ON LOCAL FUNCTIONAL EQUATION FOR SPLIT RANK ONE SPHERICAL VARIETIES

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1. Introduction

Let k be a p-adic local field of characteristic zero with odd residue characteristic.

We will study the precise local functional equations in the paper. Using the classification of the spherical subgroups for all split rank one unramified groups, we introduce a list of basic spherical varieties (Definition 5.11). For the spherical subgroups appearing in the list, they are chosen to be the smallest spherical subgroups with respect to a given spherical system.

Let's denote by $\Delta_{\tilde{\chi}}^Y$ the morphism adjoint to $S_{\tilde{\chi}}^Y$ and F_ω the Fourier transform on the basic affine space $U\backslash G$, we denote $\underline{\Delta}_\chi^Y$ the span of $\Delta_{\tilde{\chi}}^Y$. Following the strategy of Sakellaridis, we can set up the goal of local functional equations as calculating the transition matrix:

$$F_\omega \underline{\Delta}_\chi^Y = \underline{b}_\omega^Y(\chi)\underline{\Delta}_{{}^\omega\chi}^{{}^\omega Y} \tag{1.1}$$

Since $F_{\omega_1}F_{\omega_2} = F_{\omega_1\omega_2}$, the matrices $\underline{b}_\omega^Y(\chi)$ satisfy the cocycle relations:

$$\underline{b}_{\omega_1\omega_2}^Y(\chi) = \underline{b}_{\omega_1}^{{}^{\omega_2}Y}({}^{\omega_2}\chi)\underline{b}_{\omega_2}^Y(\chi)$$

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Assume the weak local functional equations, so that we can diagonalize the matrices $\underline{b}^Y_\omega$. Then

$$F_\omega \Delta^Y_{\tilde{\chi}} = b^Y_\omega(\tilde{\chi}) \Delta^{\omega Y}_{\omega\tilde{\chi}}$$

so for the calculation of $b_\omega(\tilde{\chi})$, we may reduce this calculation to the calculation for $b^Y_{\omega_\alpha}(\tilde{\chi})$, where ω_α is a simple reflection. We develop a rank-one reduction process in Proposition 5.50 for the calculation of the local coefficients $b^Y_{\omega_\alpha}(\tilde{\chi})$ which reduces the calculation of $b^Y_{\omega_\alpha}(\tilde{\chi})$ for a general parabolic pair $(\mathbf{Y}, \alpha)$ to the calculation of $b_\omega(\tilde{\chi})$ for the following list of extended basic spherical varieties, this is the first main result of our paper

Definition 1.1. (Definition 5.14) We introduce the list of extended basic spherical varieties for $\mathrm{Res}_{\ell/k}\mathrm{SL}_2$ and $\mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$ as:

- $\mathrm{Res}_{\ell/k}\mathrm{SL}_2 \backslash \mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$.
- $\mathrm{Res}_{\ell/k}\mathrm{T} \backslash \mathrm{Res}_{\ell/k}\mathrm{SL}_2$, here T is a nonsplit torus of $\mathrm{SL}_{2,\ell}$.
- $\Gamma_{r_1} \backslash \mathrm{Res}_{\ell/k}\ \mathrm{SL}_2 \times \mathbb{G}_m$, Γ_{r_1} is the following spherical subgroup: for T a maximal split torus of $\mathrm{SL}_{2,\ell}$

$$\Gamma_{r_1}:\ \mathrm{Res}_{\ell/k}\mathrm{T} \xrightarrow{(i,r_1)} \mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$$

- $\mathrm{N}(\mathrm{Res}_{\ell/k}\mathrm{T}) \backslash \mathrm{Res}_{\ell/k}\mathrm{SL}_2$.
- $\mathrm{U} \backslash \mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, U the unipotent radical of the Borel.
- $\Gamma_{r_2} \backslash \mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$, Γ_{r_2} is the following spherical subgroup: for $\mathrm{T}_1\mathrm{U}_1$ a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\ \mathrm{SL}_2$ and k'/k is an unramified extension of degree m

$$\Gamma_{r_2}:\ \mathrm{Res}_{k'/k}\mathrm{T}_1\mathrm{U}_1 \xrightarrow{(i,r_2)} \mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$$

- $\mathrm{Res}_{k'/k}\mathrm{SL}_1(D) \backslash \mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, for ℓ/k' an unramified quadratic extension and k'/k a finite unramified extension, D a division algebra over k'.
- $\mathrm{Res}_{k'/k}\mathrm{SL}_2 \backslash \mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, for ℓ/k' an unramified cubic extension and k'/k a finite unramified extension.

We introduce the list of extended basic spherical varieties for $\mathrm{Res}_{\ell/k}\ \mathrm{SU}_3$ and $\mathrm{Res}_{\ell/k}\ \mathrm{SU}_3 \times \mathbb{G}_m$ as:

- $\mathrm{Res}_{\ell/k}\mathrm{SU}_3 \backslash \mathrm{Res}_{\ell/k}\mathrm{SU}_3$.
- $\mathrm{Res}_{\ell/k}\mathrm{SO}_3 \backslash \mathrm{Res}_{\ell/k}\mathrm{SU}_3$.
- $\mathrm{U} \backslash \mathrm{Res}_{\ell/k}\mathrm{SU}_3$, U the unipotent radical of the Borel.
- $\mathrm{Res}_{\ell/k}\mathrm{SL}_1(D) \backslash \mathrm{Res}_{\ell/k}\mathrm{SU}_3$, here D is a division algebra over k.
- $\Gamma_{r_3} \backslash \mathrm{Res}_{\ell/k}\mathrm{SU}_3 \times \mathbb{G}_m$, Γ_{r_3} is the following spherical subgroup: for $\mathrm{T}_1\mathrm{U}_1$ a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\ \mathrm{SU}_3$,

$$\Gamma_{r_3}:\ \mathrm{Res}_{k'/k}\mathrm{T}_1\mathrm{U}_1 \xrightarrow{(i,r_3)} \mathrm{Res}_{\ell/k}\mathrm{SU}_3 \times \mathbb{G}_m$$

- $\Gamma_{r_4} \backslash \mathrm{Res}_{\ell/k}\ \mathrm{SU}_3 \times \mathbb{G}_m$, Γ_{r_4} is the following spherical subgroup: for $\mathrm{T}_2\mathrm{U}_2$ a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\mathrm{SU}_3$,

$$\Gamma_{r_4}:\ \mathrm{Res}_{k'/k}\mathrm{T}_2\mathrm{U}_2 \xrightarrow{(i,r_4)} \mathrm{Res}_{\ell/k}\mathrm{SU}_3 \times \mathbb{G}_m$$

For spherical varieties appearing in this list, we will call them extended basic spherical varieties.

Our second main result of this paper is the following calculation of local functional equations:

Theorem 1.2. *(cf. Theorem ??) We have the following cases of local functional equations:*

- *For $X = SL_2(k) \backslash SL_2(\ell)$ with ℓ/k an unramified quadratic extension and $\chi \in Adm_X$, we have*

$$F_\omega \Delta_\chi = (-1)\chi^{-1}(\varpi) \frac{1 - q^{-1}\chi^{-1}(\varpi)}{1 - q^{-1}\chi(\varpi)} \Delta_{\omega\chi}$$

- *For $X = SL_2(k)\backslash SL_2(\ell)$ with ℓ/k an unramified cubic extension and distinguished unramified characters $\chi \in Adm_X$ (for the definition of distinguished unramified characters see Definition 7.9), we have*

$$F_\omega \Delta_\chi = \frac{1 - q^{-\frac{1}{2}}\chi^{-1}(\varpi)}{1 - q^{-\frac{1}{2}}\chi(\varpi)} \frac{1 - q^{-\frac{3}{2}}\chi^{-1}(\varpi)}{1 - q^{-\frac{3}{2}}\chi(\varpi)} \Delta_{^\omega\chi}$$

- *For $X = SU_2\backslash SU_3$ and $\chi \in Adm_X$, we have*

$$F_\omega \Delta_\chi = (-1)\chi^{-1}(\varpi) \frac{1 - q^{-2}\chi^{-1}(\varpi)}{1 - q^{-2}\chi(\varpi)} \Delta_{^\omega\chi}$$

This theorem is proved in Sections 6, 7, 8 by reducing the calculations to explicit computations in several multiplicity free cases.

2. Some Axioms

Let k be a p-adic local field of characteristic 0, we denote $\mathfrak{o}$ the integer ring of k, ϖ the uniformizer of $\mathfrak{o}$. We let $\mathbf{G}$ be a connected unramified reductive group with k-Borel subgroup $\mathbf{B}$, denote $K = \mathbf{G}(\mathfrak{o})$. We let $\mathbf{B} = \mathbf{TU}$ be the Levi decomposition of $\mathbf{B}$ and $\mathbf{A}$ the maximal split torus of $\mathbf{T}$. Let $\mathbf{X}$ be a homogeneous affine $\mathbf{G}$-spherical variety. We denote by $\mathbf{A_X}$ the maximal split torus of $\mathbf{X}$ and set $A_X = \mathbf{A_X}(k)$. Assume that $\mathring{\mathbf{X}}(k)$ has a single B-orbit.

We let $\Lambda_X := \mathrm{Hom}(\mathbb{G}_m, \mathbf{A_X})$ be the cocharacter lattice of $\mathbf{A_X}$. Define the set of antidominant elements of Λ_X by

$$\Lambda_X^+ = \{\ \check{\lambda} \in X_*(A_X) \mid \langle \gamma, \check{\lambda} \rangle \leq 0 \text{ for all } \gamma \in \Sigma_k(X)\}$$

We have the valuation map:

$$\mathrm{val}: \quad A_X \longrightarrow \Lambda_X$$

we define the set of antidominant elements of A_X as $\mathrm{val}^{-1}(\Lambda_X^+)$.

Identify A_X as a subset of X via a chosen base point $x_0 \in \mathring{X}$. We can state the following axiom of Cartan decomposition:

Axiom 2.1. *The set A_X^+ contains a complete set of representatives for the K-orbits on X; elements of A_X^+ that map to distinct elements of Λ_X^+ belong to different K-orbits.*

Remark 2.2. This axiom is proven in [Sak12] under assumptions which hold at almost every place if the variety is defined over a global field.

We write this correspondence as $x_{\check{\lambda}} \leftrightarrow \check{\lambda}$ with $x_{\check{\lambda}} \in A_X^+$ and $\check{\lambda} \in \Lambda_X^+$.

Denote $J \subset K$ the inverse image of $\mathbf{B}(\mathbb{F}_q)$ under the reduction map $\mathbf{G}(\mathfrak{o}) \to \mathbf{G}(\mathbb{F}_q)$.

Axiom 2.3. *For $x \in A_X^+$, we have $xJ \subset x\boldsymbol{B}(\mathfrak{o})$.*

Remark 2.4. For $X = G\backslash G \times G$ this axioms follows from the Iwahori factorization.

3. Fourier transform on basic affine spaces for quasisplit groups

In this section, we recall a construction of the Fourier transform on the basic affine spaces which generalizes the classical Fourier transform on the two-dimensional affine space.

Let k be a local non-archimedean field with norm $|\cdot|$, the ring of integers $\mathcal{O}$ and a fixed uniformizer ϖ. We fix a non-trivial additive character ψ throughout the paper.

For a quadratic extension ℓ of k we denote by $\chi_{\ell/k}$ the quadratic character of $k^\times$ associated to ℓ by class field theory.

For G an unramified group, we fix B a Borel subgroup with Levi decomposition $B = TU$, T a maximal torus of G. We let A be the maximal split torus of T and denote $W_k = W(G, A)$ the k-Weyl group of G.

3.1. $\mathbf{G} = \mathbf{SL_2}$. Let $(V, \langle \cdot, \cdot \rangle)$ be a two-dimensional symplectic space with standard basis e_1, e_2 satisfying $\langle e_1, e_2 \rangle = 1$. The group G acts on V on the right and preserves the symplectic form. Let $B = TU$ be the Borel subgroup stabilizing ke_2 and T the maximal torus. The space $X = U\backslash G$ is identified with $V\backslash\{0\}$ via $[g] \mapsto e_2 g$. The G-invariant measure ω_X on X is fixed to be the self-dual measure $|dv|$ on V with respect to the additive character ψ and the symplectic form on V.

The Fourier transform $F \in \mathrm{Aut}(C_c^\infty(V))$ is defined by the formula:

$$F(f)(\omega) = \int_V f(v)\psi(\langle \omega, v \rangle_V)\, dv \tag{3.1}$$

We have the following properties of F:

Proposition 3.1. *([GK25, Proposition 3.1]) We have*

- *F extends to a unitary involution on $L^2(V, |dv|)$.*

- $F(L_a f) = L_{a^{-1}} F f$.

We fix a character ψ on U^{op} by $\psi(x(r)) = \psi(r)$ with

$$x(r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

Then the Whittaker map $W_\psi : \ C_c^\infty(X) \to C_c^\infty(T)$ is defined by

$$W_\psi(f)(t) = \int_{U^{op}} L_t f([u]) \psi^{-1}(u) \, du$$

Proposition 3.2. *([GK25, Proposition 3.5]) Let $f \in C_c^\infty(U\backslash G)$*

$$W_\psi(Ff)(a) = W_\psi(f)(a^\omega) = W_\psi(f)(a^{-1})$$

here $a^\omega := \omega^{-1} t \omega$ for $a \in T, \ \omega \in W_k$.

3.2. $\mathbf{G = SU_3}$. Let ℓ be a quadratic extension over k with the Galois involution $x \mapsto \overline{x}$, norm map $\mathrm{Nm} : x \mapsto x\overline{x}$, and trace map $\mathrm{Tr} : x \mapsto x + \overline{x}$, we write $|\cdot|_\ell$ for the absolute value on ℓ such that $|x|_\ell = |\mathrm{Nm}(x)|_k$. We fix an element $\tau \in \mathcal{O}_k$ so that $\mathcal{O}_\ell = \mathcal{O}_k + \sqrt{\tau}\mathcal{O}_k$.

We fix on ℓ a self-dual measure dx with respect to ψ and Nm. The Fourier transform on ℓ is denoted by $\mathcal{F}_{\psi,\ell}$ to distinguish it from the Fourier transform $\mathcal{F}_\psi$ with respect to ψ and the self-dual measure on k.

Let (W, h) be the following Hermitian space

$$W = \ell^3, \quad h(v_1, v_2) = x_1\overline{z_2} + y_1\overline{y_2} + z_1\overline{x_2}$$

where $v_i = (x_i, y_i, z_i)$.

The group $G = SU(W, h)$ is the group of automorphisms of W acting on the right, that preserves the Hermitian form h of determinant 1.

Let $B = TU$ denote the Borel subgroup of G, preserving the line $\ell(0, 0, 1)$ in W, the unipotent radical U is the stabilizer of the vector $(0, 0, 1)$. The space $X = U\backslash G$ can be identified with the set W^0 of h-isotropic non-zero vectors in the space W.

The k-vector space W of dimension 6 carries the k-bilinear form $\langle v_1, v_2\rangle = \mathrm{Tr}\ h(v_1, v_2)$ and the corresponding quadratic form q is given by

$$q(v) = \langle v, v\rangle/2 = \mathrm{Tr}(x\overline{z}) + \mathrm{Nm}(y), \qquad v = (x, y, z)$$

We fix the self-dual measure dw on W with respect to ψ and q, it gives rise to a measure on the cone W^0 and hence to a measure on X which we denote by ω_X.

We fix bijections

$$x : \ell \times \sqrt{\tau}k \to U^{op} \quad t : k^\times \to T$$

by

$$x(r, y) = \begin{pmatrix} 1 & 0 & 0 \\ -\overline{r} & 1 & 0 \\ -\frac{\mathrm{Nm}(r)}{2} + y & r & 1 \end{pmatrix} \quad t(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1}\overline{a} & 0 \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \quad a \in \ell^\times$$

We fix on ℓ a ψ-selfdual measure with respect to the pairing $(r, s) \mapsto \mathrm{Tr}_{\ell/k}(r\overline{s})$ and a ψ-selfdual measure on k. Under the isomorphism x we obtain a Haar measure

$$du = x_*(d\ell \ dk) \tag{3.2}$$

on U^{op}.

There exists an unitary involution $F \in \mathrm{Aut}(L^2(X, \omega_X))$, the Fourier transform ([GK25, Theorem 4.17]) that satisfies

$$L_t F(f) = F L_{\omega t}(f)$$

for $f \in C_c^\infty(X)$, $t \in T$.

We fix a character ψ on U^{op} by $\psi(x(r,y)) = \psi(r)$. The Whittaker map $W_\psi :\ C_c^\infty(X) \to C_c^\infty(T)$ is defined by

$$W_\psi(f)(t) = \int_{U^{op}} L_t f([u])\psi^{-1}(u)\ du$$

Proposition 3.3. *([GK25, Proposition 4.8]) Let $f \in C_c^\infty(U\backslash G)$,*

$$W_\psi(Ff)(a) = W_\psi(f)(t(-1)a^\omega)$$

here $a^\omega := \omega^{-1}a\omega$ for $a \in T,\ \omega \in W_k$.

3.3. **The Schwartz space.** In this section, G will be the k-points of a connected unramified group $\mathbf{G}$ over k, $B = TU$ a Borel subgroup of G and T its maximal torus. We denote W_k the k-Weyl group of G and let $\mathbf{X}$ be a $\mathbf{G}$-spherical variety.

Using the construction of the Fourier transform for SL_2 and SU_3, in the proof of [GK25, Theorem 1.2], for $\omega \in W_k$ with a reduced expression $\omega = \omega_1 \cdots \omega_n$ as a product of simple reflections, they introduce $F_\omega :\ L^2(U\backslash G) \to L^2(U\backslash G)$ as

$$F_\omega = F_{\omega_1} \cdots F_{\omega_n}$$

It is clear from the definition that F_ω is unitary and $F_{\omega_1\omega_2} = F_{\omega_1}F_{\omega_2}$ for $\omega_1,\ \omega_2 \in W_k$.

Definition 3.4. We define the Schwartz space $\mathcal{S}(U\backslash G)$ to be the smallest subspace which contains $C_c^\infty(U\backslash G)$ and is closed under all Fourier transforms.

We define the unitary action of $a \in T$ on $L^2(U\backslash G)$ as:

$$(L_a f)(Ug) = \delta^{-\frac{1}{2}}(a)f(Uag)$$

We obtain:

$$F_\omega(L_a f) = L_{\omega a}F_\omega f$$

Let's consider the Mellin transform $R_\chi : \mathcal{S}(U\backslash G) \to I(\chi)$ given by the rational continuation of the integral:

$$f \mapsto \int_T L_a f\chi^{-1}(a)\ da$$

Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{S}(U\backslash G) & \xrightarrow{F_\omega} & \mathcal{S}(U\backslash G) \\ \downarrow{\scriptstyle R_\chi} & & \downarrow{\scriptstyle R_{\omega\chi}} \\ I(\chi) & \xrightarrow{F_{\omega,\chi}} & I({}^\omega\chi) \end{array}$$

we will call $F_{\omega,\chi}$ the fiberwise Fourier transform. The family $F_{\omega,\chi}$ is rational at χ. Thus at parameters at which one of the operators has a pole, all identities involving $F_{\omega,\chi}$ are understood as identities of rational families.

For $\omega \in W_k$, χ an unramified character of T. $\mathbf{Y} \in \mathfrak{B}_k(X)$ and $\tilde{\chi} \in \mathrm{Adm}_Y$, $\Delta_{\tilde{\chi}}^Y$ the associated intertwining morphism, we want to introduce $F_\omega\Delta_{\tilde{\chi}}^Y$ which will be used later for the calculation for the local functional equations.

Recall that we have defined the standard intertwining operator T_ω with the Chevalley normalization on page **??**. Given the action of the rational family of operators $T_\omega :\ I(\chi) \to I({}^\omega\chi)$, its smooth transpose is

$$T_\omega^\vee :\ I({}^\omega\chi^{-1}) \longrightarrow I(\chi^{-1})$$

as the contragredient $I(\chi)^\vee$ of $I(\chi)$ is isomorphic to $I(\chi^{-1})$.

Now for the generalized section $I(\chi)^{-\infty} := \mathrm{Hom}_{cont}(I(\chi)^\vee, \mathbb{C})$. We define the extension

$$T_\omega^{-\infty} : \ I(\chi)^{-\infty} \longrightarrow I({}^\omega\chi)^{-\infty}$$

by $(T_\omega^{-\infty} u)(\lambda) := u(T_\omega^\vee \lambda)$.

Under the duality between $I(\chi)$ and $I(\chi^{-1})$, it is a multiple of the adjoint of $T_{\omega^{-1}}$ applied to $I({}^\omega\chi^{-1})$. We may apply T_ω to the generalized vector $ev_\xi \circ \Delta_{\tilde{\chi}}^Y \in I(\chi)^{-\infty}$ where $\xi \in X$, and we define $T_\omega \Delta_{\tilde{\chi}}^Y$ to be the morphism:

$$I({}^\omega\chi) \longrightarrow C^\infty(X)$$

for which

$$ev_\xi \circ T_\omega \Delta_{\tilde{\chi}}^Y = T_\omega(ev_\xi \circ \Delta_{\tilde{\chi}}^Y)$$

holds for every $\xi \in X$. Under the Chevalley normalization (**??**) of T_ω, we have:

$$T_\omega \Delta_{\tilde{\chi}}^Y = \Delta_{\tilde{\chi}}^Y \circ T_{\omega^{-1}}$$

For the fiberwise Fourier transform $F_{\omega,\chi} : I(\chi) \to I({}^\omega\chi)$, we can define $F_\omega \Delta_{\tilde{\chi}}^Y$ in a similar way, we have

$$F_\omega \Delta_{\tilde{\chi}}^Y = \Delta_{\tilde{\chi}}^Y \circ F_{\omega^{-1}}$$

3.4. **Comparison of Fourier transform and standard intertwining operator.** In this section, let k be a p-adic local field of characteristic 0. We fix ψ an additive character of k with conductor $\mathcal{O}$. We fix ℓ/k an unramified quadratic extension.

We assume $G = SL_2$, we denote $B = AU$ the upper triangular Borel subgroup of G with A the diagonal maximal torus and U the unipotent radical, denote U^{op} the opposite unipotent radical of U. Denote $X = U^{op}\backslash SL_2$, we introduce the Whittaker model

$$C^\infty(X, \mathcal{L}_\psi) = \{W \in C^\infty(G) \ : \ W(ug) = \psi(u)W(g) \text{ for every } u \in U^{op}\}$$

For $\varphi \in I(\chi)$, we introduce the Whittaker map $\Delta_\chi : \ I(\chi) \to C^\infty(X, \mathcal{L}_\psi)$,

$$(\Delta_\chi \varphi)(g) = \int_k \varphi(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} g)\psi(-x)\ dx$$

We denote $\ell_\chi := ev_{U^{op}1} \circ \Delta_\chi$. Let $f \in C_c^\infty(U\backslash SL_2)$, and $R_\chi : \ C_c^\infty(U\backslash SL_2) \to I(\chi)$ the Mellin map

$$R_\chi(f) = \int_A L_a(f)\chi^{-1}(a)\ da$$

By definition

$$(\Delta_\chi \phi)(1) = \int_k \phi(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix})\psi(-x)\ dx$$

Now let $\phi = R_\chi(f)$, we get in the region of absolute convergence:

$$\begin{aligned}
\ell_\chi(R_\chi(f)) &= \int_k R_\chi(f)(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix})\psi(-x)\ dx \\
&= \int_k \int_A (L_a f)(U \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix})\chi^{-1}(a)\psi(-x)\ da\ dx \\
&= \int_A \left(\int_k (L_a f)(U \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}) \psi(-x)\ dx \right) \chi^{-1}(a)\ \ da \\
&= \int_A W_\psi(f)(a)\chi^{-1}(a)\ da
\end{aligned}$$

There is no additional scalar because we use the same additive character ψ and the measure dx on $k \cong U^{op}$ in the definition for the Whittaker map Δ_χ and W_ψ.

Proposition 3.5. *In the convergent region for the Whittaker map, we have*

$$\ell_{\omega\chi} \circ F_\omega = \ell_\chi$$

Here the Fourier transform is defined as in Section 3.1.

Proof. Using the property $W_\psi(F_\omega f)(a) = W_\psi(f)(a^\omega)$ (Proposition 3.2),

$$\begin{aligned}
\ell_{\omega\chi}(F_\omega R_\chi(f)) &= \ell_{\omega\chi}(R_{\omega\chi}(Ff)) \\
&= \int_A W_\psi(Ff)(a)({}^\omega\chi^{-1})(a)\, da \\
&= \int_A W_\psi(f)(a^\omega)\chi(a^\omega)^{-1}\, da \\
&= \int_A W_\psi(f)(a)\chi^{-1}(a)\, da \\
&= \ell_\chi(R_\chi(f))
\end{aligned}$$

Hence $\ell_{\omega\chi} \circ F_\omega = \ell_\chi$. □

We recall the fiberwise Fourier transform $F_{\omega,\chi}$ as

$$\begin{array}{ccc}
\mathcal{S}(U\backslash G) & \xrightarrow{F_\omega} & \mathcal{S}(U\backslash G) \\
\downarrow{\scriptstyle R_\chi} & & \downarrow{\scriptstyle R_{\omega\chi}} \\
I(\chi) & \xrightarrow{F_{\omega,\chi}} & I({}^\omega\chi)
\end{array}$$

Corollary 3.6. *For χ an unramified character of $k^\times$, we have*

$$F_{w,\chi} = \frac{1-\chi(\varpi)}{1-q^{-1}\chi^{-1}(\varpi)} T_\omega$$

here we view both sides as rational families of operators in the variable $z = \chi(\varpi)$ and T_ω is normalized by $vol(U(\mathfrak{o})) = 1$ as in [CS80].

Proof. This is because from [CS80, Proposition 4.3], we have

$$T_\omega \Delta_\chi = \frac{1-\chi(\varpi)}{1-q^{-1}\chi(\varpi)^{-1}} \Delta_{\omega\chi}$$

Then we have

$$F_{w,\chi} = \frac{1-\chi(\varpi)}{1-q^{-1}\chi^{-1}(\varpi)} T_{\omega,\chi}$$

in the convergent region for the Whittaker map ℓ_χ.

Since we have $F_{\omega,\chi} = C(\chi)T_{\omega,\chi}$ for $C(\cdot)$ a rational function in χ, we conclude

$$F_{w,\chi} = \frac{1-\chi(\varpi)}{1-q^{-1}\chi^{-1}(\varpi)} T_{\omega,\chi}$$

for generic unramified character χ. □

From now on we assume $G = SU_3$ the unramified unitary group associated with an ℓ-Hermitian vector space, $B = TU$ the upper triangular Borel subgroup of SU_3, U^{op} the opposite unipotent subgroup.

Via the parametrization $x : \ell \times \sqrt{\tau}\, k \to U^{op}$, we equip U^{op} with the pushforward of the fixed product Haar measure $d\ell\, dk$ as in (3.2). The same measure will be used for the Whittaker map Δ_χ and W_ψ below.

Denote $X = U^{op}\backslash SU_3$, we introduce the Whittaker model

$$C^\infty(X, \mathcal{L}_\psi) = \{W \in C^\infty(G) \ : \ W(ug) = \psi(u)W(g) \text{ for every } u \in U^{op}\}$$

here $\psi : U^{op} \to \mathbb{C}^\times$ is defined in Section 3.2. Let

$$\Delta_\chi : \ I(\chi) \longrightarrow C^\infty(X, \mathcal{L}_\psi)$$

be the Whittaker map and $\ell_\chi := ev_{U^{op}1} \circ \Delta_\chi$.

In the region of absolute convergence, the Whittaker map is given by the Jacquet integral

$$(\Delta_\chi \phi)(g) = \int_{U^{\mathrm{op}}} \phi(ug)\psi^{-1}(u)\, du.$$

Let $f \in C_c^\infty(U\backslash SU_3)$ and $R_\chi : C_c^\infty(U\backslash SU_3) \to I(\chi)$ be the Mellin map

$$R_\chi(f) = \int_T (L_a f)\chi^{-1}(a)\ da$$

Consequently, evaluation at the base point $U^{\mathrm{op}} \cdot 1$ gives

$$\ell_\chi(\phi) = (\Delta_\chi \phi)(1) = \int_{U^{\mathrm{op}}} \phi(u)\psi^{-1}(u)\, du.$$

Applying this to $\phi = R_\chi(f)$ and using the definition

$$R_\chi(f)(g) = \int_T (L_t f)(Ug)\chi^{-1}(t)\, dt,$$

we obtain

$$\begin{aligned}
\ell_\chi(R_\chi(f)) &= \int_{U^{\mathrm{op}}} R_\chi(f)(u)\psi^{-1}(u)\, du \\
&= \int_{U^{\mathrm{op}}} \int_T (L_t f)([u])\chi^{-1}(t)\psi^{-1}(u)\, dt\, du \\
&= \int_T \left(\int_{U^{\mathrm{op}}} (L_t f)([u])\psi^{-1}(u)\, du \right) \chi^{-1}(t)\, dt \\
&= \int_T W_\psi(f)(t)\chi^{-1}(t)\, dt.
\end{aligned}$$

Proposition 3.7. *In the convergent region for ℓ_χ, we have*

$$\ell_{\omega\chi} \circ F_\omega = \ell_\chi$$

here F_ω is the SU_3 Fourier transform as defined in Section 3.2.

