free spaces.** Let V be a finite dimensional G-module.

Definition 2.4. We say that V is *multiplicity free* if its coordinate ring $\mathscr{P} := \mathbb{C}[V]$ contains every simple G-module at once.

We have a geometric criterion for multiplicity freeness

Theorem 2.5. *Let $B \subseteq G$ be a Borel subgroup, then V is multiplicity free if and only if B has an open orbit in V.*

Proof. If $\mathscr{P}$ contains two different but isomorphic simple submodules M_1 and M_2, let f_i be the highest weight vector of M_i, thn f_i are B-semiinvariant functions for the same weight, hence $h = f_1/f_2$ is a non-constant B-invariant rational function on V, this contradicts to the fact that B has an open orbit.

Conversely, assume there is no open B-orbit. Let f and Σ as in 2.2, then the action of L on Σ factors through $A = L/(L, L)$, suppose all weight spaces of $\mathbb{C}[\Sigma]$ are one-dimensional, then we can choose weight vectors $f_1, \cdots, f_r$ which generate $\mathbb{C}[\Sigma]$ as an algebra, choose $f_1, \cdots, f_r$ which generate $\mathbb{C}[\Sigma]$ as an algebra and $x_0 \in \Sigma$ with $f_i(x_0) \neq 0$ for all i, the algebra homomorphism $\mathbb{C}[\Sigma] \to \mathbb{C}[A]$ corresponds to the orbit map $a \mapsto ax_0$ is injective, hence the orbit map is dominant, Ax_0 is dense, a contradiction. We conclude we can find non-propotional weight vectors f_1, f_2 with the same weight, and these functions can be uniquely extended to B-semiinvariants on X_f. For $f >> 0$, we have two regular functions $f^N f_1$ and $f^N f_2$ with the same weights, the G-modules M_1 and M_2 generated by them are simple, different but isomorphic. □

Theorem 2.6. *Let $\mathscr{P} = \oplus_{\lambda\in\Lambda}\mathscr{P}_\lambda$ be the decomposition of $\mathscr{P}$ with $\mathscr{P}_\lambda$ a simple G-module with lowest weight $-\lambda$.*

Then there are linear independent weights $\lambda_1, \cdots, \lambda_r$ such that $\Lambda = \mathbb{N}\lambda_1 + \cdots + \mathbb{N}\lambda_r$.

Definition 2.7. Consider the pairs (Δ, Ψ) where Δ is the set of positive roots of a Levi subgroup L of G and Ψ is the multiset of weights of representation V of L.

We define (Δ, Ψ) to be multiplicity free if V is a multiplicity free representation of L. For a highest weight $\chi \in \Psi$, define $S_\chi = \{a \in \Delta \mid \langle\chi, \alpha^\vee\rangle > 0\ \}$.

We have the following criterion on multiplicity freeness

Theorem 2.8. *We have*

- *If $S_\chi = \emptyset$, then (Δ, Ψ) is multiplicity free if and only if Ψ is linearly independent.*
- *If there is a highest weight vector $\chi \in \Psi$ with $S_\chi \neq 0$, then we put $\Delta' = \Delta/S_\chi$ and $\Psi' := \Psi\backslash\{\chi-\alpha | \ \alpha \in S_\chi\}$, then (Δ, Ψ) is multiplicity free if and only if (Δ', Ψ') is multiplicity free.*

Example 2.9. For $G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on $V := \mathbb{C}^m \otimes \mathbb{C}^n$, we assume $m \leq n$, then

$$\Delta = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq m\} \cup \{\epsilon'_i - \epsilon'_j \mid 1 \leq i < j \leq n\}$$

and

$$\Psi = \{\epsilon_i + \epsilon'_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

we can take $\chi = \epsilon_1 + \epsilon'_1$, and use the previous theorem to reduce to the case $m = 1$ and show that $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ action on $V = \mathbb{C}^m \otimes \mathbb{C}^n$ is multiplicity free of rank $\min(m, n)$.

The tables of multiplicity free representations can be found in Knop's paper [Kno06].

2.3. **Harmonic analysis on multiplicity free spaces.** If V is multiplicity free and so is its dual space V^*. The coordinate ring $\mathscr{D} := \mathbb{C}[V^*]$ decomposes as $\oplus_{\lambda\in\Lambda}\mathscr{D}_\lambda$, where $\mathscr{D}_\lambda$ is a simple G-module with highest weight λ. The ring $\mathscr{D}$ can be identified with the set of constant coefficient differential operators, we are going to denote the coordinate ring of $V \oplus V^*$ by $\mathscr{P} \otimes \mathscr{D}$, taking invariants, we conclude $(\mathscr{P} \otimes \mathscr{D})^G$ decomposes as

$$\oplus_{\lambda,\mu\in\Lambda}(\mathscr{P}_\lambda \otimes \mathscr{D}_\mu)^G$$

we can think $V \oplus V^*$ as the cotangent bundle of V.