Proof. Using the property $W_\psi(Ff)(a) = W_\psi(f)(t(-1)a^\omega)$ (Proposition 3.3),

$$\begin{aligned}
\ell_{\omega\chi}(F_\omega R_\chi(f)) &= \ell_{\omega\chi}(R_{\omega\chi}(Ff)) \\
&= \int_T W_\psi(Ff)(a)({}^\omega\chi^{-1})(a)\ da \\
&= \int_T W_\psi(f)(t(-1)a^\omega)\chi(a^\omega)^{-1}\ da \\
&= \int_T W_\psi(f)(t(-1)a^\omega)\chi(t(-1)a^\omega)^{-1}\ da \\
&= \int_T W_\psi(f)(a)\chi^{-1}(a)\ da \\
&= \ell_\chi(R_\chi(f))
\end{aligned}$$

here we use χ is unramified, hence $\chi(t(-1)) = 1$. □

We recall the fiberwise Fourier transform $F_{\omega,\chi}$ as

$$\begin{array}{ccc} \mathcal{S}(U\backslash G) & \xrightarrow{F_\omega} & \mathcal{S}(U\backslash G) \\ \downarrow{\scriptstyle R_\chi} & & \downarrow{\scriptstyle R_{\omega_\chi}} \\ I(\chi) & \xrightarrow{F_{\omega,\chi}} & I({}^\omega\chi) \end{array}$$

Corollary 3.8. *For χ an unramified character of $\ell^\times$, we have*

$$F_{w,\chi} = \frac{1-\chi^2(\varpi)}{(1-q^{-2}\chi(\varpi)^{-1})(1+q^{-1}\chi(\varpi)^{-1})} T_{\omega,\chi} \tag{3.3}$$

here we view both sides as rational families of operators in the variable $z = \chi(\varpi)$ and T_ω is normalized by vol$(U(\mathfrak{o})) = 1$ as in [CS80].

Proof. This is because from [CS80, Proposition 4.3], we have

$$T_\omega \Delta_\chi = \frac{(1-q^{-2}\chi(\varpi)^{-1})(1+q^{-1}\chi(\varpi)^{-1})}{1-\chi(\varpi)^2} \Delta_{{}^\omega\chi}$$

Then we have

$$F_{w,\chi} = \frac{1-\chi^2(\varpi)}{(1-q^{-2}\chi(\varpi)^{-1})(1+q^{-1}\chi(\varpi)^{-1})} T_{\omega,\chi}$$

in the convergent region for the Whittaker map ℓ_χ.

Since we have $F_{\omega,\chi} = C(\chi)T_{\omega,\chi}$ for $C(\cdot)$ a rational function in χ, we conclude

$$F_{w,\chi} = \frac{1-\chi^2(\varpi)}{(1-q^{-2}\chi(\varpi)^{-1})(1+q^{-1}\chi(\varpi)^{-1})} T_{\omega,\chi}$$

for generic unramified character χ. □

4. Construction of the K-invariant vector in the Schwartz function space

In this section, let k be a p-adic field of characteristic zero, we denote $\mathfrak{o}$ the integer ring of k and ϖ its uniformizer, $\mathbb{F}_q = \mathfrak{o}/\varpi\mathfrak{o}$ the residue field. Let G be the k-points of a connected unramified group $\mathbf{G}$ over k, $K = G(\mathfrak{o})$. We fix $\mathbf{B}$ a k-Borel subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{B} = \mathbf{TU}$.

We introduce the standard Iwahori subgroup J using,

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{G}(\mathfrak{o}) \\ \uparrow & & \uparrow \\ J & \longrightarrow & \mathbf{B}(\mathbb{F}_q) \end{array}$$

We denote $X = U\backslash G$. We are going to construct a function $\Phi_K \in \mathcal{S}(X)^K$.

4.0.1. $G = SU_3$. In this subsection, let ℓ/k be an unramified quadratic extension of p-adic fields, we let $G = SU_3$ be the k-points of an unramified unitary group $\mathbf{SU}_3$ associated with an ℓ-Hermitian vector space. We fix B the upper triangular Borel subgroup of G and T its maximal torus.

Following Sakellaridis' normalization, normalize the Haar measure so that vol$(K) = 1$, then vol$(J) = \frac{1}{1+q^3}$. We denote $a = \mathrm{diag}(\varpi, 1, \varpi^{-1})$ and $z = \chi(a)$ for χ an unramified character of T.

For $\Phi \in \mathcal{S}(U\backslash G)^K$, we denote

$$e^{\check{\alpha}} \cdot \Phi := L_{e^{\check{\alpha}}(\varpi)}\Phi$$

Note that we have the following formula (3.3) which compares F_ω with T_ω: We have

$$F_{\omega,\chi} = T_{\omega,\chi}\frac{1-z^2}{(1+q^{-1}z^{-1})(1-q^{-2}z^{-1})} = T_{\omega,\chi}a(\chi)$$

We define the projection operator P_χ as

$$\begin{aligned} P_\chi :\ C_c^\infty(G) &\longrightarrow I(\chi) \\ f &\longmapsto P_\chi(f)(\cdot) = \int_B \chi^{-1}(b)\delta^{1/2}(b)f(b\cdot)\ d_\ell b \end{aligned}$$

Here $d_\ell b$ is the left Haar measure on B.

Definition 4.1. We introduce the Iwahori basis for $I(\chi)^J$ as

$$\begin{aligned} \phi_1^\chi &:= P_\chi(1_J) \\ \phi_\omega^\chi &:= P_\chi(1_{J\omega J}) \end{aligned}$$

Because $K = J \sqcup J\omega J$, we get

$$\phi_K := P_\chi(1_K) = P_\chi(1_J) + P_\chi(1_{J\omega J}) = \phi_1^\chi + \phi_\omega^\chi$$

Proposition 4.2. *([Cas80, Theorem 3.1]) Under the Chevalley normalization of the intertwining operator T_ω, we have the following identity,*

$$T_{\omega,{}^\omega\chi}(\phi_K^{{}^\omega\chi}) = c({}^\omega\chi)\phi_K^\chi$$

Here

$$c_\omega(\chi) := \frac{(1-q^{-2}z)(1+q^{-1}z)}{1-z^2}$$

Then we have the following computation on the action of the standard intertwining operator on the Iwahori basis.

Proposition 4.3. *([Cas80, Theorem 3.4]) We have*

$$T_{\omega,{}^\omega\chi}(\varphi_1^{{}^\omega\chi}) = (c({}^\omega\chi)-1)\varphi_1^\chi + q^{-3}\varphi_\omega^\chi$$

More explicitly, using the formula $c({}^\omega\chi) = \frac{(1-q^{-2}z^{-1})(1+q^{-1}z^{-1})}{1-z^{-2}}$, we get

$$T_{\omega,{}^\omega\chi}(\varphi_1^{{}^\omega\chi}) = (\frac{(1-q^{-2}z^{-1})(1+q^{-1}z^{-1})}{1-z^{-2}} - 1)\varphi_1^\chi + q^{-3}\varphi_\omega^\chi$$

We introduce a function $\Phi_K := \mathrm{vol}(J)^{-1}(\Phi_J - e^{\check{\alpha}}F\Phi_J)$, here the k-simple roots of SU_3 are $\{a, 2a\}$.

Proposition 4.4. *We have*

$$R_\chi(\Phi_K) = (1+q^{-3})\frac{1-z^2}{(1-q^{-2}z)(1+q^{-1}z)}\varphi_{K,\chi}$$

Proof. By construction, we have

$$R_\chi(\Phi_K) = \mathrm{vol}(J)^{-1}R_\chi(\Phi_J - e^{\check{\alpha}}F\Phi_J)$$

we have $R_\chi(\Phi_J) = \varphi_1$ and since we have

$$\begin{array}{ccc} \mathcal{S}(U\backslash G) & \xrightarrow{F_\omega} & \mathcal{S}(U\backslash G) \\ \downarrow{\scriptstyle R_{\omega_\chi}} & & \downarrow{\scriptstyle R_\chi} \\ I({}^\omega\chi) & \xrightarrow{a({}^\omega\chi)T_{\omega,{}^\omega\chi}} & I(\chi) \end{array}$$

We deduce

$$R_\chi(e^{\check{\alpha}}(\chi)F\Phi_J) = e^{\check{\alpha}}(\chi)\frac{(1-z^{-2})}{(1+q^{-1}z)(1-q^{-2}z)}T_{\omega,{}^\omega\chi}({}^\omega\varphi_1)$$

Using Proposition 4.3, we can calculate the coefficient of φ_1 in $\mathrm{vol}(J)R_\chi(\Phi_K)$ as

$$\varphi_1 - \frac{z^2(1-z^{-2})}{(1+q^{-1}z)(1-q^{-2}z)}\frac{(q^{-1}-q^{-2})z+(1-q^{-3})}{1-z^{-2}}\varphi_1 = \frac{q^{-3}(1-z^2)}{(1+q^{-1}z)(1-q^{-2}z)}\varphi_1$$

We can calculate the coefficient of φ_ω as

$$\frac{(1-z^2)q^{-3}}{(1-q^{-2}z)(1+q^{-1}z)}\varphi_\omega$$

hence

$$R_\chi(\Phi_K) = (1+q^3)q^{-3}\frac{(1-z^2)}{(1-q^{-2}z)(1+q^{-1}z)}(\varphi_1+\varphi_\omega) = (1+q^{-3})\frac{(1-z^2)}{(1-q^{-2}z)(1+q^{-1}z)}\varphi_{K,\chi}$$

□

Proposition 4.5. *For any $f \in \mathcal{S}(X)^L$, where $L \subset K$ is an open compact subgroup of finite index, one can write*

$$f = f_0 + Fg$$

with $f_0,\ g \in C_c^\infty(X)^L$.

Proof. Let e_L denote the projection operator

$$e_L h(x) = \frac{1}{\mathrm{vol}(L)}\int_L h(x\ell)\,d\ell$$

then we have $e_L h \in C_c^\infty(X)^L$ whenever $h \in C_c^\infty(X)$. Suppose $f \in \mathcal{S}(X)^L$. From the definition of the space, we have

$$\mathcal{S}(X) = C_c^\infty(X) + FC_c^\infty(X)$$

so we may choose $f = f' + Fg'$, since f is L-invariant, we have $f = e_L f$. Since $e_L F = F e_L$ because the Fourier transform is G-equivariant, we get

$$f = e_L f' + e_L F(g') = e_L f' + F(e_L g')$$

put $f_0 = e_L f'$ and $g = e_L g'$, then we have

$$f = f_0 + Fg$$

with $f_0,\ g \in C_c^\infty(X)^L$. □

Fix an isomorphism

$$\mathbf{T}(k)/\mathbf{T}(\mathfrak{o}) \cong \mathbb{Z}$$

for $\mathbf{T}$ the maximal torus of $\mathbf{SU}_3$, and let $n \mapsto t_\varpi^n \in \mathbf{T}(k)$ be this map. We denote $x^n = \chi\delta^{1/2}(t_\varpi^n)$.

By definition, the unramified principal series consists of locally constant functions $f : G \longrightarrow \mathbb{C}$ such that

$$f(bg) = \delta^{1/2}(b)\chi(b)f(g)$$

for all $b \in B,\ g \in G$. By the Iwasawa decomposition, $f \in I(\chi)$ is determined by its restriction to K, and we have

$$f(b_0 k) = \delta^{1/2}(b_0)\chi(b_0)f(k) = f(k)$$

for $b_0 \in B \cap K$, and hence we have

$$I(\chi)|_K \cong C^\infty(B\cap K\backslash K)$$

For $L \subset K$ an open compact subgroup of finite index, denote $V_L = C^\infty(B \cap K \backslash K / L)$, then from $I(\chi)|_K \cong C^\infty(B \cap K \backslash K)$, we have a canonical identification

$$I(\chi)^L \cong V_L$$

let $k_1, \cdots, k_r$ be a finite set of representatives for $B \cap K \backslash K / L$. Now for $f \in \mathcal{S}(X)^L$, we have $R_\chi(f)$ is determined by

$$(R_\chi(f)(k_1), \cdots R_\chi(f)(k_r))$$

Note that

$$R_\chi(f)(k_i) = \int_T L_t f([k_i]) \chi^{-1}(t) \, dt$$

under the isomorphism $\mathbf{T}(k)/\mathbf{T}(\mathfrak{o}) \cong \mathbb{Z}$ with generator t_ϖ, we may write

$$R_\chi(f)(k_i) = \sum_{n \in \mathbb{Z}} c_{n,i} x^n$$

hence we get $(R_\chi(f)(k_1), \cdots R_\chi(f)(k_r)) \in (\mathbb{C}((x)))^r$.

We have the following proposition:

Proposition 4.6. *Let $L \subset K$ be an open compact subgroup of finite index. For $f \in \mathcal{S}(X)^L$, denote $R_\chi(f) \in I(\chi)^L$, then we have $R_\chi(f)(k_i) \in \mathbb{C}(x)$.*

Proof. Let's denote

$$C_L(x) : I(\chi)^L \longrightarrow I(\chi^{-1})^L$$

the matrix of the normalized intertwining operator action (with normalization coming from the Fourier transform) on the L-invariant vectors of $I(\chi)$. We have $C_L(x) \in M_r(\mathbb{C}(x))$ by the calculation of the c-function, where $r := \dim_{\mathbb{C}} I(\chi)^L$ is finite, and we identify $I(\chi)^L$ as a $\mathbb{C}$-vector space of dimension r.

We may write $f = f_0 + Fg$ with $f_0 \in \mathcal{S}(X)^L$ and $g \in \mathcal{S}(X)^L$. Then $R_\chi(f_0)(k_i) \in \mathbb{C}[x]$ as $f_0 \in C_c^\infty(X)^L$.

Note

$$R_\chi(Fg)(k_i) = c_L(x) \cdot R_{\chi^{-1}}(g)(k_i) \in \mathbb{C}(x)$$

Altogether, we conclude $R_\chi(f)(k_i) \in \mathbb{C}(x)$. □

We deduce the following separation lemma for the Schwartz function space:

Lemma 4.7. *Let $L \subset K$ be an open compact subgroup of finite index. Suppose $f \in \mathcal{S}(U \backslash G)^L$ and f is invariant under the left $\boldsymbol{T}(\mathfrak{o})$-action, then if $R_\chi(f) = 0$ for a Zariski-dense set of unramified characters $\chi \in T^*$ at which the Mellin transform is defined, then we have $f = 0$.*

Proof. Put $T^0 = \mathbf{T}(\mathfrak{o})$ and choose representatives $k_1, \cdots, k_r$ for $B \cap K \backslash K / L$. By Proposition 4.6, the map

$$\psi(f) = (R_\chi(f)(k_1), \cdots, R_\chi(f)(k_r))$$

is a rational family on the torus of unramified characters. Since it vanishes on a Zariski-dense set, it vanishes identically as a rational family. Thus $R_\chi(f) = 0$ at every χ where R_χ is defined and in particular for every unramified unitary character χ.

We introduce

$$||R_\chi(f)|_K||^2_{L^2(K)} = \int_K |R_\chi(f)(k)|^2 \, dk$$

The Plancherel formula for $L^2(U \backslash G)$ gives

$$||f||^2_{L^2(U \backslash G)} = c \int_{\widehat{T/T^0}} ||R_\chi(f)|_K||^2_{L^2(K)} \, d\chi$$

here $c > 0$ is a constant and $d\chi$ denotes the Haar probability measure on the compact group of unitary characters T/T^0.

Since $\mathcal{S}(U\backslash G) \subset L^2(U\backslash G)$ and the integrand vanishes almost eveverywhere, we obtain $||f||_{L^2(U\backslash G)} = 0$ and hence $f = 0$ almost everywhere. Since f is right L-invariant, this implies $f = 0$. □

Corollary 4.8. *We have*

$$\Phi_K = vol(J)^{-1}(\Phi_J - e^{\check{a}} F \Phi_J) \in \mathcal{S}(U\backslash G)^K$$

Proof. Let $n \in K$ represent the nontrivial Weyl group element, since K is generated by J and n and Φ_K is J-invariant, it is enough to prove

$$R_n \Phi_K = \Phi_K$$

set $f_n = R_n\Phi_K - \Phi_K$. Then f_n is right invariant under $L = J \cap n^{-1}Jn$, now from Proposition 4.4, we know that $R_\chi(\Phi_K)$ is proportional to the spherical vector $\phi_{K,\chi} \in I(\chi)^K$. Hence

$$R_\chi(f_n) = R_\chi(R_n\Phi_K - \Phi_K) = n \cdot R_\chi(\Phi_K) - R_\chi(\Phi_K) = 0$$

for a generic family of χ.

By the separation Lemma 4.7, we get $f_n = 0$, therefore

$$R_n \Phi_K = \Phi_K$$

Because Φ_K is already J-invariant and $K = \langle J, n\rangle$, we get $\Phi_K \in \mathcal{S}(U\backslash G)^K$. □

4.1. **Construction in general.** In this section, let k be a p-adic field of characteristic zero. G will be the k-points of a connected unramified group $\mathbf{G}$. Fix a Borel subgroup $B \subset G$ with decomposition $B = TU$, and let $A \subset T$ be the maximal k-split torus. The k-root system is

$$\Phi_k(G) := \mathrm{res}_A(\Phi(G_{\overline{k}}, T_{\overline{k}}))\backslash\{0\}$$

We denote $\Phi^+(G_{\overline{k}}, T_{\overline{k}})$ the set of positive roots and $\Phi^-(G_{\overline{k}}, T_{\overline{k}})$ the set of negative roots. We denote

$$\Phi_k^+(G) = \mathrm{res}_A(\Phi^+(G_{\overline{k}}, T_{\overline{k}}))\backslash\{0\},\ \Phi_k^-(G) = -\Phi_k^+(G)$$

We introduce the reduced k-root system

$$\Phi_k^{re}(G) := \{a \in \Phi_k(G) \mid 2a \notin \Phi_k(G)\}$$

and $\Phi_k^{re,\pm}(G) := \Phi_k^{re}(G) \cap \Phi_k^{\pm}(G)$. Then $\Phi_k^{re}(G)$ is a reduced root system with Weyl group W_k.

For every k-simple root β, define

$$\tilde{\beta} = \begin{cases} \beta, & \text{if } 2\beta \notin \Phi_k(G) \\ 2\beta, & \text{if } 2\beta \in \Phi_k(G) \end{cases}$$

For every $\alpha \in \Phi_k^{re}(G)$, we denote

$$\alpha_{ind} = \begin{cases} \alpha, & \alpha/2 \notin \Phi_k(G) \\ \alpha/2, & \alpha/2 \in \Phi_k(G) \end{cases} \qquad \lambda_\alpha := (\alpha_{ind})^\vee$$

then $\lambda_{\omega\alpha} = \omega\lambda_\alpha$ and $\lambda_{\tilde{\beta}} = \check{\beta}$.

We have the following construction of K-invariant vector Φ_K, note that the expression below can also be read from the Casselman-Shalika formula for spherical functions. That is, our construction of Φ_K recovers the Casselman-Shalika formula formally.

Proposition 4.9. *Let F_ω be the Fourier transform as defined in Section 3.3. Define*

$$\Phi_K := vol(J)^{-1} \sum_{w \in W_k} \Big(\prod_{\alpha \in \Phi_k^{re,+}(G),\ w^{-1}\alpha \in \Phi_k^{re,-}} (-e^{\lambda_\alpha}) \Big) F_w \Phi_J$$

We have $\Phi_K \in \mathcal{S}(U\backslash G)^K$.

Proof. For $w \in W_k$, we denote

$$\mathrm{Inv}(w) = \{\alpha \in \Phi_k^{re,+}(G) \mid w^{-1}\alpha \in \Phi_k^{re,-}(G)\}$$

For a k-simple root β, we denote K_β the corresponding parahoric subgroup. It is enough to prove K_β-invariance for every k-simple root β, because K is generated by the Iwahori subgroup J and the rank-one parahorics K_β.

Fix β, let $W_\beta = \{1, w_\beta\}$, and let $W^\beta = [W/W_\beta]$ be the minimal-length coset representatives. If $w \in W^\beta$, then

$$\mathrm{Inv}(ww_\beta) = \mathrm{Inv}(w) \sqcup \{w\tilde{\beta}\}$$

therefore we have

$$(\prod_{\alpha \in \mathrm{Inv}(w)} (-e^{\lambda_\alpha}))(F_w \Phi_J - e^{w\lambda_{\tilde{\beta}}} F_{ww_\beta} \Phi_J)$$

Using the Fourier-transform equivariance $F_w e^{\lambda_{\tilde{\beta}}} = e^{w\lambda_{\tilde{\beta}}} F_w$, we get

$$(\prod_{\alpha \in \mathrm{Inv}(w)} (-e^{\lambda_\alpha}))F_w(\Phi_J - e^{\lambda_{\tilde{\beta}}} F_{w_\beta} \Phi_J)$$

but the vector $\Phi_J - e^{\lambda_{\tilde{\beta}}} F_{w_\beta} \Phi_J$ is K_β -invariant by our result for SU_3 (Corollary 4.8) and Sakellaridis' result for SL_2 [Sak13, §4.1]. Since F_w commutes with the right G-action, applying F_w preserves right K_β-invariance. Since Φ_K is K_β-invariant for every β, we deduce that Φ_K is K-invariant. □

5. Reduction process

5.1. **The local functional equation.** The goal of this paper is to set up the framework for computing eigenfunctions of the Hecke algebra on $C^\infty(X)$ for X a homogeneous G-spherical variety. The details of the present section are quite technical, therefore we give here an overview of the contents.

In this section, let k be a p-adic local field of characteristic 0. Let G be the k-points of a connected reductive group $\mathbf{G}$ over k, we fix $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ and $B = \mathbf{B}(k)$. Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a homogeneous spherical variety. We will state the local functional equation for $X = \mathbf{X}(k)$. We let $I(\chi) = \mathrm{Ind}_B^G(\chi\delta^{\frac{1}{2}})$ be an unramified principal series representation. We will choose ω_X a G-invariant eigenmeasure on X.

We obtain a duality,

$$C_c^\infty(X) \otimes \nu \otimes C^\infty(X) \to \mathbb{C} \tag{5.1}$$

$$\phi \otimes |\omega_X| \otimes f \mapsto \int_X \phi f |\omega_X| \tag{5.2}$$

where ν is the character of $|\omega_X|$.

For $Y \in \mathfrak{B}_k(X)$ and $\tilde{\chi} \in \mathrm{Adm}_Y$. We will denote the intertwining morphism by

$$S_{\tilde{\chi}}^Y : C_c^\infty(X) \longrightarrow I(\chi)$$

We view $I(\chi)$ as a subspace of the space of smooth functions on $U\backslash G$ and through the duality (5.1) we have an adjoint for $S^Y_{\tilde{\chi}^{-1}\nu^{-1}}$

$$\Delta_{\tilde{\chi}}^Y : \mathcal{S}(U\backslash G) \longrightarrow C^\infty(X)$$

the space $\mathcal{S}(U\backslash G)$ is the Schwartz space for $U\backslash G$ (Definition 3.4) and it is closed under the Fourier transforms F_ω, $\omega \in W_k$ (3.3).

We denote $\underline{\Delta}_\chi^Y$ to be the span of $\Delta_{\tilde{\chi}}^Y$ with $\tilde{\chi}$ extending χ. For every $\omega \in W_k$, there is a rational family of linear operators

$$\underline{b}_\omega^Y(\chi):\ \underline{\Delta}_\chi^Y \to\ \underline{\Delta}_{\omega\chi}^{\omega Y}$$

such that

$$F_\omega \Delta = \underline{b}_\omega^Y(\chi)\Delta$$

for $\Delta \in \underline{\Delta}_\chi^Y$.

Since $F_{\omega_1}F_{\omega_2} = F_{\omega_1\omega_2}$, $\underline{b}_\omega^Y(\chi)$ satisfy the cocycle relations

$$\underline{b}_{\omega_1\omega_2}^Y(\chi) = \underline{b}_{\omega_1}^{\omega_2 Y}({}^{\omega_2}\chi)\underline{b}_{\omega_2}^Y(\chi) \tag{5.3}$$

The goal of the precise local functional equation is to calculate the proportionality factors

$$F_\omega[\Delta_{\tilde{\chi}}^Y]_{\tilde{\chi}} = \underline{b}_\omega^Y(\tilde{\chi})[\Delta_{\omega\tilde{\chi}}^{\omega Y}]_{\tilde{\chi}}$$

where $\omega = \omega_\alpha$ is a simple reflection. Together with the construction of $\Phi_K \in \mathcal{S}(U\backslash G)^K$, this will give us the computation for

$$\Delta_{\tilde{\chi}}(\phi_K) \in C^\infty(X)^K$$

where $\phi_K \in I(\chi)^K$, this is the goal of computing eigenvectors of the unramified Hecke algebra. We will describe this method of computing eigenfunctions in Section **??**.

Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a spherical variety. For η an unramified character of H, we choose a basis $\alpha_1, \cdots, \alpha_r$ which forms a $\mathbb{Z}$-basis for $X_k^*(H) = \mathrm{Hom}_{\text{k-grp}}(\mathbf{H}, \mathbb{G}_m)$. We define

$$\mathbf{H}_0 = \bigcap_{1\le i\le r} \ker(\alpha_i)$$

then $\mathbf{H}/\mathbf{H}_0 \cong \mathbb{G}_m^r$ is a split torus.

We introduce $\mathbf{A}_H = \mathbf{H}/\mathbf{H}_0$ a split torus and $r : \mathbf{H} \to \mathbf{A}_H$. For η an unramified character of H, we set $\eta = \overline{\eta} \circ r$ for $\overline{\eta}$ a character of $\mathbf{A}_H$.

Definition 5.1. We introduce $\mathbf{X}_H = \Gamma_H\backslash\mathbf{G} \times \mathbf{A}_H$ with

$$\Gamma_H = \{(h, r(h))\ :\ h \in H\}$$

we have $\mathbf{A}_H$ acts on $\mathbf{X}_H$ via the inclusion $\mathbf{A}_H \hookrightarrow \mathbf{G} \times \mathbf{A}_H$.

Under the natural map $\mathbf{X}_H \to \mathbf{X}$, the open $\mathbf{B}$-orbit pulls back to an open $\mathbf{B} \times \mathbf{A}_H$-orbit of $\mathbf{X}_H$, hence $\mathbf{X}_H$ is a spherical variety for $\mathbf{G} \times \mathbf{A}_H$.

We will say $\mathbf{X}_H$ is an extended space for $\mathbf{X}$.

Under the natural map $\mathbf{X}_H \to \mathbf{X}$, for any Borel orbit $\mathbf{Y}$ in $\mathbf{X}$ and $\mathbf{Y}_H$ the corresponding Borel orbit of $\mathbf{Y}$ in $\mathbf{X}_H$, we have a short exact sequence

$$1 \longrightarrow \mathbf{A}_H \longrightarrow \mathbf{T}_{Y_H} \longrightarrow \mathbf{T}_Y \longrightarrow 1$$

Proposition 5.2. *If we denote $\boldsymbol{T}_{Y_H} = \boldsymbol{T} \times \boldsymbol{A}_H/\boldsymbol{T}^{Y_H}$ and $\boldsymbol{T}_Y = \boldsymbol{T}/\boldsymbol{T}^Y$, then we have*

$$\boldsymbol{T}^{Y_H} \cong \boldsymbol{T}^Y$$

as k-groups.

Proof. Fix $y \in Y$ and $y_H \in Y_H$ above it. The projection $\mathbf{B} \times \mathbf{A}_H \to \mathbf{B}$ restricts to an isomorphism of stabilizers

$$(\mathbf{B} \times \mathbf{A}_H)_{y_H} \cong \mathbf{B}_y$$

Write the inverse as $b \mapsto (b, r_y(b))$. When $y = Hg$ and $y_H = \Gamma_H(g, 1)$, this map is given by $r_y(b) = r(gbg^{-1}),\ b \in B_y$.

Let $\pi : \mathbf{B} \to \mathbf{T} = \mathbf{B}/\mathbf{U}$ be the quotient map, we have

$$\mathbf{T}^Y = \pi(\mathbf{B}_y), \quad \mathbf{T}^{Y_H} = \{(\pi(b), r_y(b))\ :\ b \in B_y\} \subset \mathbf{T} \times \mathbf{A}_H$$

Since an algebraic homomorphism from a unipotent group to a torus is trivial, we have r_y factors through $\overline{r}_y : \mathbf{T}^Y \to \mathbf{A}_H$.

Hence

$$\mathbf{T}^{Y_H} = \{(t, \overline{r}_y(t)) \ : \ t \in T^Y\} \cong \mathbf{T}^Y$$

□

Proposition 5.3. *For χ an unramified character of T, for $\boldsymbol{Y}$ a $\boldsymbol{B}$-orbit of $\boldsymbol{X}$ with $\boldsymbol{Y}(k) \neq \emptyset$, then we have*

$$Hom_B(C_c^\infty(Y, \mathcal{L}_\eta), \mathbb{C}_{\chi\delta^{1/2}}) \neq 0$$

if and only if

$$\chi^{-1}\delta^{1/2}|_{B_y} = \delta_{B_y}\eta_y^{-1}|_{B_y} \tag{5.4}$$

we denote the set of unramified characters of T satisfying (5.4) by $Adm_\eta Y$.

Proof. For $y \in Y = \mathbf{Y}(k)$, let B_y be the stabilizer of y. For $f \in C_c^\infty(Y, \mathcal{L}_\eta)$, it pulls back to a function on B satisfying

$$f(hb) = \eta_y(h)f(b) \quad (h \in B_y, \ b \in B)$$

If $y = Hg$, then we have $B_y = B \cap g^{-1}Hg$ and $\eta_y(b) = \eta(gbg^{-1})$ for $b \in B_y, \ gbg^{-1} \in H$.

Let $d_r h$ be a right Haar measure on B_y and $d_l b$ a left Haar measure on B, we have a surjective twisted quotient map

$$p_\eta : \ C_c^\infty(B) \longrightarrow C_c^\infty(Y, \mathcal{L}_\eta), \quad (p_\eta f)(b) = \int_{B_y} f(hb)\eta_y(h)^{-1} \ d_r h$$

For $\ell \in \mathrm{Hom}_B(C_c^\infty(Y, \mathcal{L}_\eta), \mathbb{C}_{\chi\delta^{1/2}})$, its pullback $D = \ell \circ p_\eta$ is a $\chi\delta^{1/2}$-eigendistribution on B. By uniqueness of Haar measure, it is a scalar multiple of

$$D_\chi(F) = \int_B F(b)\chi(b)^{-1}\delta(b)^{1/2} \ d_l b$$

For $h_0 \in B_y$, we set $(L_{h_0}F)(b) = F(h_0 b)$, then use $d_r(h_0 h) = \delta_{B_y}(h_0)d_r h$, we have

$$p_\eta(L_{h_0}F) = \delta_{B_y}(h_0)^{-1}\eta_y(h_0)p_\eta(F)$$

On the other hand, left invariance of $d_l b$ gives

$$D_\chi(L_{h_0}F) = \chi(h_0)\delta(h_0)^{-1/2}D_\chi(F)$$

Thus a nonzero D_χ can factor through p_η only if

$$\chi^{-1}\delta^{1/2}|_{B_y} = \delta_{B_y}\eta_y^{-1}|_{B_y}$$

which is (5.4).

Conversely, if χ satisfies (5.4), we have D_χ factors through p_η, and we get $\ell \in \mathrm{Hom}_B(C_c^\infty(Y, \mathcal{L}_\eta), \mathbb{C}_{\chi\delta^{1/2}})$ such that $D_\chi = \ell \circ p_\eta$.

□

Proposition 5.4. *Let χ be an unramified character of T and $\boldsymbol{Y}$ a Borel orbit of $\boldsymbol{X}$ such that $\chi \in Adm_\eta Y$.*

Let $\boldsymbol{Y}_H$ be the Borel orbit of $\boldsymbol{X}_H$ corresponding to $\boldsymbol{Y}$. Then for $\chi_H = (\chi, \overline{\eta}^{-1})$ as an unramified character of $T \times A_H$, we have $\chi_H \in Adm_{Y_H}$.

Proof. Fix $y \in Y$ and a point $y_H \in Y_H$ above y. If $y = Hg$, we may choose $y_H = \Gamma_H(g, 1)$, put

$$r_y(b) = r(gbg^{-1}), \quad b \in B_y$$

we obtain

$$(B \times A_H)_{y_H} = \{(b, r_y(b)) \ : \ b \in B_y\}$$

By definition we have $\eta_y(b) = \eta(gbg^{-1})$ for $b \in B_y$. Then $\overline{\eta} \circ r_y(b) = \overline{\eta}(r(gbg^{-1})) = \eta(gbg^{-1}) = \eta_y(b)$, we get $\eta_y = \overline{\eta} \circ r_y$.

For $\chi_H = (\chi, \overline{\eta}^{-1})$ and $(b, r_y(b))$ in the stabilizer, we have

$$\begin{aligned}\chi_H^{-1}\delta_{B\times A_H}^{1/2}(b, r_y(b)) &= \chi(b)^{-1}\overline{\eta}(r_y(b))\delta(b)^{1/2} \\ &= (\chi^{-1}\delta^{1/2})(b)\eta_y(b) \\ &= \delta_{B_y}(b) \\ &= \delta_{(B\times A_H)_{y_H}}((b, r_y(b)))\end{aligned}$$

where we used the twisted admissibility condition.

We proved that $\chi_H \in Adm_{Y_H}$. □

We denote $q : T \to T_Y$ the quotient map. Fix $\chi_0 \in Adm_\eta Y$. For $\chi \in Adm_\eta Y$, we can describe all the extensions of χ by (χ, λ) for $\lambda \in \mathrm{Hom}(T_Y, \mathbb{C}^\times)$ and $\lambda(q(t)) = \frac{\chi(t)}{\chi_0(t)}, \ t \in T$.

We denote $p_Y : T_{Y_H} \to T_Y$ and $q_H : T \times A_H \to T_{Y_H}$, fix $\chi_{H,0} = (\chi_0, \overline{\eta}^{-1}) \in Adm_{Y_H}$. For $\chi_H = (\chi, \overline{\eta}^{-1}) \in Adm_{Y_H}$. Given $\tilde{\chi} = (\chi, \lambda)$ an extension of χ, we get $\tilde{\chi}_H$ an extension of χ_H as $\tilde{\chi}_H = (\chi_H, \lambda_H)$, here $\lambda_H := \lambda \circ p_Y \in \mathrm{Hom}(T_{Y_H}, \mathbb{C}^\times)$ and it satisfies

$$\lambda_H(q_H(t, a)) = \frac{\chi_H(t, a)}{\chi_{H,0}(t, a)}, \ (t, a) \in T \times A_H$$

Definition 5.5. For $\mathbf{Y}$ a Borel orbit of $\mathbf{X}$, and $\mathbf{Y}_H$ the corresponding Borel orbit of $\mathbf{X}_H$, and χ an unramified character of T such that $\chi \in Adm_\eta Y$. Let $\chi_H = (\chi, \overline{\eta}^{-1})$ be an unramified character of $T \times A_H$ with $\chi_H \in Adm_{Y_H}$. We fix $\chi_0 \in Adm_\eta Y$ and use it to define the possible extensions $\tilde{\chi}$ of χ.

For $\tilde{\chi} = (\chi, \lambda)$ an extension of χ and $\tilde{\chi}_H = (\chi_H, \lambda_H)$ the corresponding extension of χ_H, we define the twisted local coefficient $b_{\omega_\alpha}^Y(\tilde{\chi}, \eta) := b_{\omega_\alpha}^{Y_H}(\tilde{\chi}_H)$,

$$F_{\omega_\alpha}\Delta_{\tilde{\chi}_H}^{Y_H} = b_{\omega_\alpha}(\tilde{\chi}_H)\Delta_{\omega_\alpha\tilde{\chi}_H}^{\omega_\alpha Y_H}$$

here $\Delta_{\tilde{\chi}_H}^{Y_H}$ is defined via the standard measure normalization on Y_H and a base point y_H above a distinguished base point $y \in Y$.

Remark 5.6. When $\eta = 1$ is the trivial character of H this definition is our usual definition of the local coefficient.

Definition 5.7. Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ and η be an unramified character of H. Write $\mathbf{X}_H = \Gamma_H\backslash\mathbf{G} \times \mathbf{A}_H$ the extended space and let $p : \mathbf{X}_H \to \mathbf{X}$ be the natural map. Fix a distinguished point $x_0 \in \mathring{\mathbf{X}}(\mathfrak{o})$ and a lift $x_{H,0}$ above it, and a Haar measure da on $\mathbf{A}_H(k)$ such that $\mathrm{Vol}(\mathbf{A}_H(\mathfrak{o}), da) = 1$.

Let $\mathbf{Y}$ be a principal Borel orbit and let $\mathbf{Y}_H = p^{-1}(\mathbf{Y})$, and let $\tilde{\chi} = (\chi, \lambda)$ be a character extending the character $\chi \in Adm_\eta Y$. Denote by $\tilde{\chi}_H = ((\chi, \eta^{-1}), \lambda \circ p_Y)$ the corresponding parameter for Y_H. Let $d\mu_{\tilde{\chi}_H}^{Y_H}$ be the standard measure on Y_H. The standard η-twisted eigenmeasure on $\mathbf{Y}$, denoted by $d\mu_{\tilde{\chi}}^{Y,\eta}$ is the $\mathcal{L}_\eta^\vee$-valued measure descent from $d\mu_{\tilde{\chi}_H}^{Y_H}$. Explicitly, if $s : \ V \to Y_H$ is a local section of p, then its scalar coefficient in the trivialization determined by s is characterized by

$$d\mu_{\tilde{\chi}_H}^{Y_H}(s(y)a) = \eta(a)da d\mu_{\tilde{\chi},s}^{Y,\eta}(y), \quad y \in V, \ a \in A_H(k)$$

Proposition 5.8. *For a distinguished integral point $y_H \in \boldsymbol{Y}_H(\mathfrak{o})$ and $y = p(y_H)$, we have*

$$\mu_{\tilde{\chi}}^{Y,\eta}(y\boldsymbol{B}(\mathfrak{o})) = 1$$

Proof. Put

$$B_0 = \mathbf{B}(\mathfrak{o}), \quad K_A = \mathbf{A}_H(\mathfrak{o}), \quad C = yB_0, \quad C_H = y_H(B_0 \times K_A)$$

Since y_H is a distinguished integral point, the standardmeasure normalization on Y_H gives

$$\mu_{\tilde{\chi}_H}^{Y_H}(C_H) = 1$$

We now describe the fiber $p^{-1}(u)$ at a point $u \in C$ for the map $p : C_H \to C$. Let $D = \mathrm{stab}_{B_0}(y)$, for $d \in D$, there is a unique $a(d) \in A_H$ such that

$$y_H d = y_H a(d)$$

the map $a : D \to A_H$ is a continuous homomorphism. Since D is a closed subgroup of the compact group K, its image is compact and thus $a(D) \subset K_A$.

For $u = yb$ with $b \in B_0$, we claim

$$C_H \cap p^{-1}(u) = y_H b K_A$$

Indeed, we can write a point on the left-hand side as $y_H b' a'$ with $b' \in B_0$ and $a' \in K_A$ where $yb' = yb$. Hence $d = b'b^{-1} \in D$ and

$$y_H b' a' = y_H dba' = y_H ba(d)a' \in y_H b K_A$$

the inclusion $y_H b K_A \subset C_H \cap p^{-1}(u)$ is clear.

Choose local sections s of p over C with values in C_H. Since η is unramified, $\eta|_{K_A} = 1$, so these sections determine the same trivialization of the associated line bundles over C. We have,

$$\begin{aligned} 1 &= \int_{C_H} d\mu_{\tilde{\chi}_H}^{Y_H} \\ &= \int_C \int_{K_A} \eta(a)\ da\ d\mu_{\tilde{\chi}}^{Y,\eta}(u) \\ &= \mu_{\tilde{\chi}}^{Y,\eta}(C) \end{aligned}$$

The local computations agree on the overlaps, so the displayed identity is independent of the choice of the local sections.

□

For $\mathbf{G}$ a semisimple group of split rank one and $\mathbf{G}_{sc}$ its simply-connected cover with the covering map $p : \mathbf{G}_{sc} \to \mathbf{G}$. For $\mathbf{H}$ a spherical subgroup of $\mathbf{G}$, and $\eta : H \to \mathbb{C}^\times$ an unramified character of H. We introduce $\mathbf{H}_{sc} = p^{-1}(\mathbf{H})$, and $\eta_{sc} : H_{sc} \xrightarrow{p} H \xrightarrow{\eta} \mathbb{C}^\times$.

Proposition 5.9. *Let $\boldsymbol{Y}$ be a principal Borel orbit for $\boldsymbol{H}\backslash\boldsymbol{G}$ and $\boldsymbol{Y}_{sc}$ the corresponding principal Borel orbit for $\boldsymbol{H}_{sc}\backslash\boldsymbol{G}_{sc}$. We denote $p_T : T_{sc} \to T$ the map between the maximal tori and the induced map $q : T_{Y_{sc}} \to T_Y$.*

For $\chi \in Adm_\eta Y$ and $\tilde{\chi} = (\chi, \lambda)$ an extension of χ, we get $\chi_{sc} := \chi \circ p_T \in Adm_{\eta_{sc}} Y_{sc}$ and $\tilde{\chi}_{sc} = (\chi_{sc}, \lambda \circ q)$. We have

$$b_{\omega_\alpha}^{Y}(\tilde{\chi}, \eta) = b_{\omega_\alpha}^{Y_{sc}}(\tilde{\chi}_{sc}, \eta_{sc})$$

here we choose compatible base points between Y_H and $(Y_{sc})_{H_{sc}}$.

Proof. Denote $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ and $\mathbf{X}_{sc} = \mathbf{H}_{sc}\backslash\mathbf{G}_{sc}$, for $\mathbf{A}_{H_{sc}} := \mathbf{H}_{sc}/(\mathbf{H}_{sc})_0$ we have a projection $p_A : \mathbf{A}_{H_{sc}} \to \mathbf{A}_H$, denote the map $p : (\mathbf{X}_{sc})_{H_{sc}} \to \mathbf{X}_H$ induced by the projection $p : \mathbf{G}_{sc} \to \mathbf{G}$ and p_A.

To simplify our notation, we will omit the admissible characters when we write the intertwining morphisms. We put $\mathbf{Z} = {}^{\omega_\alpha}\mathbf{Y}$ and $\mathbf{Z}_{sc}$ the corresponding Borel orbit in $\mathbf{X}_{sc}$, we denote

Δ_{Y_H}, $\Delta_{(Y_{sc})_{H_{sc}}}$ the intertwining morphisms associated with $\mathbf{Y}_H$ and $(\mathbf{Y}_{sc})_{H_{sc}}$ and Δ_{Z_H}, $\Delta_{(Z_{sc})_{H_{sc}}}$ the intertwining morphisms associated with Z_H and $(Z_{sc})_{H_{sc}}$. We choose compatible base points for $\mathbf{Y}$ and $\mathbf{Z}$, this induces compatible base points between $\mathbf{Y}_H$, $\mathbf{Z}_H$ and $(\mathbf{Y}_{sc})_{H_{sc}}$, $(\mathbf{Z}_{sc})_{H_{sc}}$.

We denote $p^*\Delta_{Y_H} = c_Y\Delta_{(Y_{sc})_{H_{sc}}}$ and $p^*\Delta_{Z_H} = c_Z\Delta_{(Z_{sc})_{H_{sc}}}$, then we have $c_Y = c_Z$ because of our compatible choices of base points. Since $p^*F_{\omega_\alpha} = F_{\omega_\alpha}p^*$, we deduce

$$b^Y_{\omega_\alpha}(\tilde{\chi}, \eta) = b^{Y_{sc}}_{\omega_\alpha}(\tilde{\chi}_{sc}, \eta_{sc})$$

□

For $\mathbf{G}_{sc}$ a simply connected unramified group of split rank one, let $\mathbf{H}_{sc}$ be a spherical subgroup of $\mathbf{G}_{sc}$. We let $\mathbf{H}$ be a spherical subgroup of $\mathbf{G}_{sc}$ with $\mathbf{H} \subset \mathbf{H}_{sc}$, we assume that the spherical systems of $(\mathbf{H}_{sc})_{\overline{k}}\backslash(\mathbf{G}_{sc})_{\overline{k}}$ and $(\mathbf{H})_{\overline{k}}\backslash(\mathbf{G}_{sc})_{\overline{k}}$ are of the same type, and hence there is a bijection between the Borel orbits for $\mathbf{H}\backslash\mathbf{G}_{sc}$ and $\mathbf{H}_{sc}\backslash\mathbf{G}_{sc}$. We let $\eta_{sc} : H_{sc} \to \mathbb{C}^\times$ be an unramified character of H_{sc} and $\eta = \eta_{sc}|_H : H \to \mathbb{C}^\times$ be an unramified character of H.

Let $\mathbf{Y}$ be a Borel orbit for $\mathbf{H}\backslash\mathbf{G}_{sc}$ and $\mathbf{Y}_{sc}$ the corresponding Borel orbit for $\mathbf{H}_{sc}\backslash\mathbf{G}_{sc}$. For $\tilde{\chi}_{sc}$ an extended character for Y_{sc}, we denote $\tilde{\chi}$ the corresponding extended character for Y.

Proposition 5.10. *Let $\boldsymbol{Y}$ be a principal Borel orbit for $\boldsymbol{H}\backslash\boldsymbol{G}_{sc}$ and $\boldsymbol{Y}_{sc}$ the corresponding principal Borel orbit for $\boldsymbol{H}_{sc}\backslash\boldsymbol{G}_{sc}$. We denote $p_T : T_Y \to T_{Y_{sc}}$ the map between the maximal tori.*

For $\chi \in Adm_{\eta_{sc}}Y_{sc}$ and $\tilde{\chi}_{sc} = (\chi, \lambda)$ an extension of χ, we get $\tilde{\chi}$ an extension of χ, $\tilde{\chi} = (\chi, \lambda \circ p_T)$. We have

$$b^{Y_{sc}}_{\omega_\alpha}(\tilde{\chi}_{sc}, \eta_{sc}) = b^Y_{\omega_\alpha}(\tilde{\chi}, \eta)$$

Proof. Denote $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}_{sc}$ and $\mathbf{X}_{sc} = \mathbf{H}_{sc}\backslash\mathbf{G}_{sc}$, we have a projection map $p : X \to X_{sc}$ and the induced map $p : \mathbf{X}_H \to (\mathbf{X}_{sc})_{H_{sc}}$.

To simplify our notation, we will omit the admissible characters when we write the intertwining morphisms. We put $\mathbf{Z} = {}^{\omega_\alpha}\mathbf{Y}$ and $\mathbf{Z}_{sc}$ the corresponding Borel orbit in $\mathbf{X}_{sc}$, we denote Δ_{Y_H}, $\Delta_{(Y_{sc})_{H_{sc}}}$ the intertwining morphisms associated with $\mathbf{Y}_H$ and $(\mathbf{Y}_{sc})_{H_{sc}}$ and Δ_{Z_H}, $\Delta_{(Z_{sc})_{H_{sc}}}$ the intertwining morphisms associated with $\mathbf{Z}_H$ and $(\mathbf{Z}_{sc})_{H_{sc}}$. We choose compatible base points for $\mathbf{Y}$ and $\mathbf{Z}$, this induces compatible base points between $\mathbf{Y}_H$, $\mathbf{Z}_H$ and $(\mathbf{Y}_{sc})_{H_{sc}}$, $(\mathbf{Z}_{sc})_{H_{sc}}$.

We denote $p^*\Delta_{(Y_{sc})_{H_{sc}}} = c_Y\Delta_{Y_H}$ and $p^*\Delta_{(Z_{sc})_{H_{sc}}} = c_Z\Delta_{Z_H}$, then we have $c_Y = c_Z$ because of our compatible choices of base points. Since $p^*F_{\omega_\alpha} = F_{\omega_\alpha}p^*$, we deduce

$$b^Y_{\omega_\alpha}(\tilde{\chi}, \eta) = b^{Y_{sc}}_{\omega_\alpha}(\tilde{\chi}_{sc}, \eta_{sc})$$

□

5.2. **List of basic split rank one spherical varieties.** In this section, let k be a p-adic local field of characteristic zero. We provide a list of basic split rank one spherical varieties.

Using the list of spherical systems in Section **??**, we obtain their geometric realizations over algebraic closure via Luna's classification. Applying the existence Criterion **??**, one obtains the corresponding wonderful subgroups over k, we can obtain the following list which consists of the spherical subgroups which are smallest within a given spherical closure (Proposition 5.12):

Definition 5.11. We introduce the list of basic spherical varieties for $\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$ where ℓ/k is an unramified extension of p-adic fields.

- $\mathrm{Res}_{\ell/k}\mathrm{SL}_2\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$.
- $\mathrm{Res}_{\ell/k}\mathrm{T}\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, T a maximal torus of $\mathrm{SL}_{2,\ell}$.
- $\mathrm{N}(\mathrm{Res}_{\ell/k}\mathrm{T})\backslash\mathrm{Res}_{\ell/k}\mathrm{SL}_2$.
- $\mathrm{U}\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, U the unipotent radical of the Borel.
- $\mathrm{Res}_{k'/k}\mathrm{T}_1\mathrm{U}_1\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, here $\mathrm{T}_1\mathrm{U}_1$ is a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\ \mathrm{SL}_2$ and k'/k is an unramified extension of degree m, over algebraic closure it has a spherical system of type $(\text{a-A-2-n})^m$.

- $\mathrm{Res}_{k'/k}\mathrm{SL}_1(D)\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, for ℓ/k' an unramified quadratic extension and k'/k a finite unramified extension, here D is a division algebra over k'.
- $\mathrm{Res}_{k'/k}\mathrm{SL}_2\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, for ℓ/k' an unramified cubic extension and k'/k a finite unramified extension.

We have the following list of basic spherical varieties for $\mathrm{Res}_{\ell/k}\ \mathrm{SU}_3$ where ℓ/k is an unramified extension of p-adic fields, SU_3 is the unramified unitary group associated with an ℓ'-Hermitian space for ℓ'/ℓ an unramified quadratic extension.

- $\mathrm{Res}_{\ell/k}\mathrm{SU}_3\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$.
- $\mathrm{Res}_{\ell/k}\mathrm{SO}_3\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$.
- $\mathrm{U}\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$, U the unipotent radical of the Borel.
- $\mathrm{Res}_{\ell/k}\mathrm{SL}_1(D)\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$, here D is a division algebra over ℓ.
- $\mathrm{Res}_{k'/k}\mathrm{T}_1\mathrm{U}_1\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$, here $\mathrm{T}_1\mathrm{U}_1$ is a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\ \mathrm{SU}_3$, over algebraic closure it has a spherical system of type $(\text{a-A-4-n})^m$.
- $\mathrm{Res}_{k'/k}\ \mathrm{T}_2\mathrm{U}_2\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SU}_3$, here $\mathrm{T}_2\mathrm{U}_2$ is a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\mathrm{SU}_3$, over algebraic closure it has a spherical system of $(\text{a-A-5-n})^m$ with n odd.

For spherical subgroups H appearing in this list, we will call them basic spherical subgroups.

For $T\cong\mathbb{G}_m$ a maximal split torus of SL_2, we have the short exact sequence

$$1\to\mathrm{Res}^1_{\ell/k}\ \mathbb{G}_m\to\mathrm{Res}_{\ell/k}\ \mathbb{G}_m\to\mathbb{G}_m\to 1 \tag{5.5}$$

For $\mathrm{Res}_{k'/k}\ T_1\hookrightarrow\mathrm{Res}_{\ell/k}\ SL_2$, we have $T_1\cong\mathbb{G}_m$ and the short exact sequence

$$1\to\mathrm{Res}^1_{k'/k}\ \mathbb{G}_m\to\mathrm{Res}_{k'/k}\ \mathbb{G}_m\to\mathbb{G}_m\to 1 \tag{5.6}$$

For $\mathrm{Res}^1_{k'/k}\ T_1\hookrightarrow\mathrm{Res}_{\ell/k}\ SU_3$, we have $T_1\cong\mathbb{G}_m$ and the short exact sequence

$$1\to\mathrm{Res}^1_{k'/k}\ \mathbb{G}_m\to\mathrm{Res}_{k'/k}\ \mathbb{G}_m\to\mathbb{G}_m\to 1 \tag{5.7}$$

For $\mathrm{Res}_{k'/k}\ T_2\hookrightarrow\mathrm{Res}_{\ell/k}\ SU_3$, we have $T_2\cong\mathrm{Res}_{K/k'}\ \mathbb{G}_m$ with K/k' an unramified quadratic extension, then we have the exact sequence

$$1\to\mathrm{Res}^1_{K/k}\ \mathbb{G}_m\to\mathrm{Res}_{K/k}\ \mathbb{G}_m\to\mathbb{G}_m\to 1 \tag{5.8}$$

We introduce the following maps,

- $r_1=N_{\ell/k}:\ \mathrm{Res}_{\ell/k}\mathrm{T}\longrightarrow\mathbb{G}_m$, for $\mathrm{T}\cong\mathbb{G}_m/\ell$.
- $r_2=N_{k'/k}\circ\mathrm{pr}_{T_1}:\mathrm{Res}_{k'/k}(T_1U_1)\longrightarrow\mathbb{G}_m$.
- $r_3=N_{k'/k}\circ\mathrm{pr}_{T_1}:\mathrm{Res}_{k'/k}(T_1U_1)\longrightarrow\mathbb{G}_m$.
- $r_4=N_{K/k}\circ\mathrm{pr}_{T_2}:\mathrm{Res}_{k'/k}(T_2U_2)\longrightarrow\mathbb{G}_m$.

We have the following characterization of the list of basic spherical varieties:

Proposition 5.12. *For H_1 a spherical subgroup of a simply-connected unramified group G of split rank one, denote $\overline{H_1}$ its spherical closure, let H be the spherical subgroup in our list of basic spherical varieties with spherical closure $\overline{H_1}$, then we have $H\subseteq H_1$.*

Proof. We can argue this case by case. The type G, U, N cases are clear. For the other cases, we have discussed them over algebraic closure at the end of Section **??**, and the inclusion $H\subseteq H_1$ follows from the inclusion over algebraic closure. □

For a rational $\mathbf{B}$-orbit $\mathbf{Y}$ and α a simple k-root, we denote $\mathbf{P}_\alpha$ the associated k-rank one parabolic subgroup, we define the Levi variety $\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}=\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha$, and $\overline{\mathbf{H}_\alpha}\backslash\overline{\mathbf{L}_\alpha}$. Here $\overline{\mathbf{L}_\alpha}=\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$, $\overline{\mathbf{H}_\alpha}$ is the image of $\mathbf{H}_\alpha$ inside $\overline{\mathbf{L}_\alpha}$, which is an adjoint unramified group of split rank one.

Definition 5.13. We define the type of a spherical pair $(\mathbf{Y},\alpha)$ as the type for the homogeneous spherical variety $\overline{\mathbf{H}_\alpha}\backslash\overline{\mathbf{L}_\alpha}$ (Proposition **??**).

We introduce a list of extended basic spherical varieties, this will be used in the Proposition 5.50.

Definition 5.14. We introduce the list of extended basic spherical varieties for $\mathrm{Res}_{\ell/k}\mathrm{SL}_2$ and $\mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$ as:

- $\mathrm{Res}_{\ell/k}\mathrm{SL}_2\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$.
- $\mathrm{Res}_{\ell/k}\mathrm{T}\backslash\mathrm{Res}_{\ell/k}\mathrm{SL}_2$, here T is a nonsplit torus of $\mathrm{SL}_{2,\ell}$.
- $\Gamma_{r_1}\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2 \times \mathbb{G}_m$, Γ_{r_1} is the following spherical subgroup: for T a maximal split torus of $\mathrm{SL}_{2,\ell}$

$$\Gamma_{r_1}:\ \mathrm{Res}_{\ell/k}\mathrm{T} \xrightarrow{(i,r_1)} \mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$$

- $\mathrm{N}(\mathrm{Res}_{\ell/k}\mathrm{T})\backslash\mathrm{Res}_{\ell/k}\mathrm{SL}_2$.
- $\mathrm{U}\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, U the unipotent radical of the Borel.
- $\Gamma_{r_2}\backslash\mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$, Γ_{r_2} is the following spherical subgroup: for $\mathrm{T}_1\mathrm{U}_1$ a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\ \mathrm{SL}_2$ and k'/k is an unramified extension of degree m

$$\Gamma_{r_2}:\ \mathrm{Res}_{k'/k}\mathrm{T}_1\mathrm{U}_1 \xrightarrow{(i,r_2)} \mathrm{Res}_{\ell/k}\mathrm{SL}_2 \times \mathbb{G}_m$$

- $\mathrm{Res}_{k'/k}\mathrm{SL}_1(D)\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, for ℓ/k' an unramified quadratic extension and k'/k a finite unramified extension, D a division algebra over k'.
- $\mathrm{Res}_{k'/k}\mathrm{SL}_2\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SL}_2$, for ℓ/k' an unramified cubic extension and k'/k a finite unramified extension.

We introduce the list of extended basic spherical varieties for $\mathrm{Res}_{\ell/k}\ \mathrm{SU}_3$ and $\mathrm{Res}_{\ell/k}\ \mathrm{SU}_3 \times \mathbb{G}_m$ as:

- $\mathrm{Res}_{\ell/k}\mathrm{SU}_3\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$.
- $\mathrm{Res}_{\ell/k}\mathrm{SO}_3\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$.
- $\mathrm{U}\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$, U the unipotent radical of the Borel.
- $\mathrm{Res}_{\ell/k}\mathrm{SL}_1(D)\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3$, here D is a division algebra over k.
- $\Gamma_{r_3}\backslash\mathrm{Res}_{\ell/k}\mathrm{SU}_3 \times \mathbb{G}_m$, Γ_{r_3} is the following spherical subgroup: for $\mathrm{T}_1\mathrm{U}_1$ a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\ \mathrm{SU}_3$,

$$\Gamma_{r_3}:\ \mathrm{Res}_{k'/k}\mathrm{T}_1\mathrm{U}_1 \xrightarrow{(i,r_3)} \mathrm{Res}_{\ell/k}\mathrm{SU}_3 \times \mathbb{G}_m$$

- $\Gamma_{r_4}\backslash\mathrm{Res}_{\ell/k}\ \mathrm{SU}_3 \times \mathbb{G}_m$, Γ_{r_4} is the following spherical subgroup: for $\mathrm{T}_2\mathrm{U}_2$ a k'-spherical subgroup of $\mathrm{Res}_{\ell/k'}\mathrm{SU}_3$,

$$\Gamma_{r_4}:\ \mathrm{Res}_{k'/k}\mathrm{T}_2\mathrm{U}_2 \xrightarrow{(i,r_4)} \mathrm{Res}_{\ell/k}\mathrm{SU}_3 \times \mathbb{G}_m$$

For spherical varieties appearing in this list, we will call them extended basic spherical varieties.