We have the multiplication map

$$m :\ \mathscr{P} \otimes \mathscr{D} \longrightarrow \mathscr{D}(V)$$

with $\mathscr{D}(V)$ the algebra of linear differential operators on V.

Definition 2.10. The functions $(\mathscr{P}_\lambda \otimes \mathscr{D}_\lambda)^G$ are called spherical functions of weight λ. The elements of $m(\mathscr{P}_\lambda \otimes \mathscr{D}_\lambda)^G$ are called Cappelli operators of weight λ.

Every weight one spherical function E_λ determines the Capelli operator $D_\lambda := m(E_\lambda)$.

Recall that the weight monoid Λ is freely generated by $\lambda_1, \cdots, \lambda_r$, let $\mathfrak{a}^*$ be the $\mathbb{C}$-vector space spanned by Λ, we are going to link $(\mathscr{P} \otimes \mathscr{D})^G$ and $\mathscr{D}(V)^G$ with $\mathfrak{a}^*$.

Let V_0 be the open B-orbit, we denote $\phi_f : V_0 \to V^* : v \mapsto \frac{df(v)}{f(v)}$, let $\mathfrak{a}^*(v) \subset V \oplus V^*$ be the set of points $(v, \phi_\chi v)$ for χ runs through $\mathfrak{a}^*$.

Proposition 2.11. *For every $v \in V^0$, the restriction to $\mathfrak{a}^*(v)$ map defines an injective homomorphism*

$$\overline{c} : (\mathscr{P} \otimes \mathscr{D})^G \longrightarrow \mathbb{C}[\mathfrak{a}^*] : h \mapsto h|_{\mathfrak{a}^*(v)}$$

Let $\rho \in \mathfrak{t}^*$ be the half-sum of positive roots we get a homomorphism

$$p : (\mathscr{D}(V))^G \longrightarrow \mathbb{C}[\mathfrak{a}^* + \rho] : D \mapsto p_D$$

where $p_D(\chi) := c_D(\chi - \rho)$, this means D acts on $\mathscr{P}_\lambda$ via $p_D(\lambda + \rho)$.

Theorem 2.12. *There is a subgroup W_V of the Weyl group W stabilizes $\mathfrak{a}^* + \rho$ such that the image of p is $\mathbb{C}[\mathfrak{a}^* + \rho]^{W_V}$, the image of $\overline{c}$ is $\mathbb{C}[\mathfrak{a}^*]^{W_V}$.*

Corollary 2.13. *The little Weyl group W_V are reflections group on $\mathfrak{a}^*$.*

Remark 2.14. There is a much more general version of this theorem, valid for all G-varieties. For smooth affine G-varieties it states that the center of invariant differential operators is a polynomial ring which is canonically isomorphic to the ring of invariants of a finite reflection group.

The polynomials $\overline{p}_\lambda := \overline{c}(E_\lambda)$ and $p_\lambda := p(D_\lambda)$ form a basis of $\mathbb{C}[\mathfrak{a}^*]^{W_V}$ and $\mathbb{C}[\mathfrak{a}^* + \rho]^{W_V}$.

3. Classification of smooth affine spherical varieties

The following is from the paper [KVS06]. We have the following corollary of Luna's slice theorem

Theorem 3.1. *Let X be a smooth affine spherical G-variety, then $X = G \times^H V$ where H is a reductive subgroup of G such that G/H is spherical and V is a spherical H-module.*

The G/H are homogeneous affine spherical varieties, spherical H-modules are the multiplicity free representations that we studied in the previous section 2.5. Knop and Steirteghem uses a Lie algebra approach to classify the smooth affine spherical varieties, and they have an axiomatic definition for what they called spherical triples. Their main result is

Theorem 3.2. *If $G \times^H V$ is a smooth affine spherical variety, then let $(\mathfrak{g}', \mathfrak{h}')$ be the Lie algebra of the derived subgroup of (G, H), we have $(\mathfrak{g}', \mathfrak{h}', V)$ is a spherical triple, and every spherical triple arises in this way.*

The result of Knop-Steirghem is not enough to deduce the Knop conjecture for smooth affine spherical varieties which characterizes the isomorphism class of smooth affine spherical varieties using the weight monoids.

Over algebraic closure, the spherical varieties are classified by spherical datum [Lun01], for the actual application to representation theory, it will be important to know the spherical datum for the smooth affine spherical varieties.

References

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