5.3. **Basic properties of the local functional equations.** In this section, we will establish some basic properties concerning the coefficients for split rank one groups

$$F_\omega \Delta_{\tilde{\chi}} = b_\omega(\tilde{\chi})\Delta_{\omega\tilde{\chi}}$$

For $\mathbf{G}$ a connected unramified reductive group of split rank one, for $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ a homogeneous spherical variety of split rank one such that $C_c^\infty(X)$ has generic multiplicity less than or equal to one. Then for a connected reductive unramified subgroup $\tilde{\mathbf{G}} \subseteq \mathbf{G}$ of split rank one with $\tilde{\mathbf{X}} := \tilde{\mathbf{H}}\backslash\tilde{\mathbf{G}} = \mathbf{H} \cap \tilde{\mathbf{G}}\backslash\tilde{\mathbf{G}}$, we assume that $\tilde{\mathbf{X}}$ is a spherical variety for $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{X}}$ is of split rank one, given χ an admissible character for X, ψ an admissible character for $\tilde{X}$ such that

$$\psi : T_{\tilde{X}} \xrightarrow{\iota} T_X \xrightarrow{\chi} \mathbb{C}^\times$$

We assume that $\tilde{\mathbf{X}}(k) = \tilde{H}\backslash\tilde{G}$.

We fix $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ and $\mathbf{T} \subset \mathbf{B}$ a maximal torus. Since X is of generic multiplicity less than or equal to 1, we also identify χ as an unramified character of T via the quoteint $T \to T_X$. We set $\tilde{\chi} := \chi \circ \tilde{\iota}$, here $\tilde{\iota} : \tilde{T} \to T$. ψ is an extension of $\tilde{\chi}$.

We fix a compatible base point $x \in \mathbf{X}(k)$ and $x \in \tilde{\mathbf{X}}(k)$ such that

$$\mathbf{G}_x = \mathbf{H}, \quad \tilde{\mathbf{G}}_x = \tilde{\mathbf{H}}$$

Set $V_\chi = \mathrm{Hom}_B(C_c^\infty(X), \mathbb{C}_{\chi\delta^{1/2}})$. We have the Frobenius map

$$\mathcal{F}_{G,x} : \ V_\chi \to \mathrm{Hom}_H(I(\chi^{-1}), 1)$$

which is given by $ev_x \circ \Delta_{\chi^{-1}}$, here $\Delta_{\chi^{-1}} : I(\chi^{-1}) \to C^\infty(X)$ is the adjoint of S_χ.

Proposition 5.15. *We have $\mathcal{F}_{G,x}$ is an isomorphism of vector spaces.*

Proof. This follows from

$$\mathrm{Hom}_B(C_c^\infty(X), \mathbb{C}_{\chi\delta^{1/2}}) = \mathrm{Hom}_G(I(\chi), C^\infty(X)) = \mathrm{Hom}_G(I(\chi^{-1}), \mathrm{Ind}_H^G(1)) = \mathrm{Hom}_H(I(\chi^{-1}), 1)$$

□

Similarly we can define the Frobenius map $ev_x \circ \Delta_{\tilde{\chi}^{-1}}$

$$\mathcal{F}_{\tilde{G},x} : \ V_{\tilde{\chi}} \to \mathrm{Hom}_{\tilde{H}}(I(\tilde{\chi}^{-1}), 1)$$

we have $\mathcal{F}_{\tilde{G},x}$ is also an isomorphism since $\tilde{\mathbf{X}}(k) = \tilde{H}\backslash\tilde{G}$.

We denote

$$\mathrm{res}_\chi : \ I(\chi^{-1})|_{\tilde{G}} \cong I(\tilde{\chi}^{-1})$$

the restriction isomorphism between irreducible unramified principal series.

Definition 5.16. We define a restriction map $V_\chi \to V_{\tilde{\chi}}$ as

$$\mathrm{Res}_{\chi,x} : V_\chi \to V_{\tilde{\chi}}, \ \mathrm{Res}_{\chi,x} = \mathcal{F}^{-1}_{\tilde{G},\tilde{\chi}} \circ \mathrm{res}_\chi \circ \mathcal{F}_{G,\chi}$$

Proposition 5.17. *We use the common base point x, define $\psi : T_{\tilde{X}} \xrightarrow{\iota} T_X \xrightarrow{\chi} \mathbb{C}^\times$ and use the standard normalization of the eigenmeaures on X and $\tilde{X}$ to define S_χ and S_ψ then we have*

$$Res(S_\chi) = S_\psi$$

Proof. We compute

$$\mathcal{F}_{\tilde{G},\tilde{\chi}}(\mathrm{Res}(S_\chi)) = \mathrm{res}_\chi \circ \mathcal{F}_{G,\chi}$$

which is

$$I(\chi^{-1})|_{\tilde{G}} \xrightarrow{\Delta_{\chi^{-1}}} C^\infty(X) \xrightarrow{ev_x} \mathbb{C}^\times$$

From the definition of ψ, $\psi : T_{\tilde{X}} \xrightarrow{\iota} T_X \xrightarrow{\chi} \mathbb{C}^\times$, this is exactly

$$I(\tilde{\chi}^{-1}) \xrightarrow{\Delta_{\psi^{-1}}} C^\infty(\tilde{X}) \xrightarrow{ev_x} \mathbb{C}^\times$$

and this is $\mathcal{F}_{\tilde{G},\tilde{\chi}}(S_\psi)$ by definition.

□

Note that there is a T_ω map

$$T_\omega : \ V_\chi \longrightarrow V_{\chi^{-1}}$$

defined as the composition $\mathrm{Hom}_B(C_c^\infty(X), \mathbb{C}_{\chi\delta^{1/2}}) \xrightarrow{Frob} \mathrm{Hom}_G(C_c^\infty(X), I(\chi)) \xrightarrow{T_\omega} \mathrm{Hom}_G(C_c^\infty(X), I(\chi^{-1}))$. Similarly we have $T_\omega : V_{\tilde{\chi}} \to V_{\tilde{\chi}^{-1}}$.

Proposition 5.18. *We have*

$$Res_{\chi^{-1},x} \circ T_\omega = T_\omega \circ Res_{\chi,x}$$

Proof. This follows from the following commutative diagram

$$\begin{array}{ccc} I(\chi) & \xrightarrow{T_\omega} & I(\chi^{-1}) \\ \downarrow{\scriptstyle \mathrm{res}_{\chi^{-1}}} & & \downarrow{\scriptstyle \mathrm{res}_{\chi}} \\ I(\tilde{\chi}) & \xrightarrow{T_\omega} & I(\tilde{\chi}^{-1}) \end{array}$$

□

We have the following proposition on the compatibility of $b_\omega(\chi)$ under central reduction:

Proposition 5.19. *For $\boldsymbol{G}$ a connected unramified reductive group of split rank one, for $\boldsymbol{X} = \boldsymbol{H}\backslash\boldsymbol{G}$ a homogeneous spherical variety of split rank one such that $C_c^\infty(X)$ has generic multiplicity less than or equal to one for irreducible unramified principal series. For a connected reductive unramified subgroup $\tilde{\boldsymbol{G}} \subseteq \boldsymbol{G}$ of split rank one with $\tilde{\boldsymbol{X}} := \tilde{\boldsymbol{H}}\backslash\tilde{\boldsymbol{G}} = \boldsymbol{H} \cap \tilde{\boldsymbol{G}}\backslash\tilde{\boldsymbol{G}}$, we assume that $\tilde{\boldsymbol{X}}$ is spherical and $\tilde{\boldsymbol{X}}$ is of split rank one. We also assume that $\tilde{\boldsymbol{X}}(k) = \tilde{H}\backslash\tilde{G}$. We fix a compatible base point $x \in \boldsymbol{X}(k)$ and $x \in \tilde{\boldsymbol{X}}(k)$ such that $\boldsymbol{G}_x = \boldsymbol{H}$, $\tilde{\boldsymbol{G}}_x = \tilde{\boldsymbol{H}}$.*

Given χ an admissible character for X, ψ an admissible character for $\tilde{X}$ such that

$$\psi : T_{\tilde{X}} \xrightarrow{\iota} T_X \xrightarrow{\chi} \mathbb{C}^\times$$

We fix $\boldsymbol{B}$ a Borel subgroup of $\boldsymbol{G}$ and $\boldsymbol{T} \subset \boldsymbol{B}$ a maximal torus. Since X is of generic multiplicity less than or equal to 1, we also identify χ as an unramified character of T via the quoteint $T \to T_X$. We set $\tilde{\chi} := \chi \circ \tilde{\iota}$, here $\tilde{\iota} : \tilde{T} \to T$. ψ is an extension of $\tilde{\chi}$.

Denote S_χ and S_ψ the intertwining morphisms associated with χ and ψ with

$$S_\chi \in V_\chi \quad S_\psi \in V_{\tilde{\chi}}$$

Then if S_χ satisfies the weak local functional equation,

$$T_\omega S_\chi = b_\omega(\chi) S_{\chi^{-1}}$$

we have S_ψ also satisfies the weak local functional equation with the same coefficients

$$T_\omega S_\psi = b_\omega(\psi) S_{\psi^{-1}} = b_\omega(\chi) S_{\psi^{-1}}$$

Proof. From the Proposition 5.18, we have

$$\mathrm{Res}(T_\omega \circ S_\chi) = T_\omega \circ \mathrm{Res}(S_\chi) = T_\omega \circ S_\psi$$

and

$$\mathrm{Res}(b_\omega(\chi) S_{\chi^{-1}}) = b_\omega(\chi) S_{\psi^{-1}}$$

As a consequence, we deduce

$$T_\omega S_\psi = b_\omega(\psi) S_{\psi^{-1}}$$

with $b_\omega(\psi) = b_\omega(\chi)$. □

5.4. **Reduction process.** In this section, following the strategy of Sakellaridis, we reduce the study of the local functional equations to the list of basic spherical varieties.

5.4.1. *Integral structures on Borel orbits and their maximal tori.* Let k be a p-adic local field of characteristic 0 and $\mathfrak{o}$ its integer ring. Let $\mathbf{X}$ be a spherical variety over k for an unramified group $\mathbf{G}$, we denote $\mathbf{T}_X$ its maximal torus, we denote $B_0 = \mathbf{B}(\mathfrak{o})$.

Let $\mathcal{G}$ be a smooth reductive group scheme over $\mathfrak{o}$ with generic fiber $\mathbf{G}$, and let $\mathcal{B} = \mathcal{T}\mathcal{U}$ be a Borel subgroup scheme whose generic fiber is $\mathbf{B} = \mathbf{T}\mathbf{U}$. We denote $\mathbf{P}_\alpha$ the k-parabolic subgroup associated with a k-simple root α, we denote $\mathbf{P}_\alpha = \mathbf{L}_\alpha \mathbf{U}_{P_\alpha}$ the Levi decomposition of $\mathbf{P}_\alpha$, and let $\mathcal{P}_\alpha$ be an $\mathfrak{o}$-scheme with Levi decomposition $\mathcal{P}_\alpha = \mathcal{L}_\alpha \mathcal{U}_{P_\alpha}$ and generic fiber $\mathbf{P}_\alpha$.

Definition 5.20. A smooth $\mathcal{B}$-equivariant integral model of the $\mathbf{B}$-variety $\mathbf{Y}$ consists of:

- A smooth, separated, finite type $\mathfrak{o}$-scheme $\mathcal{Y}$.
- An identification of its generic fiber with $\mathbf{Y}$:
$$\iota: \ \mathcal{Y}_k := \mathcal{Y} \times_{\mathfrak{o}} k \cong \mathbf{Y}$$
- A right action over $\mathfrak{o}$,
$$a: \ \mathcal{Y} \times_{\mathfrak{o}} \mathcal{B} \longrightarrow \mathcal{Y}$$
After passing to the generic fibers, the action
$$a_k: \ \mathcal{Y}_k \times_k \mathbf{B} \longrightarrow \mathcal{Y}_k$$
corresponds under ι to the given action $\mathbf{Y} \times \mathbf{B} \longrightarrow \mathbf{Y}$.

We have the following short exact sequence
$$1 \longrightarrow \mathbf{T}^X \longrightarrow \mathbf{T} \longrightarrow \mathbf{T}_X \longrightarrow 1$$

Assumption 5.21. We make the following assumptions:

- $\mathring{\mathbf{X}}$ has a smooth $\mathcal{B}$-equivariant model $\mathring{\mathcal{X}}$ and $\mathring{\mathcal{X}}(\mathfrak{o}) \neq \emptyset$.
- $\mathbf{T}^X$ is a subtorus of $\mathbf{T}$.
- For every principal $\mathbf{B}$-orbit $\mathbf{Y} \subset \mathbf{X}$ and every simple k-root α such that $(\mathbf{Y}, \alpha)$ is of type U, there exists a separated, finite type $\mathfrak{o}$-scheme $\mathcal{O}$ with a $\mathcal{P}_\alpha$-action extending the generic-fiber action, and $\mathcal{O}_k \cong \mathscr{O} = \mathbf{Y}\mathbf{P}_\alpha$. The $\mathfrak{o}$-scheme $\mathcal{O}/\mathcal{R}(\mathcal{P}_\alpha)$ is isomorphic to $\mathcal{S}\mathcal{U}\backslash\mathcal{G}_\alpha$ for $\mathcal{G}_\alpha = \mathrm{Res}_{\mathfrak{o}_\ell/\mathfrak{o}}\mathbf{PGL}_2$, $\mathrm{Res}_{\mathfrak{o}_\ell/\mathfrak{o}}\mathbf{PU}_3$, $\mathcal{U}$ the unipotent radical of the Borel subgroup of $\mathcal{G}_\alpha$ and $\mathcal{S} \subset \mathcal{N}(\mathcal{U})$. Moreover, the complement of $\mathbf{Y}\mathbf{P}_\alpha$ is the closure of the complement of $(\mathbf{Y}\mathbf{P}_\alpha)_k$.

Remark 5.22. According to [BZSV23, Proposition 3.7.4], our second assumption is satisfied when $M = T^*X$ is a hyperspherical variety ([BZSV23, §3.5.1]).

We denote the quotient map
$$q_R: \ \mathcal{O} \longrightarrow \mathcal{Q} = \mathcal{H}\backslash\mathcal{G}_\alpha$$
here $\mathcal{G}_\alpha$ is the adjoint group of relative rank one in Assumption 5.21, denote $\mathcal{B}_{ad}$ the Borel group scheme of $\mathcal{G}_\alpha$, we have $\mathcal{H} = \mathcal{S}\mathcal{U}_{ad} \subseteq \mathcal{B}_{ad}$. There are two $\mathcal{B}_{ad}$-orbits in $\mathcal{Q}$, we take $\mathcal{Z}$ and $\mathcal{Y}$ to be the inverse images of the closed and open Borel orbits over $\mathfrak{o}$, then the generic fiber of $\mathcal{Z}$ and $\mathcal{Y}$ is $\mathbf{Z}$ and $\mathbf{Y}$, here $\mathbf{Z}$ is the closed $\mathbf{B}$-orbit in $\mathbf{Y}\mathbf{P}_\alpha$.

The proof of the following lemma reduces to the split rank one group case. We use it to define the compatible eigenmeasure:

Proposition 5.23. *Let $(\boldsymbol{Y}, \alpha)$ be a type $\boldsymbol{U}$ spherical pair, $\boldsymbol{Z}$ the closed Borel orbit contained in $\boldsymbol{Y}\boldsymbol{P}_\alpha$. Fix a representative $\tilde{\omega}_\alpha \in N_G(A)(k)$ of the simple reflection ω_α and put*
$$\boldsymbol{U}_\alpha := \boldsymbol{U} \cap \tilde{\omega}_\alpha^{-1} \boldsymbol{U}^- \tilde{\omega}_\alpha$$
The multiplication induces an isomorphism of k-varieties,
$$m_\alpha: \ \boldsymbol{Z} \times \boldsymbol{U}_\alpha \cong \boldsymbol{Y}$$
$$(z, u) \mapsto z\tilde{\omega}_\alpha u$$

Proof. We denote $\mathcal{O} = \mathbf{YP}_\alpha$, we have $\mathcal{O} = \mathbf{Y} \cup \mathbf{Z}$ and let $z \in \mathbf{Z}(k)$, $\mathbf{P}_{\alpha,z}$ the stabilizer of z in $\mathbf{P}_\alpha$.

By the definition of the type U spherical pair, after passing to the rank one Levi $\overline{\mathbf{L}}_\alpha$, the corresponding homogeneous spherical variety is of the form $\mathbf{S}_\alpha\overline{\mathbf{U}}_\alpha\backslash\overline{\mathbf{L}}_\alpha$ with $\mathbf{S}_\alpha\overline{\mathbf{U}}_\alpha \subset \overline{\mathbf{B}}_\alpha$. Pulling back to $\mathbf{P}_\alpha$ gives $\mathbf{P}_{\alpha,z} \subset \mathbf{B}$, we get an isomorphism $\mathbf{Z} \cong \mathbf{P}_{\alpha,z}\backslash\mathbf{B}$.

The relative Bruhat decomposition of $\mathbf{P}_\alpha$ is

$$\mathbf{P}_\alpha = \mathbf{B} \sqcup \mathbf{B}\tilde{\omega}_\alpha\mathbf{B}$$

More precisely, multiplication gives an isomorphism

$$\mathbf{B} \times \mathbf{U}_\alpha \cong \mathbf{B}\tilde{\omega}_\alpha\mathbf{B}$$
$$(b, u) \mapsto b\tilde{\omega}_\alpha u$$

The existence follows from the usual Bruhat decomposition. For the uniqueness, if

$$b\tilde{\omega}_\alpha u = b'\tilde{\omega}_\alpha u'$$

then

$$b^{-1}b' \in \mathbf{B} \cap \tilde{\omega}_\alpha\mathbf{U}_\alpha\tilde{\omega}_\alpha^{-1} \subset \mathbf{B} \cap \mathbf{U}^- = \{1\}$$

hence $b' = b$ and $u' = u$.

The isomorphism $\mathbf{B} \times \mathbf{U}_\alpha \cong \mathbf{B}\tilde{\omega}_\alpha\mathbf{B}$ is equivariant for the $\mathbf{P}_{\alpha,z}$ action on the first factor, hence descends to

$$\mathbf{P}_{\alpha,z}\backslash\mathbf{B} \times \mathbf{U}_\alpha \cong \mathbf{P}_{\alpha,z}\backslash\mathbf{B}\tilde{\omega}_\alpha\mathbf{B}$$

the left hand side is $\mathbf{Z}$ and the right hand side is the open $\mathbf{B}$-orbit $\mathbf{Y}$. Under these identifications the descended map is

$$(z, u) \mapsto z\tilde{\omega}_\alpha u$$

□

We consider the quotient $\mathbf{YP}_\alpha \to \mathbf{YP}_\alpha/\mathbf{U}_{P_\alpha}$ and denote by $\bullet$ images under this map.

Proposition 5.24. *Under the Assumption 5.21, let $\boldsymbol{Y}$ be a principal $\boldsymbol{B}$-orbit. Let $(\boldsymbol{Y}, \alpha)$ be a type $\boldsymbol{U}$ spherical pair, $\boldsymbol{Z}$ the closed Borel orbit contained in $\boldsymbol{YP}_\alpha$. Fix $y_0 \in \mathcal{Y}(\mathfrak{o})$, then there is a closed subgroup $F \subseteq \mathcal{T}(\mathfrak{o})$ normalizing $\boldsymbol{U}_\alpha$ and an $\boldsymbol{L}_\alpha(\mathfrak{o})$-equivariant homeomorphism*

$$F\mathcal{U}_\alpha(\mathfrak{o})\backslash\boldsymbol{L}_\alpha(\mathfrak{o}) \cong \overline{y}_0\boldsymbol{L}_\alpha(\mathfrak{o})$$

The Iwahori subgroup J_α of $K_\alpha = \boldsymbol{L}_\alpha(\mathfrak{o})$ has two orbits in $\overline{y_0}\boldsymbol{L}_\alpha(\mathfrak{o})$, one of them is the $\boldsymbol{B}_\alpha(\mathfrak{o})$-orbit of y_0 and the other intersects the smaller $\boldsymbol{B}_\alpha$-orbit in a unique $\boldsymbol{B}_\alpha(\mathfrak{o})$-orbit $\overline{z_0B_0}$.

Proof. We denote $\mathbf{P}_\alpha = \mathbf{L}_\alpha\mathbf{U}_{P_\alpha}$, $\mathbf{B}_\alpha = \mathbf{B} \cap \mathbf{L}_\alpha = \mathbf{TU}_\alpha$ and $K_\alpha = \mathbf{L}_\alpha(\mathfrak{o})$, choose an integral representative $n \in N_{\mathcal{L}_\alpha}(\mathcal{T})(\mathfrak{o})$ of the nontrivial relative Weyl element.

We denote the projection $\rho : \mathcal{L}_\alpha \to \mathcal{G}_\alpha$, we denote $\mathcal{Q}_{cl} = \mathcal{H}\backslash\mathcal{B}_{ad}$ and $\mathcal{Q}_{big} = \mathcal{H}\backslash\mathcal{B}_{ad}\rho(n)\mathcal{B}_{ad}$ the closed and open $\mathcal{B}_{ad}$-orbits in $\mathcal{H}\backslash\mathcal{G}_\alpha$. Then we have an isomorphism

$$\mathcal{Q}_{cl} \times \mathcal{U}_\alpha \cong \mathcal{Q}_{big}, \quad (x, u) \mapsto x\rho(n)\rho(u)$$

there are unique $x_0 \in \mathcal{Q}_{cl}(\mathfrak{o})$ and $u_0 \in \mathcal{U}_\alpha(\mathfrak{o})$ such that

$$q_R(y_0) = x_0\rho(n)\rho(u_0)$$

set $z_0 = y_0u_0^{-1}n^{-1}$, then $z_0 \in \mathcal{Z}(\mathfrak{o})$.

Then after quotient by $\mathbf{U}_{P_\alpha}$, we get

$$\overline{y_0} = \overline{z_0}nu_0, \quad \overline{y_0}K_\alpha = \overline{z_0}K_\alpha \tag{5.9}$$

Let $\mathbf{J} = \mathrm{stab}_{L_\alpha}(\overline{z}_0)$, then we have

$$\mathbf{U}_\alpha \subseteq \mathbf{J} \subseteq \mathbf{B}_\alpha \tag{5.10}$$

this is because if we denote $\mathbf{Y}_\alpha$ the $\mathbf{B}_\alpha$-orbit corresponding to $\mathbf{Y}$ in $\mathbf{YP}_\alpha/\mathbf{U}_{P_\alpha}$, then $(\mathbf{Y}_\alpha, \alpha)$ is a type U pair.

Take the compact stabilizer $C = \mathrm{stab}_{K_\alpha}(\overline{z}_0) = \mathbf{J}(k) \cap K_\alpha$. We have $\mathbf{U}_\alpha(\mathfrak{o}) \subseteq C \subseteq \mathbf{B}_\alpha(\mathfrak{o})$ because of (5.10).

Define $F = C \cap \mathcal{T}(\mathfrak{o})$, then we have $C = F\mathbf{U}_\alpha(\mathfrak{o})$, hence an isomorphism

$$F\mathbf{U}_\alpha(\mathfrak{o})\backslash\mathbf{L}_\alpha(\mathfrak{o}) \cong \overline{z_0}K_\alpha = \overline{y_0}K_\alpha$$

From the Iwahori decomposition

$$K_\alpha = J_\alpha \sqcup \mathcal{U}_\alpha(\mathfrak{o}) n \mathcal{B}_\alpha(\mathfrak{o})$$

since $\mathcal{U}_\alpha(\mathfrak{o}) \subseteq C \subseteq J_\alpha$, we have

$$K_\alpha = CJ_\alpha \sqcup CnJ_\alpha$$

from the isomorphism $C\backslash K_\alpha$, we get the two J_α-orbits in $\overline{y_0}K_\alpha$ are $\overline{y_0}J_\alpha$ and $\overline{z_0}J_\alpha$. □

Definition 5.25. Under the Assumption 5.21, $\mathbf{Y}$ a principal Borel orbit. For a distinguished integral point $y_0 \in \mathcal{Y}(\mathfrak{o})$ with B_0-orbit y_0B_0, we will say the B_0-orbit z_0B_0 the B_0-orbit corresponding to y_0B_0.

Proposition 5.26. *For a principal Borel orbit $\boldsymbol{Y}$ and a k-oriented path*

$$\gamma: \quad \boldsymbol{Y} = \boldsymbol{Y}_0 \xrightarrow{\alpha_1} \boldsymbol{Y}_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} \boldsymbol{Y}_r = \mathring{X}$$

such that $(\boldsymbol{Y}_i, \alpha_{i+1})$ is of type U for $0 \leq i \leq r-1$.

Under the Assumption 5.21, we have a short exact sequence of tori over $\mathfrak{o}$:

$$1 \longrightarrow \mathcal{T}^Y \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_Y \longrightarrow 1$$

whose generic fiber is $1 \to \boldsymbol{T}^Y \to \boldsymbol{T} \to \boldsymbol{T}_Y \to 1$.

Proof. We denote $\mathbb{F}_q \cong \mathfrak{o}/\varpi\mathfrak{o}$ the residue field of k. Under the Assumption 5.21, we have $\mathbf{T}^X$ is a subtorus of $\mathbf{T}$. For $X^*(T)$ the geometric character lattice of $\mathbf{T}$, since $\mathbf{T}$ is defined over k it has a natural action of $\mathrm{Gal}(\overline{k}/k)$, and since $\mathbf{T}$ is unramified the action of $\mathrm{Gal}(\overline{k}/k)$ factors through $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ here $\mathbb{F}_q$ is the residue field of k. Similar discussion holds for $X^*(T_X)$ and $X^*(T^X)$.

Character duality gives an exact sequence of finite free $\mathbb{Z}$-modules with $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-action:

$$0 \longrightarrow X^*(T_X) \longrightarrow X^*(T) \longrightarrow X^*(T^X) \longrightarrow 0 \tag{5.11}$$

The equivalence between the tori over $\mathfrak{o}$ and finite free $\mathbb{Z}$-modules with continuous $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-action, we get a short exact sequence

$$1 \longrightarrow \mathcal{T}^X \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_X \longrightarrow 1$$

For $(\mathbf{Y}, \alpha)$ a spherical pair of type U, $\mathbf{Z}$ the closed Borel orbit in $\mathbf{YP}_\alpha$. Suppose we have a short exact sequence of finite free $\mathbb{Z}$-modules

$$0 \longrightarrow X^*(T_Y) \longrightarrow X^*(T) \longrightarrow X^*(T^Y) \longrightarrow 0$$

then since $X^*(T_Y) = \omega_\alpha X^*(T_Z)$ and $X^*(T^Y) = \omega_\alpha X^*(T^Z)$, we get

$$0 \longrightarrow X^*(T_Z) \longrightarrow X^*(T) \longrightarrow X^*(T^Z) \longrightarrow 0$$

Since all α_i are of type U, we can repeat the previous argument and the short exact sequence (5.11) gives us

$$0 \longrightarrow X^*(T_Y) \longrightarrow X^*(T) \longrightarrow X^*(T^Y) \longrightarrow 0$$

this gives us a short exact sequence

$$1 \longrightarrow \mathcal{T}^Y \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_Y \longrightarrow 1$$

□

Proposition 5.27. *Let*

$$1 \longrightarrow \mathcal{T}^Y \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_Y \longrightarrow 1$$

be a short exact sequence of tori over $\mathfrak{o}$ *with generic fiber,*

$$1 \longrightarrow \boldsymbol{T}^Y \longrightarrow \boldsymbol{T} \longrightarrow \boldsymbol{T}_Y \longrightarrow 1$$

Let ℓ/k *be a finite unramified extension, we denote the short exact sequence*

$$1 \longrightarrow \boldsymbol{T}^{Y_\ell} \longrightarrow \boldsymbol{T}_\ell \longrightarrow \boldsymbol{T}_{Y_\ell} \longrightarrow 1$$

with $\boldsymbol{T}^{Y_\ell} = (\boldsymbol{T}^Y)_\ell$ *and* $\boldsymbol{T}_{Y_\ell} = (\boldsymbol{T}_Y)_\ell$.

We introduce $T_{Y,0} := im(\boldsymbol{T}(k) \cap K \to \boldsymbol{T}_Y(k))$ *and* $T_{Y_\ell,0} := im(\boldsymbol{T}_\ell(\ell) \cap K_\ell \to \boldsymbol{T}_{Y_\ell}(\ell))$, *then we have* $T_{Y,0} = T_{Y_\ell,0} \cap T_Y$.

Proof. We denote κ the residue field of k. First we want to prove $q : \mathcal{T}(\mathfrak{o}) \twoheadrightarrow \mathcal{T}_Y(\mathfrak{o})$. Take any $a \in \mathcal{T}_Y(\mathfrak{o})$, consider the inverse image $P_a = q^{-1}(a)$. We consider the reduction of q

$$q_\kappa : \mathcal{T}(\kappa) \longrightarrow \mathcal{T}_Y(\kappa)$$

Because κ is finite and $\mathcal{T}_\kappa^Y$ is connected, Lang's theorem ([Lan56, Theorem 2]) tells us that q_κ is surjective

$$q_\kappa : \mathcal{T}(\kappa) \twoheadrightarrow \mathcal{T}_Y(\kappa)$$

Now P_a is smooth over $\mathfrak{o}$, since $\mathfrak{o}$ is Henselian, Hensel's Lemma [Sta26, Tag 0H74] for smooth schemes gives

$$P_a(\mathfrak{o}) \twoheadrightarrow (P_a)_\kappa(\kappa)$$

this proves q is surjective.

For the choosen integral models $\mathbf{T}(k) \cap K = \mathcal{T}(\mathfrak{o})$ and hence $T_{Y,0} = q(\mathcal{T}(\mathfrak{o})) = \mathcal{T}_Y(\mathfrak{o})$. Similarly

$$T_{Y_\ell,0} = q(\mathcal{T}(\mathfrak{o}_\ell)) = \mathcal{T}_Y(\mathfrak{o}_\ell)$$

So it remains to prove

$$\mathcal{T}_Y(\mathfrak{o}) = \mathcal{T}_Y(\mathfrak{o}_\ell) \cap T_Y$$

The torus $\mathcal{T}_Y$ is affine, we write $\mathcal{T}_Y = \operatorname{Spec} A$, suppose $x \in \mathcal{T}_Y(\mathfrak{o}_\ell) \cap T_Y$, then since $x \in T_Y$ we get a homomorphism

$$x^\# : A \longrightarrow k$$

since $x \in \mathcal{T}_Y(\mathfrak{o}_\ell)$, we get $x^\#(f) \in k \cap \mathfrak{o}_\ell$ for every $f \in A$. But $k \cap \mathfrak{o}_\ell = \mathfrak{o}$, we deduce $x^\# \in \mathfrak{o}$, so it defines an element of $\mathcal{T}_Y(\mathfrak{o})$. This proves

$$\mathcal{T}_Y(\mathfrak{o}_\ell) \cap T_Y = \mathcal{T}_Y(\mathfrak{o})$$

□

5.4.2. *Compatible family of measures.* In this section, let k be a p-adic local field of characteristic 0. G the k-points of a connected unramified group $\mathbf{G}$ over k, we fix $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ and $B = \mathbf{B}(k)$, $B_0 = \mathbf{B}(\mathfrak{o})$ with $\mathfrak{o}$ the integer ring of k. Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a homogeneous spherical variety, $\mathbf{Y} \in \mathfrak{B}_k(X)$.

Let $(\mathbf{Y}, \alpha)$ be a type $\mathbf{U}$ spherical pair, $\mathbf{Z}$ the closed Borel orbit contained in $\mathbf{YP}_\alpha$. Assume that we are given a $\mathbf{B}$-eigenmeasure on $\mathbf{Z}$, we show how to associate a $\mathbf{B}$-eigenmeasure on $\mathbf{Y}$.

Pick a representative $\tilde{\omega}_\alpha$ for ω_α in $N_G(A)(\mathfrak{o})$ as in Lemma 5.23, then as $\mathfrak{o}$-schemes $\mathbf{Z} \times \mathbf{U}_\alpha \cong \mathbf{Y}$ under the map $(z, u) \mapsto z\tilde{\omega}_\alpha u$, thus to a $\mathbf{B}$-eigenvolume form ω_Z on $\mathbf{Z}$ we can associate the volume form $\omega_Y = \omega_Z \wedge du$ on $\mathbf{Y}$, where du denotes the eigenform on $\mathbf{U}_\alpha$ with the Chevalley normalization.

Definition 5.28. We will say $\omega_Y = \omega_Z \wedge du$ an eigenform compatible with ω_Z.

Lemma 5.29. *Let $(\boldsymbol{Y}, \alpha)$ be a type $\boldsymbol{U}$ spherical pair, $\boldsymbol{Z}$ the closed Borel orbit contained in $\boldsymbol{YP}_\alpha$ and assume that they are equipped with compatible eigenmeasures (Definition 5.28). Using these eigenmeasures and the compatible choices of base points to define the morphisms $S^Y_{\omega_\alpha \tilde{\chi}}$, $S^Z_{\tilde{\chi}}$, let $T_{\omega_\alpha} : I(\chi) \to I({}^{\omega_\alpha}\chi)$ denote the standard intertwining operator between the principal series representations as defined in Page ?? and the given choice of $\tilde{\omega}_\alpha$, then*

$$T_{\omega_\alpha} \circ S^Z_{\tilde{\chi}} = S^Y_{\omega_\alpha \tilde{\chi}}$$

Proof. The proof is the same as Sakellaridis' proof in the split case. We already know that one side has to be a rational multiple of the other from the weak local functional equation (Theorem **??**). To compute the proportionality constant, it is enough to do it when both sides, viewed as functionals by Frobenius reciprocity, are applied to functions with support in $\mathbf{Z} \cup \mathbf{Y}$. For those one can write down explicitly the integral expressions for both sides, when $\tilde{\chi}$ is in the region of convergence of T_{ω_α}. □

Definition 5.30. Let $\mathbf{Y}$ be a $\mathbf{B}$-orbit with a k-point $y \in \mathbf{Y}(k)$, we put

$$\mathbf{T}^Y := \mathrm{im}(\mathbf{B}_y \longrightarrow \mathbf{B}/\mathbf{U} = \mathbf{T}), \quad \mathbf{T}_Y := \mathbf{T}/\mathbf{T}^Y$$

The choice of y gives a $\mathbf{T}$-equivariant trivialization

$$q_y : \ \mathbf{Y}/\mathbf{U} \cong \mathbf{T}_Y, \quad q_y(y) = 1$$

We also denote by $q_y : \ \mathbf{Y} \to \mathbf{T}_Y$ the resulting $\mathbf{U}$-invariant map.

Fix a nonzero k-rational top degree $\mathbf{B}$-eigenform ω_Y on $\mathbf{Y}$, and write $\mathfrak{c}_Y$ for its B-eigencharacter

$$R_b^* \omega_Y = \mathfrak{c}_Y(b)\omega_Y$$

Let

$$\tilde{\chi}' : \ \mathbf{T}_Y(k) \longrightarrow \mathbb{C}^\times$$

be a character whose pullback to $\mathbf{T}(k)$ is unramified.

The associated pseudo-rational eigenmeasure on $\mathbf{Y}$ is the complex Radon measure

$$d\mu^Y_{\tilde{\chi}'}(y_1) := (\tilde{\chi}' \circ q_y(y_1))^{-1} |\omega_Y|(y_1)$$

If we set $\tilde{\chi}' = \delta^{1/2}\tilde{\chi}|\mathfrak{c}_Y|$, then

$$R_b^* d\mu^Y_{\tilde{\chi}'} = \tilde{\chi}(b)^{-1}\delta(b)^{-1/2} d\mu^Y_{\tilde{\chi}'}$$

Definition 5.31. Let

$$\mathfrak{X}_Y = \{\tilde{\chi}' : \ \mathbf{T}_Y(k) \longrightarrow \mathbb{C}^\times \mid \tilde{\chi}' \circ \pi_Y \text{ is unramified}\}$$

where $\pi_Y : \ T \to T_Y$ is the quotient map.

The choice of $y \in Y$ gives a $\mathbf{T}$-equivariant trivialization

$$q_y : \ \mathbf{Y}/\mathbf{U} \cong \mathbf{T}_Y, \quad q_y(y) = 1$$

for the $\mathbf{B}$-eigenform ω_Y fixed above, define

$$\mathscr{C}(y, \omega_Y) = \{d\mu_{\tilde{\chi}'} = (\tilde{\chi}' \circ q_y)^{-1} |\omega_Y|\}_{\tilde{\chi}' \in \mathfrak{X}_Y}$$

such a family is called a compatible family of pseudo-rational $\mathbf{B}$-eigenmeasures on $\mathbf{Y}$.

Two compatible families $\mathscr{C} = (d\mu_{\tilde{\chi}'})_{\tilde{\chi}'}$ and $\mathscr{C}' = (d\mu'_{\tilde{\chi}'})_{\tilde{\chi}'}$ are equivalent if there exists a scalar $c \in \mathbb{C}^\times$ independent of $\tilde{\chi}'$ such that

$$\mu'_{\tilde{\chi}'} = c\mu_{\tilde{\chi}'}$$

for every $\tilde{\chi}' \in \mathfrak{X}_Y$. The resulting equivalence class is called a class of compatible family of pseudo-rational eigenmeasures and is denoted by $[\mathscr{C}]$.

For $\mathbf{Y}$ a $\mathbf{B}$-orbit over k with $Y \neq \emptyset$, we denote $\mathrm{Comp}(\mathbf{Y}_k)$ the set of equivalence classes of compatible family of pseudo-rational eigenmeasures on $\mathbf{Y}_k$.

Let $T_{Y,0} := \mathrm{im}(\mathbf{T}(k) \cap K \to \mathbf{T}_Y(k))$. Now let ℓ/k be a finite unramified extension such that $\mathbf{G}_\ell$ is split and similarly we denote $T_{Y_\ell,0} := \mathrm{im}(\mathbf{T}(\ell) \cap K_\ell \to \mathbf{T}_{Y_\ell}(\ell))$.

There is a canonical map,

$$j_{\ell/k} : \ \mathbf{T}_Y(k)/T_{Y,0} \longrightarrow \mathbf{T}_{Y_\ell}(\ell)/T_{Y_\ell,0}$$
$$[t] \longmapsto [t_\ell]$$

this is induced from the inclusion $\mathbf{T}_Y(k) \hookrightarrow \mathbf{T}_{Y_\ell}(\ell)$ and it is well defined since $T_{Y,0} \subset T_{Y_\ell,0}$.

Put

$$X^{nr}(T_Y) := \mathrm{Hom}(\mathbf{T}_Y(k)/T_{Y,0}, \mathbb{C}^\times)$$

the set of unramified characters of T_Y and $\mathfrak{X}_Y$ the set of characters $\tilde{\chi}'$ as in Definition 5.31. Fix a point $y_* \in \mathbf{Y}(k)$ and let

$$q_* : \ \mathbf{Y}/\mathbf{U} \cong \mathbf{T}_Y$$

a point $t \in \mathbf{T}_Y(k)$ defines another trivialization

$$q_t := t^{-1} q_* : \ \mathbf{Y}/\mathbf{U} \cong \mathbf{T}_Y$$

The compatible family of eigenmeasures associated with t is

$$\mathscr{C}_t = \{ d\mu^Y_{\tilde{\chi}',t}(y) := \tilde{\chi}'(q_t(yU))^{-1} |\omega_Y(y)| \}_{\tilde{\chi}' \in \mathfrak{X}_Y}$$

we have

$$\begin{aligned} d\mu^Y_{\tilde{\chi}',t}(y) &= \tilde{\chi}'(q_t(yU))^{-1} |\omega_Y(y)| \\ &= \tilde{\chi}'(t^{-1} q_*(yU))^{-1} |\omega_Y(y)| \\ &= \tilde{\chi}'(t) \tilde{\chi}'(q_*(yU))^{-1} |\omega_Y(y)| \\ &= \tilde{\chi}'(t) d\mu^Y_{\tilde{\chi}',1}(y) \end{aligned}$$

Lemma 5.32. *Fix $y_* \in \boldsymbol{Y}(k)$ and let*

$$q_* = q_{y_*} : \ \boldsymbol{Y}/\boldsymbol{U} \cong \boldsymbol{T}_Y$$

Then the map

$$Y/U \longrightarrow T_Y, \quad [y] \mapsto q_*(yU)$$

is a bijection. If $t = q_*(yU)$ then

$$q_y = t^{-1} q_* = q_t$$

Proof. Let $p : \ \mathbf{Y} \to \mathbf{Y}/\mathbf{U}$ be the quotient map. For $y \in Y$ put

$$t_y = q_*(p(y)) \in T_Y$$

the trivialization $t_y^{-1} q_*$ is $\mathbf{T}_Y$-equivariant and satisfies $t_y^{-1} q_*(p(y)) = 1$. It follows the definition that

$$q_y = t_y^{-1} q_* = q_{t_y}$$

Conversely, let $t \in T_Y$ then

$$x_t := q_*^{-1}(t) \in (\mathbf{Y}/\mathbf{U})(k)$$

the fiber $p^{-1}(x_t)$ is a homogeneous space under the unipotent group U, since k has characteristic 0, every such homogeneous U-space has a k-point. We may therefore choose $y_t \in Y$ such that $p(y_t) = x_t$. Hence $q_*(y_t U) = t$, this gives

$$q_{y_t} = t^{-1} q_* = q_t$$

Any two such lifts y_t lie in the same U-orbit, so the class of y_t in Y/U is uniquely determined. This proves the assertion.

□

Proposition 5.33. *Let $\boldsymbol{Y}$ be a principal Borel orbit. Fix a point $y_* \in Y$ and use $q_* = q_{y_*}$ to define $\mathscr{C}_t$ for $t \in T_Y$ and an eigenform ω_Y on $\boldsymbol{Y}$. Then the assignment*

$$[t] \mapsto [\mathscr{C}_t]$$

induces a bijection

$$\Psi_k : \ \boldsymbol{T}_Y(k)/T_{Y,0} \cong Comp(\boldsymbol{Y}_k)$$

consequently, $\Phi_k = \Psi_k^{-1}$ is a bijection

$$\Phi_k : \ Comp(\boldsymbol{Y}_k) \cong \boldsymbol{T}_Y(k)/T_{Y,0}$$

Proof. We first prove that Ψ_k is well defined. Fix $\tilde{\chi}'_0 \in \mathfrak{X}_Y$, then we have

$$\mathfrak{X}_Y = \tilde{\chi}'_0 X^{nr}(T_Y)$$

this is because if $\tilde{\chi}' \in \mathfrak{X}_Y$, then $\eta := \tilde{\chi}'(\tilde{\chi}'_0)^{-1}$ has unramified pullback to T, hence η is trivial on $T_{Y,0}$ and therefore $\eta \in X^{nr}(T_Y)$.

For $a_0 \in T_{Y,0}$, we have

$$q_{ta_0} = a_0^{-1} q_t$$

We have

$$d\mu^Y_{\tilde{\chi}',ta_0} = \tilde{\chi}'(a_0) d\mu^Y_{\tilde{\chi}',t} \tag{5.12}$$

write $\tilde{\chi}' = \tilde{\chi}'_0 \eta$ with $\eta \in X^{nr}(T_Y)$, we have $\eta(a_0) = 1$ by definition. The equation (5.12) now becomes

$$d\mu^Y_{\tilde{\chi}',ta_0} = \tilde{\chi}'_0(a_0) d\mu^Y_{\tilde{\chi}',t}$$

The scalar $\tilde{\chi}'_0(a_0)$ is independent of $\tilde{\chi}'$. Hence

$$[\mathscr{C}_{ta_0}] = [\mathscr{C}_t]$$

so Ψ_k only depend on the class of t in $\mathbf{T}_Y(k)/T_{Y,0}$.

Next we prove the surjectivity. Let $[\mathscr{C}] \in \mathrm{Comp}(\mathbf{Y}_k)$ and choose a representative $\mathscr{C}(y, \omega_Y)$. Set $t = q_*(yU)$, by the preceding Lemma 5.32

$$q_y = t^{-1} q_* = q_t$$

and therefore

$$\mathscr{C}(y, \omega_Y) = \mathscr{C}_t$$

Thus $[\mathscr{C}] = [\mathscr{C}_t]$.

It remains to prove injectivity. Suppose that

$$[\mathscr{C}_{t_1}] = [\mathscr{C}_{t_2}]$$

then there exists a scalar $c \in \mathbb{C}^\times$, independent of $\tilde{\chi}'$ such that

$$d\mu^Y_{\tilde{\chi}',t_1} = c d\mu^Y_{\tilde{\chi}',t_2}$$

Since

$$d\mu^Y_{\tilde{\chi}',t} = \tilde{\chi}'(t) d\mu^Y_{\tilde{\chi}',1}$$

this gives

$$\tilde{\chi}'(t_1 t_2^{-1}) = c$$

for every $\tilde{\chi}' \in \mathfrak{X}_Y$.

Set $d = t_1 t_2^{-1}$, then

$$\tilde{\chi}'(d) = c, \ \tilde{\chi}'\eta(d) = c$$

for all $\eta \in X^{nr}(T_Y)$. Since the characters of $T_Y/T_{Y,0}$ separate points, it follows that $d \in T_{Y,0}$, thus $[t_1] = [t_2]$ in $\mathbf{T}_Y(k)/T_{Y,0}$. Therefore Ψ_k is injective. □

Over ℓ, fix $y_* \in \mathbf{Y}(k) \hookrightarrow \mathbf{Y}_\ell(\ell)$, apply the same argument as in Proposition 5.33, we obtain a bijection

$$\Phi_\ell : \ \mathrm{Comp}(\mathbf{Y}_\ell) \cong \mathbf{T}_{Y_\ell}(\ell)/T_{Y_\ell,0}$$

Definition 5.34. We define the base change map $\mathrm{bc}_{\ell/k} : \ \mathrm{Comp}(\mathbf{Y}_k) \to \mathrm{Comp}(\mathbf{Y}_\ell)$ as,

$$\mathrm{bc}_{\ell/k} := \Phi_\ell^{-1} \circ j_{\ell/k} \circ \Phi_k$$

Proposition 5.35. *Under the Assumption 5.21. Given $\boldsymbol{Y}$ a principal Borel orbit, for $[\mathscr{C}_1],\ [\mathscr{C}_2] \in Comp(\boldsymbol{Y})$, we have $bc_{\ell/k}([\mathscr{C}_1]) = bc_{\ell/k}([\mathscr{C}_2])$ implies $[\mathscr{C}_1] = [\mathscr{C}_2]$.*

Proof. It follows from the Proposition 5.26 and Proposition 5.27 that

$$T_Y \cap T_{Y_\ell,0} = T_{Y,0}$$

Denote $\Phi_k([\mathscr{C}_i]) = [t_i]$, under base change $\mathrm{bc}_{\ell/k}([t_i]) = [t_{i,\ell}]$. If $\mathrm{bc}_{\ell/k}(\mathscr{C}_1) = \mathrm{bc}_{\ell/k}(\mathscr{C}_2)$, then $[t_{1,\ell}] = [t_{2,\ell}]$. We may assume there is $a \in \mathbf{T}_Y(k)$ such that $t_2 = t_1 a$. If $[t_{1,\ell}] = [t_{2,\ell}]$, then we have $a \in T_{Y_{\ell,0}}$, therefore $a \in \mathbf{T}_Y(k) \cap T_{Y_\ell,0} = T_{Y,0}$. We get $[t_1] = [t_2]$ in $\mathbf{T}_Y(k)/T_{Y,0}$.

Applying Φ_k^{-1}, we obtain $[\mathscr{C}_1] = [\mathscr{C}_2]$. □

Let $\mathscr{C}_Y$ be a family of pseudo-rational B-eigenmeasures on $\mathbf{Y}$, define its transport $\tau_\alpha(\mathscr{C}_Y)$ to $\mathbf{Z}$ using the decomposition of eigenforms

$$m_\alpha^* \omega_Y = \omega_Z \wedge du_\alpha$$

here m_α is the isomorphism

$$m_\alpha : \ \mathbf{Z} \times \mathbf{U}_\alpha \cong \mathbf{Y}$$

We next define the transport of the trivialization occurring in a compatible pseudo-rational family. Let

$$q_Y : \ \mathbf{Y}/\mathbf{U} \longrightarrow \mathbf{T}_Y$$

be a $\mathbf{T}$-equivariant trivialization. Choose $y \in \mathbf{Y}(k)$ such that $q_Y(yU) = 1$ and write

$$m_\alpha^{-1}(y) = (z, u)$$

Since $u \in \mathbf{U}_\alpha(k)$, we have

$$y\mathbf{U} = z\tilde{\omega}_\alpha \mathbf{U}$$

we define $q_Z : \ \mathbf{Z}/\mathbf{U} \cong \mathbf{T}_Z$ to be the unique $\mathbf{T}$-equivariant trivialization satisfying $q_Z(zU) = 1$.

Proposition 5.36. *Given $\mathscr{C}_Y$ a family of pseudo-rational eigenmeasures on $\boldsymbol{Y}$, for a representative $\mathscr{C}(y, \omega_Y) \in \mathscr{C}_Y$, we can define a family of pseudo-rational eigenmeasures on $\boldsymbol{Z}$ via $\mathscr{C}(z, \omega_Z)$, here ω_Y is an eigenform compatible with ω_Z. The resulting class $[\mathscr{C}(z, \omega_Z)]$ of compatible pseudo-rational eigenmeasures on Z is independent of the representative $\mathscr{C}(y, \omega_Y)$, and hence independent of the resulting choice of z.*

Proof. Let $q_{Y,1}$ and $q_{Y,2}$ be two trivializations representing the same class $\mathscr{C}_Y$. By Proposition 5.33 there exists $a_0 \in T_{Y,0}$ such that

$$q_{Y,2} = a_0^{-1} q_{Y,1}$$

we choose $a \in T(\mathfrak{o})$ such that a maps to a_0. If y_1 satisfies $q_{Y,1}(y_1 U) = 1$, then we can take $y_2 = y_1 a$ as $q_{Y,2}(y_2 U) = 1$.

Write $m_\alpha^{-1}(y_1) = (z_1, u_1),\ y_1 = z_1 \tilde{\omega}_\alpha u_1$, then we have

$$\begin{aligned} y_2 &= z_1 \tilde{\omega}_\alpha u_1 a \\ &= z_1 \tilde{\omega}_\alpha a (a^{-1} u_1 a) \\ &= z_1 \omega_\alpha(a) \tilde{\omega}_\alpha (a^{-1} u_1 a) \end{aligned}$$

we can take $z_2 = z_1 \omega_\alpha(a)$.

Denote $\mathbf{T}_Y = \mathbf{T}/\mathbf{T}^Y$ and $\mathbf{T}_Z = \mathbf{T}/\mathbf{T}^Z$, we have

$$\omega_\alpha(\mathbf{T}^{\mathbf{Y}}) \cong \mathbf{T}^Z$$

this induces an isomorphism

$$\overline{\omega}_\alpha : \ T_Y \cong T_Z$$

If $q_{Z,i}$ denotes the T_Z-equivariant trivilization normalized by

$$q_{Z,i}(z_i U) = 1$$

then $q_{Z,2} = \overline{\omega}_\alpha(a_0)^{-1} q_{Z,1}$. We have $\overline{\omega}_\alpha(a_0) \in T_{Z,0}$. Now applied Proposition 5.33 to $\mathbf{Z}$, we get

$$[\mathscr{C}(z_1, \omega_Z)] = [\mathscr{C}(z_2, \omega_Z)]$$

Hence the resulting class of compatible pseudo-rational eigenmeasures on $\mathbf{Z}$ is independent of all choices.

□

Definition 5.37. Given a class of pseudo-rational eigenmeasures $[\mathscr{C}_Y]$ on $\mathbf{Y}$, we call the class $[\mathscr{C}_Z]$ of pseudo-rational eigenmeasures on $\mathbf{Z}$ the transported class of $[\mathscr{C}_Y]$.

Fix on $\mathring{\mathbf{X}}$ a class $[\mathscr{C}_{\mathring{X}}]$ of pseudo-rational eigenmeasures. For a principal Borel orbit $\mathbf{Y}$ and a k-oriented path

$$\gamma : \ \mathbf{Y} = \mathbf{Y}_0 \xrightarrow{\alpha_1} \mathbf{Y}_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} \mathbf{Y}_r = \mathring{X}$$

We get a class of pseudo-rational eigenmeasures on $\mathbf{Y}$ as

$$[\mathscr{C}_Y^\gamma] := \tau_{\alpha_1} \circ \cdots \circ \tau_{\alpha_r}([\mathscr{C}_{\mathring{X}}]) \tag{5.13}$$

Lemma 5.38. *Let $\boldsymbol{Z} \xrightarrow{\alpha} \boldsymbol{Y}$ be a type U edge. Let ℓ/k be a finite unramified extension, suppose that after base change this edge unfolds into a sequence of type U edges*

$$\boldsymbol{Z}_\ell = \boldsymbol{Z}_0 \xrightarrow{\beta_1} \boldsymbol{Z}_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} \boldsymbol{Z}_m = \boldsymbol{Y}_\ell$$

Then for every class of compatible family $[\mathscr{C}]$ on $\boldsymbol{Y}$

$$bc_{\ell/k}^{Z_\ell}(\tau_\alpha([\mathscr{C}])) = \tau_{\beta_1} \circ \cdots \circ \tau_{\beta_m}(bc_{\ell/k}^{Y_\ell}([\mathscr{C}]))$$

as classes of compatible pseudo-rational families on $\boldsymbol{Z}_\ell$.

Proof. We first record what the base change map does to the representatives. Fix the base point $y_* \in \mathbf{Y}(k)$ and let $q_* = q_{y_*}$. Fix an eigenform ω_Y on $\mathbf{Y}$, $\omega_{Y,\ell}$ the eigenform obtained via extending scalars to ℓ.

For $y \in \mathbf{Y}(k)$, denote y_ℓ the image of y under the natural map $\mathbf{Y}(k) \to \mathbf{Y}_\ell(\ell)$, then we have

$$\mathrm{bc}_{\ell/k}^{Y_\ell}([\mathscr{C}(y, \omega_Y)]) = [\mathscr{C}(y_\ell, \omega_{Y,\ell})]$$

as a consequence of the Definition of the base-change Map 5.35.

Choose an integral representatives n of ω_α used in the isomorphism

$$m_\alpha : \ \mathbf{Z} \times \mathbf{U}_\alpha \longrightarrow \mathbf{Y}$$

We write the reduced expression for ω_α as

$$\omega_\alpha = s_{\beta_1} \cdots s_{\beta_m}$$

choose integral representatives $n_i \in N_{G_\ell}(T_\ell)(\mathfrak{o}_\ell)$ of s_{β_i}. Set $N = n_1 \cdots n_m$, since N and n have the same image in the absolute Weyl group, we have $N = t_0 n$ for $t_0 \in \mathbf{T}(\mathfrak{o}_\ell)$.

Apply the Proposition 5.23 repeatedly, we get

$$M: \mathbf{Z}_\ell \times \prod_{i=1}^{m} \mathbf{U}_{\beta_i} \longrightarrow \mathbf{Y}_\ell$$
$$M(z, u_1, \cdots, u_m) = z n_1 u_1 \cdots n_m u_m$$

Reduced root factorization gives an isomorphism

$$F: \prod_{i=1}^{m} U_{\beta_i} \cong U_{\alpha,\ell}$$
$$F(u_1, \cdots, u_m) = \prod_{i=1}^{m} h_i^{-1} u_i h_i$$

we have $n_1 u_1 \cdots n_m u_m = NF(u_1, \cdots, u_m)$.

We have

$$M = m_{\alpha,\ell} \circ (R_{t_0} \times F) \tag{5.14}$$

For $R_{t_0}(z) = zt_0$. We write

$$m_{\alpha,\ell}^{-1}(y_\ell) = (z_{rel}, u), \quad M^{-1}(y_\ell) = (z_{abs}, u_1, \cdots, u_m)$$

since $M = m_{\alpha,\ell} \circ (R_{t_0} \times F)$, we get $z_{rel} = z_{abs} t_0$.

Let a_0 be the image of t_0 in $T_{Z_\ell}(\ell)$, we get $a_0 \in T_{Z_\ell,0}$, and we have $q_{rel} = a_0^{-1} q_{abs}$. Every character $\eta \in \mathfrak{X}_{Z_\ell}$ is trivial on $T_{Z_\ell,0}$, and hence $\eta(q_{rel}) = \eta(q_{abs})$.

It remains to compare the eigenforms. We have

$$m_\alpha^* \omega_Y = \omega_{Z,rel} \wedge du_\alpha \tag{5.15}$$
$$M^* \omega_{Y,\ell} = \omega_{Z,abs} \wedge du_{\beta_1} \wedge \cdots \wedge du_{\beta_m} \tag{5.16}$$

and

$$F^*(du_{\alpha,\ell}) = a du_{\beta_1} \wedge \cdots \wedge du_{\beta_m}, \quad a \in \ell^\times \tag{5.17}$$

Let c_Z be the eigencharacter of $\omega_{Z,rel,\ell}$, from equations (5.14), (5.15) and (5.17), we have

$$\omega_{Z,abs} = a c_Z(t_0) \omega_{Z,rel,\ell}$$

Therefore for every $\eta \in \mathfrak{X}_{Z_\ell}$, we have

$$\eta(q_{abs})^{-1} |\omega_{Z,abs}|_\ell = |a c_Z(t_0)|_\ell \eta(q_{rel})^{-1} |\omega_{z,rel,\ell}|_\ell$$

since the factor $|ac_Z(t_0)|_\ell$ is independent of η, so the two families define the same class. Now since

$$[\mathscr{C}(z_{rel}, \omega_{Z,rel,\ell})] = bc_{\ell/k}^{Z_\ell}(\tau_\alpha([\mathscr{C}(y, \omega_Y)]))$$
$$[\mathscr{C}(z_{abs}, \omega_{Z,abs})] = \tau_{\beta_1} \circ \cdots \circ \tau_{\beta_m}(\mathrm{bc}_{\ell/k}^{Y_\ell}([\mathscr{C}(y, \omega_Y)]))$$

we get

$$\mathrm{bc}_{\ell/k}^{Z_\ell}(\tau_\alpha([\mathscr{C}])) = \tau_{\beta_1} \circ \cdots \circ \tau_{\beta_m}(\mathrm{bc}_{\ell/k}^{Y_\ell}([\mathscr{C}]))$$

□

Definition 5.39. We define an equivalence relation among

$$\mathscr{C}(y, \omega_Y) = \{d\mu_\eta = (\eta \circ q_y)^{-1} |\omega_Y| =: d\mu_{y,\omega_Y,\eta}\}_{\eta \in \mathfrak{X}_Y}$$

Let $\mathscr{C}_i = \mathscr{C}(y_i, \omega_i)$, let $\lambda \in X^*(T_Y)$ be the character of the rational B-eigenfunction $f = \omega_2/\omega_1$, so that

$$f(xt) = \lambda(\pi_Y(t)) f(x)$$

The two families are said to be equivalent if there is a scalar $c \in C^\times$, independent of η, satisfies

$$d\mu_{y_2,\omega_2,\eta|\lambda|} = cd\mu_{y_1,\omega_1,\eta}$$

here $|\lambda|_k(t) = |\lambda(t)|_k$.

For $\mathbf{Y}$ a principal Borel orbit, we denote the resulting set of classes $[\mathscr{C}(y,\omega_Y)]$ as $\overline{\mathrm{Comp}}(\mathbf{Y})$

Remark 5.40. We introduce this definition as the different k-paths may introduce different eigenforms and base points on $\mathbf{Y}$.

Proposition 5.41. *Fix a point $y_* \in Y$ and use $q_* = q_{y_*}$ to define $\mathscr{C}_t$ for $t \in T_Y$ and fix the eigenform ω_0 on $\boldsymbol{Y}$. Then the assignment*

$$[t] \mapsto [\mathscr{C}_t]$$

induces a bijection

$$\Psi_k : \ \boldsymbol{T}_Y(k)/T_{Y,0} \cong \overline{Comp}(\boldsymbol{Y}_k)$$

consequently, $\Phi_k = \Psi_k^{-1}$ is a bijection

$$\Phi_k : \ \overline{Comp}(\boldsymbol{Y}_k) \cong \boldsymbol{T}_Y(k)/T_{Y,0}$$

Proof. We only need to prove $[\mathscr{C}(y,\omega)] = [\mathscr{C}(y,\omega_0)] \in \overline{\mathrm{Comp}}(\mathbf{Y})$. Denote $f = \omega/\omega_0$ a nonzero k-rational $\mathbf{B}$-eigenfunction. Let $\lambda \in X^*(T_Y)$ be its character, then $f/\lambda \circ q_y$ is a $\mathbf{B}$-invariant function on the homogeneous space $\mathbf{Y}$, hence a constant. Consequently,

$$f = d(\lambda \circ q_y), \quad d \in k^\times$$

For $\eta \in \mathfrak{X}_Y$, we have

$$d\mu_{y,\omega,\eta|\lambda|_k} = |d|_k d\mu_{y,\omega_0,\eta}$$

hence $[\mathscr{C}(y,\omega)] = [\mathscr{C}(y,\omega_0)]$.

For the rest of the proof, we can repeat the argument in Proposition 5.33. □

Using the bijection $\mathrm{Comp}(\mathbf{Y}) \cong \mathbf{T}_Y(k)/T_{Y,0} \cong \overline{\mathrm{Comp}}(\mathbf{Y})$, from the Proposition 5.35, we can define the base change map

$$\mathrm{bc}_{\ell/k} : \ \overline{\mathrm{Comp}}(\mathbf{Y}) = \mathrm{Comp}(\mathbf{Y}) \longrightarrow \overline{\mathrm{Comp}}(\mathbf{Y}_\ell) = \mathrm{Comp}(\mathbf{Y}_\ell)$$

From the Proposition 5.36, we can define the transport of classes,

$$\tau_\alpha : \overline{\mathrm{Comp}}(\mathbf{Y}) \longrightarrow \overline{\mathrm{Comp}}(\mathbf{Z})$$

We choose an integral point $x_0 \in \mathring{\mathbf{X}}(\mathfrak{o})$, we fix a class of pseudo-rational eigenmeasure $[\mathscr{C}_{\mathring{X}}] = [\mathscr{C}(x_0,\omega)]$. For γ a path from $\mathbf{Y}$ to $\mathring{\mathbf{X}}$

$$\gamma : \ \mathbf{Y} = \mathbf{Y}_0 \xrightarrow{\alpha_1} \mathbf{Y}_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} \mathbf{Y}_r = \mathring{X}$$

such that $(\mathbf{Y}_i, \alpha_{i+1})$ is of type U for $0 \le i \le r-1$. We can get a class of pseudo-rational eigenmeasures on Y as

$$[\mathscr{C}_Y^\gamma] := \tau_{\alpha_1} \circ \cdots \circ \tau_{\alpha_r}([\mathscr{C}_{\mathring{X}}])$$

Proposition 5.42. *Under the Assumption 5.21, for $\boldsymbol{G}$ a split group and for $\boldsymbol{Y}$ a Borel orbit of maximal rank, the resulting class $[\mathscr{C}_Y^\gamma] \in \overline{Comp}(\boldsymbol{Y})$ is independent of the choice of the k-oriented path γ.*

Proof. This is [Sak13, Proposition 3.3.6]. □

Proposition 5.43. *Under the Assumption 5.21, for $\boldsymbol{Y}$ a principal Borel orbit, the class $[\mathscr{C}_Y^\gamma]$ is independent of the choice of the k-oriented path γ.*

Proof. Let $\mathbf{Y}$ be a principal Borel orbit, we denote γ_1, γ_2 two oriented paths in the k-Knop graph from $\mathbf{Y}$ to $\mathring{\mathbf{X}}$

$$\gamma_i : \ \mathbf{Y} = \mathbf{Y}_{i,0} \longrightarrow \cdots \longrightarrow \mathbf{Y}_{i,r_i} = \mathring{\mathbf{X}}, \quad i = 1, 2$$

we put $\mathscr{C}_Y^{\gamma_1} := \mathscr{C}_1$, $\mathscr{C}_Y^{\gamma_2} := \mathscr{C}_2$ the corresponding class of pseudo-rational eigenmeasures via (5.13).

Let α be a k-simple root. Consider the type U edge $\mathbf{Z} \xrightarrow{\alpha} \mathbf{Y}'$. Let ω_α be the absolute Weyl group representative of the relative simple reflection and choose a reduced expression

$$\omega_\alpha = s_{\beta_m} \cdots s_{\beta_1}$$

We may apply Lemma 5.23 to get the isomorphism

$$\mathbf{Y}' \cong \mathbf{Z} \times \mathbf{U}_\alpha$$

we have $\dim \mathbf{Y}'_\ell - \dim \mathbf{Z}_\ell = \dim \mathbf{U}_{\alpha,\ell} = m$.

Apply successively $s_{\beta_1}, \cdots, s_{\beta_m}$ to $\mathbf{Z}_\ell$, the type U edge unfolds into an ℓ-path

$$\mathbf{Z}_\ell = \mathbf{Z}_0 \xrightarrow{\beta_1} \mathbf{Z}_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} \mathbf{Z}_m = \mathbf{Y}'_\ell$$

we have

$$\dim \mathbf{Z}_m - \dim \mathbf{Z}_0 = \sum_{i=1}^{m} (\dim \mathbf{Z}_i - \dim \mathbf{Z}_{i-1}) \leq m$$

and hence $\dim \mathbf{Z}_i - \dim \mathbf{Z}_{i-1} = 1$ for all $1 \leq i \leq m$, this implies that every ℓ-edge $\mathbf{Z}_{i-1} \xrightarrow{\beta_i} \mathbf{Z}_i$ is of type U.

From the Lemma 5.38, we get

$$\mathrm{bc}_{\ell/k}^{Z_\ell}(\tau_\alpha(\mathscr{C})) = \tau_{\beta_1} \circ \cdots \circ \tau_{\beta_m}(\mathrm{bc}_{\ell/k}^{Y'_\ell}(\mathscr{C})) \tag{5.18}$$

for $\mathscr{C}$ a class of pseudo-rational eigenmeasures on $\mathbf{Y}'$.

Unfold the k-oriented paths γ_1, γ_2 gives us two ℓ-oriented paths $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ from $\mathbf{Y}_\ell$ to $\mathring{\mathbf{X}}_\ell$. Applying the equation (5.18) repeatedly, we get

$$\mathrm{bc}(\mathscr{C}_i) = \mathscr{C}_{Y_\ell}^{\widetilde{\gamma}_i}$$

Since $\mathbf{G}_\ell$ is a split group, from Sakellaridis' result (Proposition 5.42), we get $[\mathscr{C}_{Y_\ell}^{\widetilde{\gamma}_1}] = [\mathscr{C}_{Y_\ell}^{\widetilde{\gamma}_2}]$. We have

$$\mathrm{bc}([\mathscr{C}_1]) = [\mathscr{C}_{Y_\ell}^{\widetilde{\gamma}_1}] = [\mathscr{C}_{Y_\ell}^{\widetilde{\gamma}_2}] = \mathrm{bc}([\mathscr{C}_2])$$

hence we deduce $[\mathscr{C}_1] = [\mathscr{C}_2]$ from the Proposition 5.35. □

This proposition allows us to define the notion of standard measures on the set of principal Borel orbits which depends only on the choice of standard eigenmeasures on $\mathring{\mathbf{X}}$.

Having chosen a point $x_0 \in \mathring{\mathbf{X}}(\mathfrak{o})$, we normalize our family of standard pseudo-rational eigenmeasures on $\mathring{X}$ by $\mu(x_0 J) = 1$. Recall that by Axiom 2.3, $x_0 J \subset x_0 \mathbf{B}(\mathfrak{o})$ therefore such a normalization is possible as all eigenmeasures in the family differs by unramified B-eigenfunctions.

Definition 5.44. Under the Axiom 2.3 and Assumption 5.21, the pseudo-rational B-eigenmeasures on $\mathring{\mathbf{X}}$ with base point x_0 normalized according to the property $\mu(x_0 J) = 1$ will be called *standard*. The corresponding eigenmeasures on any principal Borel orbit $\mathbf{Y}$, obtained from Proposition 5.43 and normalized by the property that $\mu(y_0 J) = 1$, will also be called *standard*. Here $y_0 B_0$ is a distinguished B_0-orbit corresponds to $x_0 B_0$ under Definition 5.25.

From now on we will only be using the morphisms $S_{\tilde{\chi}}^Y$ defined using the standard eigenmeasures.

5.5. **Reduction to basic spherical varieties.** In this section, following the strategy of Sakellaridis, we reduce the local functional equation in general to our list of basic spherical varieties.

We let $\mathbf{G}$ be an unramified group and $\mathbf{X}$ a homogeneous $\mathbf{G}$-spherical variety. We fix a $\mathbf{G}$-eigenform ω_X on $\mathbf{X}$, we denote by $\mathfrak{n}$ the eigencharacter of the underlying eigenform ω_X and by ν the eigencharacter of $|\omega_X|$.

Denote $\mathcal{S}(U\backslash G)$ the Schwartz space, we introduce

$$C^\infty_{temp}(U\backslash G) = \{f \in C^\infty(U\backslash G) \mid fdg \in \mathcal{S}(U\backslash G)^\vee\}$$

where dg is a fixed invariant measure on $U\backslash G$ and $\mathcal{S}(U\backslash G)^\vee$ denotes the continuous dual of the Schwartz space.

For a generic unramified character χ of the maximal torus $\mathbf{T}$ of $\mathbf{G}$, there is an inclusion

$$I(\chi) \hookrightarrow C^\infty_{temp}(U\backslash G)$$

Hence for generic $\tilde{\chi}$

$$S^Y_{\tilde{\chi}^{-1}\nu^{-1}}(C^\infty_c(X)) \subset I(\chi^{-1}\nu^{-1}) \subset C^\infty_{temp}(U\backslash G)$$

For a rank-one parabolic subgroup $P_\alpha = L_\alpha U_{P_\alpha}$, let U_α be the subgroup of Lemma 5.23. Since $U = U_\alpha \ltimes U_{P_\alpha}$, projection onto L_α induces $U\backslash P_\alpha \simeq U_\alpha\backslash L_\alpha$. Hence we define

$$C^\infty_{temp}(U\backslash P_\alpha) := C^\infty_{temp}(U_\alpha\backslash L_\alpha)$$

Assume $(\mathbf{Y}, \alpha)$ is not of type G and type N. To reduce the general local functional equation to the case of $\mathbf{SL}_2$ or $\mathbf{SU}_3$ cases, we proceed as follows: we can compose $S^Y_{\tilde{\chi}^{-1}\nu^{-1}}$ with the restriction morphism to get a sequence of P_α-morphisms

$$C^\infty_c(X) \xrightarrow{S^Y_{\tilde{\chi}^{-1}\nu^{-1}}} C^\infty_{temp}(U\backslash G) \xrightarrow{Res} C^\infty_{temp}(U\backslash P_\alpha)$$

we denote by $S'^Y_{\tilde{\chi}^{-1}\nu^{-1}}$ the composition.

We have a commutative diagram of P_α-morphisms

$$\begin{array}{ccc} C^\infty_{temp}(U\backslash G) & \xrightarrow{Res} & C^\infty_{temp}(U\backslash P_\alpha) \\ \downarrow{\scriptstyle F_{\omega_\alpha}} & & \downarrow{\scriptstyle F_{\omega_\alpha}} \\ C^\infty_{temp}(U\backslash G) & \xrightarrow{Res} & C^\infty_{temp}(U\backslash P_\alpha) \end{array}$$

Using the isomorphism $U\backslash P_\alpha \cong U_\alpha\backslash L_\alpha$, we have

$$\begin{array}{ccc} C^\infty_{temp}(U\backslash G) & \xrightarrow{Res} & C^\infty_{temp}(U_\alpha\backslash L_\alpha) \\ \downarrow{\scriptstyle F_{\omega_\alpha}} & & \downarrow{\scriptstyle F_{\omega_\alpha}} \\ C^\infty_{temp}(U\backslash G) & \xrightarrow{Res} & C^\infty_{temp}(U_\alpha\backslash L_\alpha) \end{array}$$

Therefore, the morphisms $S'^Y_{\tilde{\chi}^{-1}\nu^{-1}}$ satisfy the same functional equations with respect to ω_α as the morphisms $S^Y_{\tilde{\chi}^{-1}\nu^{-1}}$. That is

$$F_{\omega_\alpha} \circ S'^Y_{\tilde{\chi}^{-1}\nu^{-1}} = b^Y_{\omega_\alpha}(\tilde{\chi}^{-1}\nu^{-1}) S'^{\omega_\alpha Y}_{\omega_\alpha(\tilde{\chi}^{-1}\nu^{-1})} \tag{5.19}$$

From the definition of $S^Y_{\tilde{\chi}^{-1}\nu^{-1}}$, the morphism $S'^Y_{\tilde{\chi}^{-1}}$ factors through the restriction of elements of $C^\infty_c(X)$ to $\overline{YP_\alpha}$, to compute the scalar proportionality factors, it is enough to restrict our P_α morphisms to the P_α-stable subspace of $C^\infty_c(X)$ of those elements whose restriction to $\overline{YP_\alpha}$ is supported in YP_α.

Fix $\xi \in \mathbf{Y}(k)$ (to be chosen carefully later) and set

$$S_{\xi,\alpha} := G_\xi \cap P_\alpha, \quad V_{\xi,\alpha} := G_\xi \cap U_{P_\alpha}$$

Let $\mathbf{H}_\alpha$ denote the image of $\mathbf{G}_\xi \cap \mathbf{P}_\alpha$ in $\mathbf{L}_\alpha = \mathbf{P}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}$.

The projection $P_\alpha \to L_\alpha$ induces an L_α-equivariant morphism

$$q_\alpha : YP_\alpha \cong S_{\xi,\alpha}\backslash P_\alpha \longrightarrow H_\alpha\backslash L_\alpha, \quad q_\alpha(S_{\xi,\alpha}p) = H_\alpha\overline{p}$$

its fibers are isomorphic $V_{\xi,\alpha}\backslash U_{P_\alpha}$. Fix a quotient measure $d\dot{u}$ on $V_{\xi,\alpha}\backslash U_{P_\alpha}$. Put $\eta_{\xi,\alpha}^{-1} := \delta_{V_{\xi,\alpha}\backslash U_{P_\alpha}}$. Let $\mathcal{L}_{\eta_{\xi,\alpha}}$ be the corresponding L_α-equivariant line bundle on $H_\alpha\backslash L_\alpha$. With our convention, a section is represented by a function Φ on L_α satisfying

$$\Phi(h\ell) = \eta_{\xi,\alpha}^{-1}(h)\Phi(\ell)$$

We have the pushforward along q_α:

$$q_{\alpha!} : \ C_c^\infty(YP_\alpha) \longrightarrow C_c^\infty(H_\alpha\backslash L_\alpha, \mathcal{L}_{\eta_{\xi,\alpha}^{-1}})$$

given by

$$(q_{\alpha!}f)(H_\alpha\ell) = \int_{V_{\xi,\alpha}\backslash U_{P_\alpha}} f(\xi u\tilde{\ell})\ d\dot{u}$$

where $\tilde{\ell} \in P_\alpha$ is any lift of ℓ. The integral is independent of the lift, its transformation under H_α is precisely $\delta_{\xi,\alpha}$.

From [Sak13, §3.2 equation (3.2)], we have

$$ev_1 \circ S^Y_{\tilde{\chi}^{-1}\nu^{-1}}(f) = \int_Y f(y)d\mu^Y_{\tilde{\chi}^{-1}\nu^{-1}}(y)$$

with $d\mu^Y_{\tilde{\chi}^{-1}}$ a pseudo-rational measure on Y.

We denote $\mathbf{Y}_\alpha \subseteq \mathbf{H}_\alpha\backslash\mathbf{L}_\alpha$ the $\mathbf{B}_\alpha$-orbit corresponding to $\mathbf{Y}$. For $z \in Y_\alpha$, choose $y_z \in q_\alpha^{-1}(z)$, the fiber is

$$q_\alpha^{-1}(z) = y_zU_{P_\alpha} \cong G_{y_z} \cap U_{P_\alpha}\backslash U_{P_\alpha}$$

If $z = H_\alpha\ell$ and $y_z = \xi\tilde{\ell}$ with $\tilde{\ell} \in P_\alpha$ then

$$G_{y_z} \cap U_{P_\alpha} = \tilde{\ell}^{-1}(G_\xi \cap U_{P_\alpha})\tilde{\ell}$$

For the eigenmeasure $d\mu^Y_{\tilde{\chi}^{-1}\nu^{-1}}$ measure on Y, there exists an eigenmeasure $d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$ on Y_α and

$$\int_Y f d\mu^Y_{\tilde{\chi}^{-1}\nu^{-1}} = \int_{Y_\alpha} q_{\alpha!}f d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$$

We denote $N = U_{P_\alpha}$ and $V = V_{\xi,\alpha} = G_\xi \cap U_{P_\alpha}$. For $y_z = \xi\tilde{\ell}$, then

$$V_z = G_{y_z} \cap N = \tilde{\ell}^{-1}N\tilde{\ell}$$

Conjugation gives an identification,

$$c_\ell : \ V\backslash N \longrightarrow V_z\backslash N \tag{5.20}$$

$$Vu \longmapsto V_z\tilde{\ell}^{-1}u\tilde{\ell} \tag{5.21}$$

Define $d\nu_z := (c_\ell)_* d\dot{u}$, then

$$\begin{aligned}(q_{\alpha!}f)(H_\alpha\ell) &= \int_{V\backslash N} f(\xi u\tilde{\ell})\ d\dot{u} \\ &= \int_{V_z\backslash N} f(y_zu')\ d\nu_z(u')\end{aligned}$$

here $u' = \tilde{\ell}^{-1}u\tilde{\ell}$.

We have the Weil formula,

$$\int_Y f(y)d\mu^Y_{\tilde{\chi}^{-1}\nu^{-1}} = \int_{Y_\alpha}\int_{(G_{y_z}\cap U_{P_\alpha})\backslash U_{P_\alpha}} f(y_z u)\ d\nu_z(u)d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}} \tag{5.22}$$

We will state a precise normalization on the measure $d\nu_z$ later which might not coincide with the usual Chevalley normalization.

We have a map

$$S^{Y,\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}:\ C_c^\infty(H_\alpha\backslash L_\alpha,\mathcal{L}_{\eta_\alpha^{-1}})\otimes\nu\longrightarrow \mathrm{Ind}_{B_\alpha}^{L_\alpha}(\chi^{-1}\nu^{-1}\delta^{1/2})\otimes\nu\cong \mathrm{Ind}_{B_\alpha}^{L_\alpha}(\chi^{-1}\delta^{1/2})\subset C^\infty_{temp}(U_\alpha\backslash L_\alpha)$$

in the domain of convergence it is given by

$$(S^{Y,\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}\Phi)(\ell)=\int_{Y_\alpha}\Phi(y\ell)d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}(y)$$

Hence for functions $f\in C_c^\infty(X)$ whose restriction to $\overline{YP_\alpha}$ is supported on YP_α, we have the following factorization of morphisms

$$S'^Y_{\tilde{\chi}^{-1}\nu^{-1}}:\ C_c^\infty(YP_\alpha)\otimes\nu\xrightarrow{q_{\alpha!}}C_c^\infty(H_\alpha\backslash L_\alpha,\mathcal{L}_{\eta_{\xi,\alpha}})\otimes\nu\xrightarrow{S^{Y,\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}}\mathrm{Ind}_{B_\alpha}^{L_\alpha}(\chi^{-1}\delta^{1/2})$$

that is

$$S'^Y_{\tilde{\chi}^{-1}\nu^{-1}}=S^{Y,\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}\circ q_{\alpha!} \tag{5.23}$$

Fix invariant measures on $U_\alpha\backslash L_\alpha$ and $(\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k)$, and denote by $\Delta^{Y,\alpha}_{\tilde{\chi}}$ the adjoint of $S^{Y,\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$

$$\Delta^{Y,\alpha}_{\tilde{\chi}}:\ \mathcal{S}(U_\alpha\backslash L_\alpha)\to C^\infty((\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k),\delta^{-1}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}})\cong C^\infty((\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k),\delta^{-1}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}) \tag{5.24}$$

Notice that this factors through the dual of $\mathrm{Ind}_{B_\alpha}^{L_\alpha}(\chi^{-1}\delta^{1/2})$, which is canonically identified with $\mathrm{Ind}_{B_\alpha}^{L_\alpha}(\chi\delta_{B_\alpha}\delta^{-1/2})=I_{L_\alpha}(\chi_\alpha)$, for $\chi_\alpha=\chi\delta_{B_\alpha}^{1/2}\delta^{-1/2}$.

We can view $\Delta^{Y,\alpha}_{\tilde{\chi}}$ as a twisted intertwining morphism associated with $\mathbf{Y}_\alpha$,

$$\Delta^{Y_\alpha}_{\tilde{\chi}_\alpha,\eta_{\xi,\alpha}}:\ \mathcal{S}(U_\alpha\backslash L_\alpha)\longrightarrow C^\infty((\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k),\delta^{-1}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}})\otimes\nu^{-1}=C^\infty((\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k),\mathcal{L}_{\eta_{\xi,\alpha}\otimes\nu^{-1}})$$

here $\eta_{\xi,\alpha}=\delta^{-1}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}$.

Proposition 5.45. *For a distinguished integral point $\xi\in\boldsymbol{Y}(\mathfrak{o})$, denote $z_0=q_\alpha(\xi)$, put $C=\xi\boldsymbol{B}(\mathfrak{o})$ and $C_\alpha=z_0\boldsymbol{B}_\alpha(\mathfrak{o})$, we have $q_\alpha(C)=C_\alpha$, and we can normalize the measure ν_z so that*

$$\nu_z(C\cap q_\alpha^{-1}(z))=1,\quad z\in C_\alpha$$

Proof. Put $\mathbf{N}=\mathbf{U}_{P_\alpha}$ and $B_{\alpha,0}=\mathbf{B}_\alpha(\mathfrak{o})$ and write

$$F_z=q_\alpha^{-1}(z),\ A_z=C\cap F_z,\ F_0=F_{z_0},\ A_0=A_{z_0}$$

Let $\pi:\ P_\alpha\to L_\alpha$ be the Levi projection. In the compatible integral models used for the Chevalley normalization, we have

$$\mathbf{B}(\mathfrak{o})=\mathbf{N}(\mathfrak{o})B_{\alpha,0},\quad \pi(\mathbf{B}(\mathfrak{o}))=B_{\alpha,0}$$

By the equivariance property of q_α, we have

$$q_\alpha(C)=q_\alpha(\xi\mathbf{B}(\mathfrak{o}))=z_0\pi(\mathbf{B}(\mathfrak{o}))=z_0B_{\alpha,0}=C_\alpha$$

We have

$$F_0=\xi\mathbf{N}(k)\cong V_{\xi,\alpha}(k)\backslash\mathbf{N}(k)$$

since both $\mathbf{N}(k)$ and $V_{\xi,\alpha}(k)$ are unimodular, F_0 admits a nonzero right $\mathbf{N}(k)$-invariant Haar measure unique up to a positive scalar, we choose such a measure and denote it by m_0. We have

$A_0 = C \cap F_0$ is open and compact in F_0, and hence $c := m_0(A_0) \in (0, \infty)$, we denote $\nu_0 = c^{-1} m_0$, then $\nu_0(A_0) = 1$.

Let

$$D = \mathrm{Stab}_{B_{\alpha,0}}(z_0)$$

we have ν_0 is D-invariant by the uniqueness of the right invariant Haar measure.

For $z \in C_\alpha$, choose $b \in B_{\alpha,0}$ and $z = z_0 b$, for $r_b : \ y \mapsto yb$, we define

$$\nu_z = (r_b)_* \nu_0$$

then this is a right $\mathbf{N}(k)$-invariant Haar measure on F_z and this is independent of the choice of b by the D-invariance of ν_0. Finally, since $A_z = r_b(A_0)$, we have

$$\nu_z(C \cap q_\alpha^{-1}(z)) = \nu_z(A_z) = \nu_0(A_0) = 1$$

for $z \in C_\alpha$.

□

Remark 5.46. From the proof, we can see that ν_z coincide with $m_0 = d\dot{u}$ with the usual Chevalley normalization if and only if $d\dot{u}(A_0) = 1$, this is not true in general.

Proposition 5.47. *Keep the normalization of the measure ν as in Proposition 5.45 so that*

$$\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}(C_\alpha) = 1$$

For $\tilde{\chi}$ an admissible character of Y_α extending the character χ, we define an admissible character $\tilde{\kappa}$ of Y_α extending the character $\kappa = \chi^{-1}\nu^{-1}\delta_{P_\alpha}^{1/2}$ using the same extension data as $\tilde{\chi}$. Then for the twisted eigenmeasure $d\mu^{Y_\alpha, \eta_{\xi,\alpha}^{-1}}_{\tilde{\kappa}}$ with $\kappa = \chi^{-1}\nu^{-1}\delta_{P_\alpha}^{1/2}$ as defined in Definition 5.7 and the eigenmeasure $d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$ as defined in the Weil Formula 5.22. We have $d\mu^{Y_\alpha, \eta_{\xi,\alpha}^{-1}}_{\tilde{\kappa}} = d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$ as $H_\alpha \times B_\alpha$ eigenmeasures on $\boldsymbol{Y}_\alpha$.

Proof. Set $\eta = \eta_{\xi,\alpha}$, we have:

$$r_b^* d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}} = \chi(b)\nu(b)\delta^{-1/2}(b) d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$$

and

$$r_b^* d\mu^{Y_\alpha, \eta^{-1}}_{\tilde{\kappa}} = \kappa(b)^{-1}\delta_{B_\alpha}^{-1/2}(b) d\mu^{Y_\alpha, \eta^{-1}}_{\tilde{\kappa}}$$

we have

$$\kappa^{-1}\delta_{B_\alpha}^{-1/2} = \chi\nu\delta^{-1/2}$$

hence $d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$ and $d\mu^{Y_\alpha, \eta^{-1}}_{\tilde{\kappa}}$ have the same B_α-equivarance property.

We have

$$d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}(h\ell) = \eta_{\xi,\alpha}(h) d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}(\ell)$$

and

$$d\mu^{Y_\alpha, \eta^{-1}}_{\tilde{\kappa}}(h\ell) = \eta(h) d\mu^{Y_\alpha, \eta^{-1}}_{\tilde{\kappa}}(\ell)$$

Note that we have

$$d\mu^{Y_\alpha, \eta^{-1}}_{\tilde{\kappa}}(C_\alpha) = \mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}(C_\alpha) = 1$$

from Proposition 5.8 and Proposition 5.45. We conclude

$$d\mu^{Y_\alpha, \eta^{-1}}_{\tilde{\kappa}} = d\mu^{Y_\alpha}_{\tilde{\chi}^{-1}\nu^{-1}}$$

□

The L_α-morphism $\Delta^{Y_\alpha}_{\tilde{\chi}_\alpha,\eta_{\xi,\alpha}}$ satisfy the same functional equations as the morphisms $\Delta^{Y}_{\tilde{\chi}}$, that is

$$b^{Y}_{\omega_\alpha}(\tilde{\chi}) = b^{Y_\alpha}_{\omega_\alpha}(\tilde{\chi}_\alpha, \eta_{\xi,\alpha}\nu^{-1})$$

To simplify the notation, we temporarily denote the distribution $ev_1 \circ \Delta^{Y,\alpha}_{\tilde{\chi}}$ on $U_\alpha\backslash L_\alpha$ by $\Delta'_{\tilde{\chi}}$, hence

$$\Delta'_{\tilde{\chi}} : \ \mathcal{S}(U_\alpha\backslash L_\alpha) \to \mathbb{C}$$

is a $T \times H_\alpha$-eigendistribution with eigencharacter

$$\chi^{-1}\delta^{-1}_{B_\alpha}\delta^{1/2} \times \delta_{G_\xi \cap U_{P_\alpha}\backslash U_{P_\alpha}}\nu$$

Write $\mathbf{L}_\alpha = \mathbf{Z}_\alpha \mathbf{L}^{der}_\alpha$, for $\mathbf{Z}_\alpha$ center of $\mathbf{L}_\alpha$ and $\mathbf{L}^{der}_\alpha$ a semisimple group. We denote $\mathbf{L}^{sc}_\alpha$ the simply-connected cover of $\mathbf{L}^{der}_\alpha$. Then we may also view $\Delta'_{\tilde{\chi}}$ as an eigendistribution for $T \times Z_\alpha H_\alpha$ since $Z_\alpha \subset T$.

Now for $L_\alpha = Z_\alpha L^{der}_\alpha$. We can pick a small subgroup-neighborhood N of $1 \in Z_\alpha$ such that:

- A neighborhood of $U_\alpha\backslash L^{der}_\alpha$ is isomorphic to $U_\alpha\backslash L^{der}_\alpha \times N$.
- The restriction of $\Delta'_{\tilde{\chi}}$ to this neighborhood can be written as a Haar measure on $N \times$ a distribution $\Delta^2_{\tilde{\chi}}$ on $U_\alpha\backslash L^{der}_\alpha$.

Note that we have $\Delta^2_{\tilde{\chi}}$ is a $T^{der} \times (H_\alpha Z_\alpha \cap L^{der}_\alpha)$ eigendistribution, and we have $\mathbf{H}_\alpha\mathbf{Z}_\alpha \cap \mathbf{L}^{der}_\alpha$ is a spherical subgroup of $\mathbf{L}^{der}_\alpha$.

Let $\mathbf{T}^{der} = \mathbf{T} \cap \mathbf{L}^{der}_\alpha$, $\mathbf{H}_{\alpha,2} := (\mathbf{Z}_\alpha\mathbf{H}_\alpha) \cap \mathbf{L}^{der}_\alpha$ and put

$$\kappa_\alpha = \chi^{-1}\delta^{-1}_{B_\alpha}\delta^{1/2}$$

Given $h_2 = zh \in H_{\alpha,2}$, we introduce a central correction for η_α as

$$\eta_{\alpha,2}(h_2) = \kappa_\alpha(z)^{-1}\eta_{\xi,\alpha}(h)$$

Then $\Delta^2_{\tilde{\chi}}$ can be viewed as a $T^{der} \times H_{\alpha,2}$ eigendistribution with eigencharacter $\kappa_\alpha \times \eta_\alpha$.

Proposition 5.48. *For $\phi \in C^\infty_c(U_\alpha\backslash L_\alpha)$ and χ in a suitable region of convergence we have*

$$ev_1 \circ \Delta^{Y,\alpha}_{\tilde{\chi}}(\phi) = \int_{\hat{Y}_\alpha} \phi(y)\tilde{\chi}'^{-1}(y)\ |\omega_{\hat{Y}_\alpha}|(y)$$

for a suitable $T \times H_\alpha$-eigenform $\omega_{\hat{Y}_\alpha}$ on $\hat{Y}_\alpha$ with eigencharacter

$$\mathfrak{c}_\alpha \times \gamma_{G_\xi \cap U_{P_\alpha}\backslash U_{P_\alpha}}\gamma$$

where $\tilde{\chi}'$ is a character of T_Y satisfying

$$\tilde{\chi}'|\mathfrak{c}_\alpha|^{-1} = e^\alpha\delta^{-\frac{1}{2}}\tilde{\chi}$$

Proof. We can apply the same argument for [Sak13, Lemma 3.4.2] here, the original result is only proved for split groups but the same argument works for general unramified groups. □

We choose $y_{der} = H_{\alpha,2} \cdot 1$ and $\mathbf{Y}_{der} = y_{der}\mathbf{B}^{der}_\alpha$ for $\mathbf{B}^{der}_\alpha := \mathbf{B}_\alpha \cap \mathbf{L}^{der}_\alpha$. We fix the identification

$$Y_{der}/U_\alpha \cong T^{der}_Y$$

using the base point $y_{der} \in Y_{der}$.

Proposition 5.49. *There exists a character $\tilde{\chi}'_{der}$ of T^{der}_Y such that*

$$\Delta^2_{\tilde{\chi}}(\phi) = \int_{\hat{Y}_{der}} \phi(y)\tilde{\chi}'^{-1}_{der}(y)\ |\omega_{\hat{Y}_{der}}|(y)$$

We get a character $\tilde{\chi}_{der}$ of T^{der}_Y using the equality

$$\tilde{\chi}'_{der}|\mathfrak{c}_\alpha| = e^\alpha\delta^{-\frac{1}{2}}\tilde{\chi}_{der}$$

Proof. We put $M = U_\alpha \backslash L_\alpha$ and $M_{der} = U_\alpha \backslash L_\alpha^{der}$ and denote the inclusion by $M_{der} \hookrightarrow M$. Define

$$\hat{Y}_{der} = \hat{Y}_\alpha \cap M_{der}, \quad F_{\tilde{\chi},der}(y) := \tilde{\chi}'(i(y))$$

where $\tilde{\chi}'(y)$ is the function appearing in Proposition 5.48.

Let $\mathbf{Z} = \mathbf{Z}(\mathbf{L}_\alpha)^0$, since the multiplication

$$\mathbf{L}_\alpha^{der} \times \mathbf{Z} \longrightarrow \mathbf{L}_\alpha$$

is a central isogeny, there exists a sufficiently small compact open subgroup $N \subset Z$ such that

$$m: \ M_{der} \times N \cong \Omega \subset M, \quad m(U_\alpha g, z) = U_\alpha g z$$

We have

$$m^{-1}(\hat{Y} \cap \Omega) = \hat{Y}_{der} \times N$$

Choose an invariant top form ω_Z and use $dz = |\omega_Z|$ in the definition of $\Delta^2_{\tilde{\chi}}$. Then we have

$$m^* \omega_{\hat{Y}_\alpha} = \mathfrak{a}^* \mathrm{pr}_1^* \omega_{\hat{Y}_{der}} \wedge \mathrm{pr}_2^* \omega_Z \tag{5.25}$$

where $\mathfrak{a}$ is the eigencharacter under right multiplication by $\mathbf{Z}$. We choose N small enough so that $|\mathfrak{a}(z)| = 1$ on N.

Now let $\phi \in C_c^\infty(M_{der})$ and choose $\psi \in C_c^\infty(N)$ with $\int_N \ \psi(z)\ dz = 1$. Define $\Phi \in C_c^\infty(M)$ by

$$\Phi(m(y,z)) = \phi(y)\psi(z)$$

on Ω and extending it by zero outside Ω.

We have

$$\begin{aligned} \Delta^2_{\tilde{\chi}}(\phi) &= \int_N \ \psi(z)\ dz \int_{\hat{Y}_{der}} \ \phi(y) F_{\tilde{\chi},der}(y)^{-1}\ |\omega_{\hat{Y}_{der}}|(y) \\ &= \int_{\hat{Y}_{der}} \ \phi(y) F_{\tilde{\chi},der}(y)^{-1} |\omega_{\hat{Y}_{der}}|(y) \end{aligned}$$

We set

$$\tilde{\chi}'_{der}(y) = F_{\tilde{\chi},der}(y)$$

□

We denote $p: \ \mathbf{L}_\alpha^{sc} \longrightarrow \mathbf{L}_\alpha^{der}$ the covering map. We denote $\mathbf{H}_\alpha^{sc} = p^{-1}(\mathbf{H}_{\alpha,2})$. Then $p^*\Delta^2_{\tilde{\chi}}$ is a $T^{sc} \times H_\alpha^{sc}$-eigendistribution with eigencharacter $\kappa_{sc} \times \eta_{sc}$, here

$$\kappa_{sc} = \kappa_\alpha \circ p, \ \eta_{sc} = \eta_{\alpha,2} \circ p$$

We denote $\mathbf{Y}_{sc}$ the Borel orbits corresponding to $\mathbf{Y}_{der}$ and introduce the character $\tilde{\chi}_{sc}$ of T_Y^{sc} as

$$T_Y^{sc} \to T_Y^{der} \xrightarrow{\tilde{\chi}_{der}} \mathbb{C}^\times$$

Let's summarize what we have achieved so far: Let $\mathbf{Y}$ be a principal Borel orbit and $\alpha \in \Sigma_k(G)$, we assume that $(\mathbf{Y}, \alpha)$ is not of type G and not of type N. We let $\mathbf{P}_\alpha = \mathbf{L}_\alpha \mathbf{U}_{P_\alpha}$ be its Levi decomposition. We first reduce the local functional equation for $(\mathbf{Y}, \alpha)$ to the local functional equation for $\mathbf{Y}_\alpha$, here $\mathbf{Y}_\alpha$ is a Borel orbit in $\mathbf{H}_\alpha \backslash \mathbf{L}_\alpha$ corresponding to $\mathbf{Y}$. We denote $\mathbf{L}_\alpha^{der}$ the derived subgroup for $\mathbf{L}_\alpha$, we reduce the local functional equation for $\mathbf{Y}_\alpha$ to $\mathbf{Y}_{der}$, for $\mathbf{Y}_{der}$ a Borel orbit in $\mathbf{H}_\alpha \cap \mathbf{L}_\alpha^{der} \backslash \mathbf{L}_\alpha^{der}$ corresponding to $\mathbf{Y}_\alpha$. We denote $\mathbf{L}_\alpha^{sc}$ the simply-connected cover of $\mathbf{L}_\alpha^{der}$, we reduce the local functional equation further to the spherical variety $\mathbf{H}_\alpha^{sc} \backslash \mathbf{L}_\alpha^{sc}$ for the Borel orbit $\mathbf{Y}_{sc}$ corresponding to $\mathbf{Y}_{der}$.

We show in the following proposition that we can reduce the calculation for the local coefficient $b^Y_{\omega_\alpha}(\tilde{\chi})$ for a rank one pair $(\mathbf{Y}, \alpha)$ to the twisted local coefficient of a basic spherical variety $\mathbf{X}_{bs}$ which is an ordinary local coefficient of an extended basic spherical variety by definition.

Proposition 5.50. *Assume that there exists a G-invariant eigenmeasure on $\boldsymbol{X}$. Under the Assumption 5.21, let $\boldsymbol{Y} \in \mathfrak{B}_k^0(X)$ and $(\boldsymbol{Y}, \alpha)$ be a parabolic pair which is not of type G and not of type N (Proposition ??). We choose $\xi \in \boldsymbol{Y}(\mathfrak{o})$ with $\xi \in y_0 \boldsymbol{B}(\mathfrak{o})$ a distinguished $\boldsymbol{B}(\mathfrak{o})$-orbit obtained from the distinguished $\boldsymbol{B}(\mathfrak{o})$-orbit $x_0\boldsymbol{B}(\mathfrak{o})$ via Proposition 5.23. We use this base point and the standard eigenmeasures on $\boldsymbol{Y}$ to define the intertwining morphisms $S^Y_{\tilde{\chi}}$, for $\tilde{\chi}$ extending the unramified character $\chi \in Adm_Y$. We assume the existence of weak local functional equation for $\tilde{\chi}$*

$$F_{\omega_\alpha} \Delta^Y_{\tilde{\chi}} = b^Y_{\omega_\alpha}(\tilde{\chi}) \Delta^{\omega_\alpha Y}_{\omega_\alpha \tilde{\chi}}$$

Let $\boldsymbol{P}_\alpha$ be the rank one parabolic subgroup associated with α with Levi decomposition $\boldsymbol{P}_\alpha = \boldsymbol{L}_\alpha \boldsymbol{U}_{\boldsymbol{P}_\alpha}$ and $\boldsymbol{L}_\alpha$ its Levi subgroup. Denote $\boldsymbol{H}_\alpha$ the image of $\boldsymbol{G}_\xi \cap \boldsymbol{P}_\alpha$ in $\boldsymbol{L}_\alpha$. Write $\boldsymbol{L}_\alpha = \boldsymbol{Z}_\alpha \boldsymbol{L}^{der}_\alpha$ for $\boldsymbol{L}^{der}_\alpha$ the derived subgroup of $\boldsymbol{L}_\alpha$. Denote $\boldsymbol{L}^{sc}_\alpha =: \boldsymbol{G}_{bs}$ the simply connected cover of $\boldsymbol{L}^{der}_\alpha$. We introduce

$$\boldsymbol{H}_{\alpha,2} = \boldsymbol{Z}_\alpha \boldsymbol{H}_\alpha \cap \boldsymbol{L}^{der}_\alpha, \quad \boldsymbol{H}^{sc}_\alpha = p^{-1}(\boldsymbol{H}_{\alpha,2})$$

where $p : \boldsymbol{L}^{sc}_\alpha \to \boldsymbol{L}^{der}_\alpha$ the covering map. Denote $\boldsymbol{H}^{sc}_\alpha = p^{-1}(\boldsymbol{H}_{\alpha,2})$ and identify $\boldsymbol{L}^{sc}_\alpha$ with the $Res_{\ell/k}\, \boldsymbol{SL}_2$ and $Res_{\ell/k}\, \boldsymbol{SU}_3$ for ℓ/k an unramified extension. We denote $\boldsymbol{H}_{bs}$ the basic spherical subgroup in our list such that $\boldsymbol{H}_{bs} \subseteq \boldsymbol{H}^{sc}_\alpha$. We denote $\boldsymbol{Y}_{bs} \subset \boldsymbol{H}_{bs} \backslash \boldsymbol{G}_{bs}$ the Borel orbit corresponding to $\boldsymbol{Y}_{sc}$.

Now we introduce the eigencharacters for $T^{der}_\alpha \times H_{\alpha,2}$, let $\boldsymbol{T}^{der}_\alpha = \boldsymbol{T}_\alpha \cap \boldsymbol{L}^{der}_\alpha$, $\boldsymbol{H}_{\alpha,2} := (\boldsymbol{Z}_\alpha \boldsymbol{H}_\alpha) \cap \boldsymbol{L}^{der}_\alpha$ and put

$$\kappa_\alpha = \chi^{-1} \delta^{-1}_{B_\alpha} \delta^{1/2}, \quad \eta_{\xi,\alpha} = \delta_{G_\xi \cap U_{P_\alpha} \backslash U_{P_\alpha}}$$

We introduce a central correction for $\eta_{\xi,\alpha}$ as

$$\eta_{\alpha,2}(h_2) = \kappa_\alpha(z)^{-1} \eta_{\xi,\alpha}(h)$$

We also introduce eigencharacters for $T^{sc} \times H_{\alpha,2}$ as

$$\kappa_{sc} = \kappa_\alpha \circ p, \quad \eta_{sc} = \eta_{\alpha,2} \circ p$$

we introduce $\eta_{bs} = \eta_{sc}|_{H_{bs}}$ as an eigencharacter of H_{bs}.

Denote by $\Delta^2_{\tilde{\chi}}$ the eigendistribution obtained via the central reduction 5.48, and $\tilde{\chi}_{der}$ the character obtained from $\Delta^2_{\tilde{\chi}}$ via Proposition 5.48. We assume the weak local functional equation for all extended basic spherical varieties.

Then we have

$$b^Y_{\omega_\alpha}(\tilde{\chi}) = b^{Y_{bs}}_{\omega_\alpha}(\tilde{\chi}_{bs}, \eta_{bs})$$

Proof. We have the reduction to split rank one Levi subgroup $\mathbf{L}_\alpha$,

$$b^Y_{\omega_\alpha}(\tilde{\chi}) = b^{Y_\alpha}_{\omega_\alpha}(\tilde{\chi}_\alpha, \eta_{\xi,\alpha}) \tag{5.26}$$

this is because the standard eigenmeasure used in the definition of $S^Y_{\tilde{\chi}^{-1}\nu^{-1}}$ gives rise to a twisted standard eigenmeasure on Y_α (Proposition 5.47).

We have the reduction to the derived subgroup $\mathbf{L}^{der}_\alpha$ of $\mathbf{L}_\alpha$,

$$b^{Y_{der}}_{\omega_\alpha}(\tilde{\chi}_{der}, \eta_{\alpha,2}) = b^{Y_\alpha}_{\omega_\alpha}(\tilde{\chi}_\alpha, \eta_{\xi,\alpha}) \tag{5.27}$$

We have the reduction to the simply-connected cover $\mathbf{L}^{sc}_\alpha$ of $\mathbf{L}^{der}_\alpha$ by Proposition 5.9,

$$b^{Y_{sc}}_{\omega_\alpha}(\tilde{\chi}_{sc}, \eta_{sc}) = b^{Y_{der}}_{\omega_\alpha}(\tilde{\chi}_{der}, \eta_{\alpha,2}) \tag{5.28}$$

Finally, we have the reduction to the basic spherical subgroups,

$$b^{Y_{bs}}_{\omega_\alpha}(\tilde{\chi}_{bs}, \eta_{bs}) = b^{Y_{sc}}_{\omega_\alpha}(\tilde{\chi}_{sc}, \eta_{sc}) \tag{5.29}$$

by our Proposition 5.10.

We conclude $b^{Y_{bs}}_{\omega_\alpha}(\tilde{\chi}_{bs}, \eta_{bs}) = b^Y_{\omega_\alpha}(\tilde{\chi})$. □

6. $\mathbf{X} = \mathbf{SL}_2\backslash \mathrm{Res}_{\ell/k}\mathbf{SL}_2$, ℓ/k QUADRATIC

In this section, we study the precise local functional equation for $\mathbf{X} = \mathbf{SL}_2\backslash \mathrm{Res}_{\ell/k}\mathbf{SL}_2$ where ℓ/k is an unramified quadratic extension of p-adic local fields. We denote $\mathfrak{o}_k$, $\mathfrak{o}_\ell$ the integer rings of k and ℓ, ϖ the uniformizer of $\mathfrak{o}_k$.

Let $H = \mathbf{GL}_2(k)$ and $G = \mathrm{Res}_{\ell/k}\mathbf{GL}_2(k) = \mathbf{G}(k) = GL_2(\ell)$, $\mathbf{B}$ the Borel subgroup of $\mathbf{G}$, we also assume that the residue characteristic p of k satisfies $p \neq 2$, we denote $\mathbf{A} \subset \mathbf{B}$ the maximal torus of $\mathbf{B}$. $K = GL_2(\mathfrak{o}_\ell)$ the maximal open compact subgroup of G. We write $|\cdot|$ and $||\cdot||$ for the absolute values on k and ℓ, let q be the order of residue field of k and $q_\ell = q^2$. Fix $i \in \mathfrak{o}_\ell^*$ such that $\mathrm{Tr}_{\ell/k}(i) = 0$ and set

$$\eta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

then $\eta\overline{\eta}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the subgroup $T = H \cap \eta^{-1}\mathbf{B}(k)\eta$ is isomorphic to ℓ^*.

For the later calculations, we also introduce $\mathbf{X}^{GL} = \mathbf{GL}_2\backslash \mathrm{Res}_{\ell/k}\, \mathbf{GL}_2$. We denote the base point $x_0 = H\eta^{-1} \in \mathring{\mathbf{X}}^{GL}(\mathfrak{o}_k)$.

Given χ an unramified character of $\ell^\times$, we want to compute under a suitable chosen measure dh,

$$\int_{T\backslash H} \phi_K(\eta h)\, dh$$

where $\phi_K \in I(\chi, \chi^{-1})^K = \mathrm{Ind}_B^G((\chi, \chi^{-1})\delta^{\frac{1}{2}})^K$, $\phi_K(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} k) = ||\frac{a}{b}||^{\frac{\lambda+1}{2}}$. We introduce the following parameter $z = \chi(\varpi) = q^{-\lambda}$.

We fix the quotient measure on $T\backslash H$ using the following lemma:

Lemma 6.1. *Every $h \in H$ can be written uniquely in the form*

$$h = t\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

with $a \in k^\times, b \in k$. In these coordinates, a right invariant measure on $T\backslash H$ is given by $\frac{da}{|a|}db$.

Proof. This follows from $\mathbf{H} = \mathbf{TB}_H$. □

Assume that the measures da and db assign measure 1 to the ring of integers $\mathcal{O}_k$, and we normalize $d^\times a$ as

$$d^\times a = (1 - q^{-1})^{-1}\frac{da}{|a|}$$

we introduce the quotient measure dh on $T\backslash H$ as $dh = d^\times a\, db$.

With this normalization, we have the following result:

Proposition 6.2. *([JLR99, Proposition 37]) We have*

$$\int_{T\backslash H} \phi_K(\eta h)\, dh = ||i||^{\frac{1}{2}}\frac{1 + q^{-\lambda-1}}{1 - q^{-\lambda}}$$

View χ as an admissible character for $\mathbf{GL}_2\backslash \mathrm{Res}_{\ell/k}\, \mathbf{GL}_2$, and denote S_χ the corresponding intertwining operator, we let Δ_χ be its adjoint.

Proposition 6.3. *Denote Φ_K the K-invariant vector we defined in Proposition 4.9, then in the domain of convergence of Δ_χ we have*

$$(\Delta_\chi\Phi_K)(x_0) = C\frac{1 + z}{1 - q^{-1}z}$$

Here C is a constant independent of χ.

Proof. Write $G_{x_0} = \eta H \eta^{-1}$ the stabilizer of x_0 in G and $B_{x_0} = B \cap G_{x_0}$, for $\widehat{\mathring{X}^{GL}} = U\backslash BG_{x_0} = A \times^{B_{x_0}} G_{x_0}$.

We fix $\omega_{\mathring{X}^{GL}}$ a standard eigenmeasure on $\mathring{X}^{GL}$, then from [Sak13, Lemma 3.4.2] we get

$$(\Delta_\chi \Phi_K)(x_0) = \int_{B_{x_0}\backslash G_{x_0}} R_\chi \Phi_K(g)\ dg^{st}$$

here R_χ is the Mellin transform, dg^{st} is the invariant measure on $B_{x_0}\backslash G_{x_0}$ compatible with $\omega_{\mathring{X}^{GL}}$.

Write $g = \eta h \eta^{-1}$, and denote

$$\iota :\ T\backslash H \longrightarrow B_{x_0}\backslash G_{x_0}, \quad Th \longmapsto B_{x_0}\eta h \eta^{-1}$$

define $dh^{st} = \iota^*(dg^{st})$.

We have

$$(\Delta_\chi \Phi_K)(x_0) = \int_{T\backslash H} R_\chi \Phi_K(\eta h \eta^{-1})\ dh^{st}$$

note that since any invariant measure on $T\backslash H$ is unique up to a constant, we have $dh^{st} = c\ dh$ for c a constant independent of χ.

Now apply the formula in [Sak13, §4.3], we get

$$R_\chi(\Phi_K) = Q\frac{1-z^2}{1-q^{-2}z^2}\phi_K$$

Here Q is a constant independent of χ.

Since $\eta \in K$, apply the calculation in Proposition 6.2, we get

$$\begin{aligned}(\Delta_\chi \Phi_K)(x_0) &= \int_{T\backslash H} (R_\chi \Phi_K)(\eta h)\ dh^{st} = Q\frac{1-z^2}{1-q^{-2}z^2}\int_{T\backslash H} \phi_K(\eta h)\ dh^{st} \\ &= cQ\frac{1-z^2}{1-q^{-2}z^2}\frac{1+q^{-1}z}{1-z} = C\frac{1+z}{1-q^{-1}z}\end{aligned}$$

Here $C = cQ$ is a constant independent of χ. □

Proposition 6.4. *We have $b_\omega(\chi) = \frac{F_\omega \Delta_\chi}{\Delta_{\omega_\chi}} = (-1)z^{-1}\frac{1-q^{-1}z^{-1}}{1-q^{-1}z}$. Here we view $F_\omega\Delta_\chi$ and Δ_{ω_χ} as distributions on $\mathcal{S}(U\backslash G)$.*

Proof. We want to compute $\frac{F_\omega\Delta_\chi(\Phi_K)}{\Delta_{\omega_\chi}(\Phi_K)}(x_0)$ as meromorphic continuations of $F_\omega\Delta_\chi(\Phi_K)(x_0)$ and $\Delta_{\omega_\chi}(\Phi_K)(x_0)$ in terms of the z-variable.

From the expression $\Phi_K = \mathrm{Vol}(J)^{-1}(\Phi_J - e^{\check{\alpha}}F_\omega\Phi_J)$, we have

$$F_\omega\Phi_K = \mathrm{Vol}(J)^{-1}F_\omega(\Phi_J - e^{\check{\alpha}}F_\omega\Phi_J) = \mathrm{Vol}(J)^{-1}(F_\omega\Phi_J - e^{-\check{\alpha}}\Phi_J) = -e^{-\check{\alpha}}\Phi_K$$

From the Proposition 6.3, we have

$$F_\omega\Delta_\chi(\Phi_K)(x_0) = \Delta_\chi(F_\omega\Phi_K)(x_0) = \Delta_\chi(-e^{-\check{\alpha}}\Phi_K)(x_0) = -Cz^{-2}\frac{1+z}{1-q^{-1}z}$$

From the Proposition 6.3, we also have

$$\Delta_{\omega_\chi}(\Phi_K)(x_0) = C\frac{1+z^{-1}}{1-q^{-1}z^{-1}}$$

hence

$$\frac{F_\omega\Delta_\chi(\Phi_K)}{\Delta_{\omega_\chi}(\Phi_K)}(x_0) = -z^{-2}\frac{1+z}{1-q^{-1}z}\frac{1-q^{-1}z^{-1}}{1+z^{-1}} = -z^{-1}\frac{1-q^{-1}z^{-1}}{1-q^{-1}z} = (-1)q^{\lambda}\frac{1-q^{-1}q^{\lambda}}{1-q^{-1}q^{-\lambda}}$$

□

We fix an isomorphism $T_{H\backslash G} \cong \ell^\times$ as in Example **??**.

Definition 6.5. For χ viewed as an unramified character of $T_{H\backslash G}$, we define $\beta(\chi)$ as

$$\beta(\chi) = \frac{1+\chi(\varpi)}{1-q^{-1}\chi(\varpi)}$$

Define $B_\omega(\chi) = b_\omega(\chi)(-e^{\check{\alpha}}(\chi))$, here $\alpha \in \Sigma_k(G)$, then:

Proposition 6.6. *We have*

$$B_\omega(\chi) = \frac{\beta(\chi)}{\beta({}^\omega\chi)}$$

Proof. Put $z = \chi(\varpi)$, for the inducing character (χ, χ^{-1}), we have

$$e^{\check{\alpha}}(\chi) = (\chi, \chi^{-1}) \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix} = z^2$$

Proposition 6.4 gives

$$\begin{aligned} B_\omega(\chi) &= (-z^2) b_\omega(\chi) \\ &= (-z^2)(-z^{-1}) \frac{1-q^{-1}z^{-1}}{1-q^{-1}z} \\ &= z \frac{1-q^{-1}z^{-1}}{1-q^{-1}z} \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\beta(\chi)}{\beta({}^\omega\chi)} &= \frac{1+z}{1-q^{-1}z} \frac{1-q^{-1}z^{-1}}{1+z^{-1}} \\ &= z \frac{1-q^{-1}z^{-1}}{1-q^{-1}z} = B_\omega(\chi) \end{aligned}$$

□

Next we consider the local functional equation for $\mathbf{X} = \mathbf{SL}_2 \backslash \mathrm{Res}_{\ell/k}\, \mathbf{SL}_2$.

Definition 6.7. Let χ be an unramified character of $\ell^\times$. For an unramified character $\widetilde{\chi}$ of $T_X \cong k^\times$ (Example **??**) satisfying

$$\widetilde{\chi} \circ N_{\ell/k} = \chi$$

we define

$$\beta(\widetilde{\chi}) = \frac{1+\widetilde{\chi}(\varpi)}{1-q^{-1}\widetilde{\chi}(\varpi)}$$

Proposition 6.8. *For every extension $\widetilde{\chi}$ of χ as above, we have*

$$B_\omega(\widetilde{\chi}) = \frac{\beta(\widetilde{\chi})}{\beta({}^\omega\widetilde{\chi})}$$

here for the definition of $\Delta_{\widetilde{\chi}}$ we use the compatible base points and the measure normalizations as in the statement of Proposition 5.19.

Proof. Put $s = \widetilde{\chi}(\varpi)$, since $N_{\ell/k}(\varpi) = \varpi^2$, we get $\chi(\varpi) = s^2$. Let θ be an unramified character of $\ell^\times$ so that $\theta(\varpi) = s$, then $\theta^2 = \chi$ and the restriction of $I(\theta, \theta^{-1})$ to $SL_2(\ell)$ is $I(\chi)$. By Proposition 6.4 and central reduction Proposition 5.19, we have

$$b_\omega(\widetilde{\chi}) = -s^{-1} \frac{1-q^{-1}s^{-1}}{1-q^{-1}s}$$

Consequently

$$B_\omega(\widetilde{\chi}) = (-e^{\check{\alpha}}(\widetilde{\chi}))b_\omega(\widetilde{\chi}) = \frac{\beta(\widetilde{\chi})}{\beta({}^\omega\widetilde{\chi})}$$

□

7. $\mathbf{X} = \mathbf{SL}_2\backslash \mathrm{Res}_{\ell/k}\mathbf{SL}_2$, ℓ/k CUBIC

In this section, I will study the precise local functional equation for the strongly tempered spherical variety $\mathbf{X} = \mathbf{SL}_2\backslash\mathrm{Res}_{\ell/k}\mathbf{SL}_2$ where ℓ/k is an unramified cubic extension. We denote $\mathfrak{o}_k,\ \mathfrak{o}_\ell$ the integer rings of k and ℓ, ϖ the uniformizer of $\mathfrak{o}_k$. We assume that the characteristic of the residue field of k is not 2, and we denote $\ell_1,\ \ell_2,\ \ell_3$ the three quadratic extensions of ℓ.

Let's first consider the case $H = \mathbf{GL}_2(k)$ and $G = \mathrm{Res}_{\ell/k}\mathbf{GL}_2(k)$, we denote $K = GL_2(\mathfrak{o}_\ell)$ the maximal open compact subgroup of G and $K_H = K \cap H$. We will write $|\cdot|$ and $||\cdot||$ for the absolute values on k and on ℓ, q the size of the residue field of k, $q_\ell = q^3$. Fix $i \in \mathfrak{o}_\ell^*\backslash\mathfrak{o}_k^*$ and set

$$\eta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

Then the subgroup $H_\eta = H \cap \eta^{-1}\mathbf{B}(k)\eta \cong Z(H)$. We want to compute the local period integral

$$\int_{Z(G)\cap H\backslash H} \phi_K(\eta h)\ dh$$

for $\phi_K \in I(\chi, \chi^{-1})^K$, χ is an unramified character of $\ell^\times$ and $z := \chi(\varpi) = q^{-\frac{3}{2}\lambda}$.

For the later calculations, we also introduce $\mathbf{X}^{GL} = \mathbf{GL}_2\backslash\mathrm{Res}_{\ell/k}\ \mathbf{GL}_2$. We denote the base point $x_0 = H\eta^{-1} \in \mathring{\mathbf{X}}^{GL}(\mathfrak{o}_k)$.

We set

$$Z = Z(H),\ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ t(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},\ b \in k,\ a \in k^\times$$

We put $\overline{H} = Z\backslash H$. We pick additive Haar measure db on $n(b),\ b \in k$ and the Haar measure $d^\times a$ on $t(a),\ a \in k^\times$, we normalize them by $\mathrm{vol}(\mathfrak{o}_k) = 1$ and $\mathrm{vol}(\mathfrak{o}_k^\times) = 1$. We fix a Haar measure dk on K_H and normalize it by $\mathrm{vol}(K_H) = 1$. We choose a Haar measure dh on H and normalize it by $\mathrm{vol}(K_H) = 1$, we normalize the Haar measure dz on $Z(H)$ by $\mathrm{vol}(Z \cap K_H) = 1$.

Lemma 7.1. *Every $h \in H$ can be written as*

$$h = Z(H)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}K$$

for K_H the maximal open compact subgroup of H, in these coordinates, a right invariant measure on $\overline{H}$ can be taken to be $\frac{d^\times a}{|a|}dbdk$.

Proof. For $\overline{H} = Z(H)\backslash H$, $\overline{B}, \overline{K}$ the image of $B,\ K_H$ in $\overline{H}$. We fix $d\mu$ a quotient Haar measure on $\overline{H}$.

In the notation of [GH24, Proposition 3.2.2], we set $G = \overline{H} = Z(H)\backslash H$, $S = \overline{B}$, $T = \overline{K}_H$. The assumptions are satisfied since $\overline{H}$ is unimodular, $\overline{B} \cap \overline{K}$ is compact and the multiplication map is surjective by Iwasawa decomposition.

The proposition gives the decomposition

$$\int_{\overline{H}} F(g)\ d\mu(g) = \int_{\overline{B}}\int_{\overline{K}} F(bk)\ dkd_\ell b$$

we omit the details on why our choice of the measure gives the exact equality here.

A left invariant Haar measure on $\overline{B}$ can be taken as

$$\frac{d^\times a}{|a|}db$$

and the result follows. □

We have the following result:

Proposition 7.2. *For Re* $\lambda > -\frac{1}{3}$*, we have*

$$\int_{Z(H)\backslash H} \phi_K(\eta h)\ dh = \frac{1+q^{-\frac{3}{2}(\lambda+1)}}{1-q^{-\frac{3}{2}\lambda-\frac{1}{2}}}.$$

Proof. From the decomposition in Lemma 7.1, we have

$$\int_{Z(H)\backslash H} \phi_K(\eta h)\ dh = \int_{k^*} f(a)||a||^{\frac{\lambda+1}{2}} \frac{d^\times a}{|a|}$$

here

$$f(a) = \int_k \max\{||a||,\ ||i-b||\}^{-\lambda-1}\ db$$

here $f(a)$ is calculated in the following Lemma 7.3.

We can calculate

$$\begin{aligned}\int_{k^*} f(a)||a||^{\frac{\lambda+1}{2}} \frac{da}{|a|} &= \sum_{-\infty}^{\infty} f(\varpi^n) q^{-\frac{3}{2}n(\lambda+1)} \\ &= X\sum_{n>0} q^{-\frac{3}{2}n(\lambda+1)}q^n + X\sum_{n\leq 0} q^{n(3\lambda+2)}q^{-\frac{3}{2}n(\lambda+1)}q^n \\ &= \frac{1+q^{-\frac{3}{2}(\lambda+1)}}{1-q^{-(\frac{3}{2}\lambda+\frac{1}{2})}}\end{aligned}$$

□

Lemma 7.3. *For Re* $\lambda > -\frac{2}{3}$*, we set*

$$X = 1 + \int_{|b|>1} |b|^{-3\lambda-2}\frac{db}{|b|} = \frac{1-q^{-3\lambda-3}}{1-q^{-3\lambda-2}}$$

then we have $f(a) = |a|^{-3\lambda-2}X$ *if* $||a|| \geq ||i||$ *and* $f(a) = |i|^{-3\lambda-2}X$ *if* $||a|| < ||i||$.

Remark 7.4. We have $||i|| = |i| = 1$ under our choice of i.

Proof. If $||a|| \geq ||i||$, then

$$\begin{aligned}f(a) &= \int_{|b|\leq|a|} ||a||^{-\lambda-1}\ db + \int_{|b|>|a|} ||b||^{-\lambda-1}\ db \\ &= |a|^{-3\lambda-3}\mathrm{vol}(|b|\leq|a|) + \int_{|b|>|a|} |b|^{-3\lambda-3}\ db \\ &= |a|^{-3\lambda-2}X\end{aligned}$$

If $||a|| < ||i||$, then

$$\begin{aligned} f(a) &= \int_F ||i-b||^{-\lambda-1}\, db \\ &= ||i||^{-\lambda-1}\mathrm{vol}(b : ||b|| \le ||i||) + \int_{||b||>||i||} |b|^{-3\lambda-2}\frac{db}{|b|} \\ &= ||i||^{-\lambda-\frac{2}{3}} + ||i||^{-\lambda-\frac{2}{3}} \int_{|b|>1} |b|^{-3\lambda-2}\frac{db}{|b|} \\ &= |i|^{-3\lambda-2}X \end{aligned}$$

□

Proposition 7.5. *Denote Φ_K the K-invariant vector we defined in Proposition 4.9, then in the domain of convergence of Δ_χ we have*

$$(\Delta_\chi \Phi_K)(x_0) = C\frac{1-z^2}{(1-q^{-\frac{3}{2}}z)(1-q^{-\frac{1}{2}}z)}$$

Here C is a constant independent of χ.

Proof. Write $G_{x_0} = \eta H \eta^{-1}$ the stabilizer of x_0 in G and $B_{x_0} = B \cap G_{x_0}$, for $\widehat{\mathring{X}^{GL}} = U\backslash BG_{x_0} = A \times^{B_{x_0}} G_{x_0}$.

We fix $\omega_{\mathring{X}^{GL}}$ a standard eigenmeasure on $\mathring{X}^{GL}$, then from [Sak13, Lemma 3.4.2] we get

$$(\Delta_\chi \Phi_K)(x_0) = \int_{B_{x_0}\backslash G_{x_0}} R_\chi \Phi_K(g)\, dg^{st}$$

here R_χ is the Mellin transform, dg^{st} is the invariant measure on $B_{x_0}\backslash G_{x_0}$ compatible with $\omega_{\mathring{X}^{GL}}$.

Write $g = \eta h \eta^{-1}$, and denote

$$\iota :\ Z(H)\backslash H \longrightarrow B_{x_0}\backslash G_{x_0}, \quad Z(H)h \longmapsto B_{x_0}\eta h \eta^{-1}$$

define $dh^{st} = \iota^*(dg^{st})$.

We have

$$(\Delta_\chi \Phi_K)(x_0) = \int_{T\backslash H} R_\chi \Phi_K(\eta h \eta^{-1})\, dh^{st}$$

note that since any invariant measure on $Z(H)\backslash H$ is unique up to a constant, we have $dh^{st} = c\, dh$ for c a constant independent of χ.

Now apply the formula in [Sak13, §4.3], we get

$$R_\chi(\Phi_K) = Q\frac{1-z^2}{1-q^{-3}z^2}\phi_K$$

Here Q is a constant independent of χ.

Since $\eta \in K$, apply the calculation in Proposition 7.2, we get

$$\begin{aligned} (\Delta_\chi \Phi_K)(x_0) &= \int_{Z(H)\backslash H} (R_\chi \Phi_K)(\eta h)\, dh^{st} = Q\frac{1-z^2}{1-q^{-3}z^2}\int_{Z(H)\backslash H} \phi_K(\eta h)\, dh^{st} \\ &= cQ\frac{1-z^2}{1-q^{-3}z^2}\frac{1+q^{-\frac{3}{2}}z}{1-q^{-\frac{1}{2}}z} = C\frac{1-z^2}{(1-q^{-\frac{3}{2}}z)(1-q^{-\frac{1}{2}}z)} \end{aligned}$$

Here $C = cQ$ is a constant independent of χ. □

Proposition 7.6. *We have $b_\omega(\chi) = \frac{F_\omega \Delta_\chi}{\Delta_{\omega_\chi}} = \frac{(1-q^{-\frac{3}{2}}z^{-1})(1-q^{-\frac{1}{2}}z^{-1})}{(1-q^{-\frac{3}{2}}z)(1-q^{-\frac{1}{2}}z)}$. Here we view $F_\omega\Delta_\chi$ and Δ_{ω_χ} as distributions on $\mathcal{S}(U\backslash G)$.*

Proof. We want to compute $\frac{F_\omega \Delta_\chi(\Phi_K)}{\Delta_{\omega\chi}(\Phi_K)}(x_0)$ as meromorphic continuations of $F_\omega\Delta_\chi(\Phi_K)(x_0)$ and $\Delta_{\omega\chi}(\Phi_K)(x_0)$ in terms of the z-variable.

From the expression $\Phi_K = \mathrm{Vol}(J)^{-1}(\Phi_J - e^{\check{\alpha}} F_\omega \Phi_J)$, we have

$$F_\omega \Phi_K = \mathrm{Vol}(J)^{-1}(F_\omega(\Phi_J - e^{\check{\alpha}} F_\omega \Phi_J)) = \mathrm{Vol}(J)^{-1}(F_\omega \Phi_J - e^{-\check{\alpha}}\Phi_J) = -e^{-\check{\alpha}}\Phi_K$$

From the Proposition 7.5, we have

$$F_\omega \Delta_\chi(\Phi_K)(x_0) = \Delta_\chi(F_\omega \Phi_K)(x_0) = \Delta_\chi(-e^{-\check{\alpha}}\Phi_K)(x_0) = -Cz^{-2}\frac{1-z^2}{(1-q^{-\frac{3}{2}}z)(1-q^{-\frac{1}{2}}z)}$$

From the Proposition 7.5, we also have

$$\Delta_{\omega\chi}(\Phi_K)(x_0) = C\frac{1-z^{-2}}{(1-q^{-\frac{3}{2}}z^{-1})(1-q^{-\frac{1}{2}}z^{-1})}$$

hence

$$\frac{F_\omega \Delta_\chi(\Phi_K)}{\Delta_{\omega\chi}(\Phi_K)}(x_0) = -z^{-2}\frac{1-z^2}{(1-q^{-\frac{3}{2}}z)(1-q^{-\frac{1}{2}}z)}\frac{(1-q^{-\frac{3}{2}}z^{-1})(1-q^{-\frac{1}{2}}z^{-1})}{1-z^{-2}} = \frac{(1-q^{-\frac{3}{2}}z^{-1})(1-q^{-\frac{1}{2}}z^{-1})}{(1-q^{-\frac{3}{2}}z)(1-q^{-\frac{1}{2}}z)}$$

□

Definition 7.7. For χ an unramified character of $T_{H\backslash G}$, we can define $\beta(\chi)$ as

$$\beta(\chi) = \frac{1-\chi^2(\varpi)}{(1-q^{-\frac{1}{2}}\chi(\varpi))(1-q^{-\frac{3}{2}}\chi(\varpi))}$$

Proposition 7.8. *We have*

$$B_\omega(\chi) = \frac{\beta(\chi)}{\beta({}^\omega\chi)}$$

Proof. This follows from Proposition 7.6. □

Now we consider the local functional equation for $\mathbf{X}$, we fix χ an unramified character of $\ell^\times$. For the irreducible unramified representation $\pi = I(\chi^2)$ of $SL_2(\ell)$, here we fix the uppertriangular Borel subgroup of $SL_2(\ell)$ with maximal torus $\cong \ell^\times$, note that π is the restriction of $\Pi_1 = I(\chi, \chi^{-1})$, $\Pi_2 = I(\chi\omega_{\ell_1/\ell}, \chi^{-1}\omega_{\ell_1/\ell}^{-1})$, $\Pi_3 = I(\chi\omega_{\ell_2/\ell}, \chi^{-1}\omega_{\ell_2/\ell}^{-1})$, $\Pi_4 = I(\chi\omega_{\ell_3/\ell}, \chi^{-1}\omega_{\ell_3/\ell}^{-1})$ of $GL_2(\ell)$.

From the exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Res}_{\ell/k}\,\mathbb{G}_m \longrightarrow \mathbf{T}_X \longrightarrow 1$$

we get

$$1 \longrightarrow \mu_2(k) \longrightarrow \ell^\times \xrightarrow{q} T_X \xrightarrow{\delta} H^1(k, \mu_2) \longrightarrow H^1(k, \mathrm{Res}_{\ell/k}\mathbb{G}_m)$$

Here

$$q : \ell^\times \longrightarrow T_X$$

is the quotient map. We also get

$$T_X/q(\ell^\times) \cong H^1(k, \mu_2) \cong k^\times/k^{\times 2}$$

Since k has odd residue characteristic, we have

$$k^\times/k^{\times 2} \cong \{1, \epsilon, \varpi, \epsilon\varpi\}$$

where $\epsilon \in \mathfrak{o}_k^\times$ is a nonsquare unit.

Let χ be an unramified character of $\ell^\times$, trivial on $\mu_2(k)$, the four characters of $T(k)$ restricting to the same χ on $q(\ell^\times)$ can be described as

$$\chi_\eta(t) = \chi(t)\eta(\delta(t)), \quad \eta \in \mathrm{Hom}(k^\times/k^{\times 2}, \mathbb{C}^\times)$$

here

$$\delta:\ T_X \longrightarrow k^\times/k^{\times 2}$$

is the connecting homomorphism.

Fix $\mathscr{D} = \{1, \epsilon, \varpi, \epsilon\varpi\} \subset k^\times$ as representatives of $k^\times/k^{\times 2}$. Write $\iota : k^\times \hookrightarrow \ell^\times$. For $d \in \mathscr{D}$, let

$$\omega_d:\ \ell^\times \longrightarrow \{\pm 1\}$$

the quadratic character corresponding to $L_d = \ell(\sqrt{d})$, set $\omega_1 = 1$. Define

$$\eta_d := \omega_d(\iota(c)), \quad [c] \in k^\times/k^{\times 2}$$

Choose the base extension of χ by

$$\tilde{\chi}_1(q(t)) = \chi^2(t), \quad \tilde{\chi}_1(\xi_c) = \chi(\iota(c))$$

We obtain four extensions to T_X of the unramified character χ^2 of T

$$\tilde{\chi}_d = \tilde{\chi}_1(\eta_d \circ \delta)$$

Definition 7.9. We call the characters $\tilde{\chi}_1$ and $\tilde{\chi}_2 = \tilde{\chi}_1 \cdot \eta_\epsilon \circ \delta$ the distinguished unramified characters in Adm_X extending the unramified character χ^2 of $\ell^\times$.

Definition 7.10. Let χ be an unramified character of $T \cong \ell^\times$ and let $\widetilde{\chi} \in \mathrm{Adm}_X$ be a distinguished unramified extension of χ to T_X.

Write $p : T \longrightarrow T_X = T/\mu_2$ be the quotient map. Let $\xi_\varpi \in T_X$ be the image of T_X under p of the element $\sqrt{\varpi} \in \overline{k}$. ξ_ϖ is independent of the choice of square root and satisfy $\xi_\varpi^2 = p(\varpi)$. Put $s = \widetilde{\chi}(\xi_\varpi)$ and we define

$$\beta(\widetilde{\chi}) = \frac{1 - s^2}{(1 - q^{-\frac{1}{2}}s)(1 - q^{-\frac{3}{2}}s)}$$

Proposition 7.11. *For every distinguished unramified extension $\widetilde{\chi}$ of χ, we have*

$$B_\omega(\widetilde{\chi}) = \frac{\beta(\widetilde{\chi})}{\beta({}^\omega\widetilde{\chi})}$$

here for the definition of $\Delta_{\widetilde{\chi}}$ we use the compatible base points and the measure normalizations as in the statement of Proposition 5.19.

Proof. This follows from Proposition 7.8 and central reduction Proposition 5.19. □

8. $\mathbf{X} = \mathbf{SU}_2\backslash\mathbf{SU}_3$

In this section, let k be a p-adic local field and ℓ/k an unramified quadratic extension. We denote $\mathfrak{o}_k$, $\mathfrak{o}_\ell$ the integer rings of k and ℓ, ϖ a uniformizer of $\mathfrak{o}_k$. Let $a \mapsto \overline{a}$ be the nontrivial involution. Let ϖ be an uniformizer of k and let q be the cardinality of the residue field. For $g \in M_3(\ell)$, we write $g^* = \overline{g}^t$.

We study the precise local functional equation for the spherical variety $\mathbf{X} = \mathbf{SU}_2\backslash\mathbf{SU}_3$ where $\mathbf{SU}_3$ is the unramified unitary group over k associated with a Hermitian space over ℓ. Note that this is a generic multiplicity free pair (Proposition **??**).

Let χ be an unramified character of $\ell^\times \cong T_X$ with $z := \chi(\varpi) = q^{-2s_0}$. Because of the local multiplicity one property, we have the weak local functional equation

$$F_\omega \Delta_\chi = b_\omega(\chi)\Delta_{{}^\omega\chi}$$

for $\chi \in \mathrm{Adm}_X$.

We fix

$$G = \{g \in \mathrm{SL}_3(\ell)\ :\ gJg^* = J\}$$

$K = G \cap SL_3(\mathfrak{o}_\ell)$. Let $H = \mathrm{SU}(\ell e_1 \oplus \ell e_3) \cong \mathrm{SU}_2$ and put $K_H = H \cap K$.

We choose $\eta \in \mathfrak{o}_\ell$ with $\eta + \overline{\eta} = -1$ and set

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \eta & 1 & 1 \end{pmatrix}$$

and set $r = (\eta, 1, 1)$, we can check that $\gamma J \gamma^* = J$ and $\det \gamma = 1$ and hence $\gamma \in G$. We can compute $H \cap \gamma^{-1} B \gamma = 1$. We set $x_0 = H\gamma^{-1} \in \mathring{\mathbf{X}}(\mathfrak{o})$.

For $g = (g_{ij}) \in G$, put

$$M(g) = \max_{1 \le j \le 3} |g_{3j}|_\ell$$

If $g = t(a)uk$ is the Iwasawa decomposition of the element g,

$$\phi_K(1) = 1, \quad \phi_K(t(a)uk) = |a|_\ell^{s_0+1}$$

then its bottom row is $\overline{a}^{-1}$ times the bottom row of k, hence

$$\phi_K(g) = M(g)^{-s_0-1} = |a|_\ell^{s_0+1}$$

We have the decomposition

$$H = \bigsqcup_{n \ge 0} K_H a_n K_H, \quad a_n = \mathrm{diag}(\varpi^n, 1, \varpi^{-n})$$

We fix a Haar measure dh on H so that $\mathrm{Vol}(K_H) = 1$.

Lemma 8.1. *We have*

$$Vol(K_H a_n K_H) = \begin{cases} 1, & n = 0 \\ q^{2n} + q^{2n-1}, & n \ge 1 \end{cases}$$

Proof. This follows from [MA26, Proposition 3.2.15]. □

For $k_1 \in K_H$, we may write $rk_1 = (A, 1, C)$ with $A, C \in \mathfrak{o}_\ell$, since $\gamma k_1 \in G$, we can deduce

$$A\overline{C} + \overline{A}C = -1$$

consequently $A, C \in \mathfrak{o}_\ell^\times$.

Let $h = k_1 a_n k_2$ with $k_1, k_2 \in K_H$, then

$$\begin{aligned} \phi_K(\gamma h) &= \phi_K(\gamma k_1 a_n) \\ &= \max\{|A\varpi^n|_\ell, 1, |C\varpi^{-n}|_\ell\}^{-s_0-1} \\ &= q^{-2n(s_0+1)} \end{aligned}$$

Proposition 8.2. *For $|z| = |\chi(\varpi)| < 1$ or equivalently $s_0 > 0$, we have*

$$ev_{x_0} \circ \Delta_\chi(\Phi_K) = C \frac{1+z}{1-q^{-2}z}$$

C is a constant independent of χ.

Proof. Write $G_{x_0} = \gamma H \gamma^{-1}$ the stabilizer of x_0 in G, then $B_{x_0} = B \cap G_{x_0} = B \cap \gamma H \gamma^{-1} = 1$, for $\mathring{\widehat{X}} = T \times G_{x_0}$, we fix a standard eigenmeasure $\omega_{\mathring{\widehat{X}}}$ on $\mathring{\widehat{X}}$, then from [Sak13, Lemma 3.4.2], we get

$$\Delta_\chi(\Phi_K)(x_0) = \int_{G_{x_0}} R_\chi \Phi_K(g)\, dg^{st}$$

here R_χ is the Mellin transform and dg^{st} is the invariant measure on G_{x_0} compatible with $\omega_{\mathring{\widehat{X}}}$.

Denote

$$\iota : \ H \longrightarrow G_{x_0}, \ h \longmapsto \gamma h \gamma^{-1}$$

and define $dh^{st} = \iota^*(dg^{st})$.

We have

$$\Delta_\chi(\Phi_K)(x_0) = \int_H R_\chi \Phi_K(\gamma h \gamma^{-1})\ dh^{st}$$

note that since the invariant measure on H is unique up to a constant, we have $dh^{st} = c\ dh$ for c a constant independent of χ.

Now apply the formula, we get

$$R_\chi(\Phi_K) = Q\frac{1-z^2}{(1-q^{-2}z)(1+q^{-1}z)}\phi_K$$

here $Q = 1+q^{-3}$.

$$\begin{aligned}
(\mathrm{ev}_{x_0}\circ\Delta_\chi)(\Phi_K) &= cQ\frac{1-z^2}{(1-q^{-2}z)(1+q^{-1}z)}\int_H \phi_K(\gamma h)\ dh \\
&= cQ\frac{1-z^2}{(1-q^{-2}z)(1+q^{-1}z)}\sum_{n\geq 0}\int_{K_H a_n K_H} \phi_K(\gamma h)\ dh^{st} \\
&= \frac{1-z^2}{(1-q^{-2}z)(1+q^{-1}z)}(cQ + cQ\sum_{n\geq 1}(q^{2n}+q^{2n-1})q^{-2n(s_0+1)}) \\
&= \frac{1-z^2}{(1-q^{-2}z)(1+q^{-1}z)}cQ(1+(1+q^{-1})\sum_{n\geq 1}z^n) \\
&= cQ\frac{1-z^2}{(1-q^{-2}z)(1+q^{-1}z)}(1+(1+q^{-1})\frac{z}{1-z}) \\
&= cQ\frac{1+z}{1-q^{-2}z}
\end{aligned}$$

We set $C = cQ$, then

$$\mathrm{ev}_{x_0}\circ\Delta_\chi(\Phi_K) = C\frac{1+z}{1-q^{-2}z}$$

□

Proposition 8.3. *We have* $b_\omega(\chi) = \frac{F_\omega\Delta_\chi}{\Delta_{\omega_\chi}} = -z^{-1}\frac{1-q^{-2}z^{-1}}{1-q^{-2}z}$. *Here* $F_\omega\Delta_\chi$ *and* Δ_{ω_χ} *are viewed as families of distributions on* $\mathcal{S}(U\backslash G)$.

Proof. We want to compute $\frac{F_\omega\Delta_\chi(\Phi_K)}{\Delta_{\omega_\chi}(\Phi_K)}(x_0)$ as meromorphic continuations of $F_\omega\Delta_\chi(\Phi_K)(x_0)$ and $\Delta_{\omega_\chi}(\Phi_K)(x_0)$ in terms of the z-variable.

From the expression $\Phi_K = \mathrm{Vol}(J)^{-1}(\Phi_J - e^{\check{a}}F_\omega\Phi_J)$, we have $F_\omega\Phi_K = -e^{-\check{a}}\Phi_K$. From the Proposition 8.2, we have

$$F_\omega\Delta_\chi(\Phi_K)(x_0) = \Delta_\chi(-e^{-\check{a}}\Phi_K)(x_0) = -Cz^{-2}\frac{1+z}{1-q^{-2}z}$$

From the Proposition 8.2, we also have

$$\Delta_{\omega_\chi}(\Phi_K)(x_0) = C\frac{1+z^{-1}}{1-q^{-2}z^{-1}}$$

hence

$$\frac{F_\omega\Delta_\chi(\Phi_K)}{\Delta_{\omega_\chi}(\Phi_K)}(x_0) = -z^{-2}\frac{1+z}{1-q^{-2}z}\cdot\frac{1-q^{-2}z^{-1}}{1+z^{-1}} = -z^{-1}\frac{1-q^{-2}z^{-1}}{1-q^{-2}z}$$

□

We can define $B_\omega(\chi)$ as

$$B_\omega(\chi) = -e^{\lambda_\alpha}(\chi) b_\omega(\chi) = z^2 z^{-1} \frac{1 - q^{-2} z^{-1}}{1 - q^{-2} z} = q^{-2s_0} \frac{1 - q^{-2} q^{2s_0}}{1 - q^{-2} q^{-2s_0}}$$

Definition 8.4. For χ an unramified character of $T_X \cong \ell^\times$, we define

$$\beta(\chi) = \frac{1 + \chi(\varpi)}{1 - q^{-2}\chi(\varpi)}$$

Proposition 8.5. *We have*

$$B_\omega(\chi) = q^{-2s_0} \frac{1 - q^{-2} q^{2s_0}}{1 - q^{-2} q^{-2s_0}} = \frac{\beta(\chi)}{\beta({}^\omega\chi)}$$

Proof. This follows from the computation for $b_\omega(\chi)$ in Proposition 8.3. □

Declaration of generative AI use

The first draft of this paper is part of my thesis and it is purely human written. We used ChatGPT-6 Astra and a verification framework OSVerify [NYL+26] with ChatGPT-6 Astra to polish the draft and improve writing. The author verified all the output. The author is responsible for the correctness of this paper.

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