ON THE UNRAMIFIED SPECTRUM FOR SPHERICAL VARIETIES FOR UNRAMIFIED GROUPS

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1. Introduction

Let k be a field of characteristic zero and $G = \mathbf{G}(k)$ denote the k-points of an unramified group $\mathbf{G}$ over k, that is $\mathbf{G}$ is quasisplit and split over an unramified extension of k. Let $\mathbf{X}$ be a $\mathbf{G}$-spherical variety, $X := \mathbf{X}(k)$. In this paper, we study the unramified spectrum for the G-spherical variety X, generalizing part of the result of [Sak08] from split groups to unramified groups.

We first recall the Satake isomorphism for G. Then we will define k-Knop action and k-Brion path, and study their properties. Following the strategy of Sakellaridis for split groups, we introduce intertwining homomorphisms S^Y_χ, where Y denotes the k-points of a $\mathbf{B}$-orbit $\mathbf{Y}$.

The first main result of this paper is the following necessary theorem:

Theorem 1.1. *(cf. Theorem 5.19) Let $\boldsymbol{X}$ be a homogeneous spherical variety for an unramified group $\boldsymbol{G}$, assume that $\mathring{X} = \mathring{\boldsymbol{X}}(k)$ carries a B-invariant measure. A necessary condition for the existence of a non-zero morphism $C_c^\infty(X) \to I(\chi)$ is*

$$\chi \in {}^{\omega}(\delta^{-1/2} A_X^*) \tag{1.1}$$

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for some $\omega \in W_k$. For almost all χ satisfying (1.1), the space of morphisms $C_c^\infty(X) \to I(\chi)$ admits a basis consisting of all S_χ^Y with $\chi \in Adm_Y$.

This theorem is proved using the reduction to rank-one cases.

Given $(\mathbf{Y}, \alpha)$ a parabolic pair, we denote $\mathbf{P}_\alpha$ the parabolic subgroup associated with α and $\mathbf{L}_\alpha$ its Levi subgroup. Then $\mathbf{Y}\mathbf{P}_\alpha/R(\mathbf{P}_\alpha)$ is a spherical variety for $\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$ which is a spherical variety for an adjoint unramified group of split rank one. We define the type of the parabolic pair $(\mathbf{Y}, \alpha)$ to be the corresponding type of spherical subgroups of $\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$, which is defined in Theorem 4.6 and Theorem 4.7.

The second main result of this paper is the following.

Theorem 1.2. *(cf. Theorem 5.21) Let $\boldsymbol{Y}$ be a k-rational $\boldsymbol{B}$-orbit on $\boldsymbol{X}$, and let α be a simple root such that $\boldsymbol{Y}$ is open in $\boldsymbol{Y}\cdot\boldsymbol{P}_\alpha$. The following theorem describes the composition of T_{ω_α} with $\underline{S}_\chi^Y$ for $\chi \in Adm_Y$:*

- *if $(\boldsymbol{Y}, \alpha)$ is of type G, then $T_{\omega_\alpha} \circ S_1 = 0$ for every $S_1 \in \underline{S}_\chi^Y$.*
- *if $(\boldsymbol{Y}, \alpha)$ is of type U, then $T_{\omega_\alpha} \circ S_{\tilde{\chi}}^Y \sim S_{\omega_\alpha \tilde{\chi}}^{\omega_\alpha Y}$.*
- *if $(\boldsymbol{Y}, \alpha)$ is of strongly tempered type, then $T_{\omega_\alpha} \circ S_{\tilde{\chi}}^Y \sim S_{\omega_\alpha \tilde{\chi}}^{\omega_\alpha Y}$.*

Remark 1.3. Put $\mathbf{G}_\alpha = \mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$, and let $\mathbf{H}_\alpha$ be the image in $\mathbf{G}_\alpha$ of the stabilizer $(\mathbf{P}_\alpha)_y$, for $y \in \mathbf{Y}(k)$. We call $(\mathbf{Y}, \alpha)$ of type G if $\mathbf{H}_\alpha = \mathbf{G}_\alpha$. This convention includes all the type G cases in Theorem 4.6 and Theorem 4.7.

For the strongly tempered case, the proof directly generalizes Sakellaridis' proof in the split type T case.

Finally, we prove the weak local functional equations for the Galois symmetric variety $\mathbf{X} = \mathbf{SL}_2\backslash \mathrm{Res}_{\ell/k}\mathbf{SL}_2$ with ℓ/k an unramified quadratic extension.

Proposition 1.4. *(cf. Proposition 7.9) For $X = SL_2(k)\backslash SL_2(\ell)$, let χ be an unramified character of $\ell^\times$, and denote by $\mathring{X}$ the open Borel orbit of X, let $S_{\tilde{\chi}_1}$, $S_{\tilde{\chi}_2}$ be two intertwining operators in $\underline{S}_\chi$ associated with two distinct admissible characters $\tilde{\chi}_i$ which extend χ. Then the intertwining functionals $S_{\tilde{\chi}_i}$ satisfy the weak local functional equations:*

$$T_\omega \circ S_{\tilde{\chi}_i} \sim S_{\omega_{\tilde{\chi}_i}}$$

2. Satake isomorphism

In this section, we recall the Satake isomorphism for unramified p-adic groups following [Sha25, Chapter 2]. Throughout this section, k is a p-adic local field of characteristic zero, we denote q the cardinality of the residue field of k. Let $\mathbf{G}$ be a connected reductive group over k, we fix $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$. Let $\mathbf{T} \subset \mathbf{B}$ be a maximal torus of $\mathbf{G}$, and let $\mathbf{A}$ be a maximal split torus of $\mathbf{T}$. We use $X^*(\mathbf{T}) = \mathrm{Hom}_{\overline{k}-\mathrm{groups}}((\mathbf{T})_{\overline{k}}, (\mathbb{G}_m)_{\overline{k}})$ and $X^*(\mathbf{A}) = \mathrm{Hom}_{\overline{k}-\mathrm{groups}}((\mathbf{A})_{\overline{k}}, (\mathbb{G}_m)_{\overline{k}})$ to denote the character lattices of $\mathbf{T}$ and $\mathbf{A}$. Let $X^*(\mathbf{T})_k = X^*(\mathbf{T})^{\Gamma_k}$ be the subgroup of k-rational characters of $\mathbf{T}$, that is, those defined over k. Observe that $X^*(\mathbf{A}) = X^*(\mathbf{A})^{\Gamma_k}$. The k-simple roots of $\mathbf{G}$ with respect to $\mathbf{B}$ will be denoted by Σ_k.

Next let $X_*(\mathbf{T}) = \mathrm{Hom}(X^*(\mathbf{T}), \mathbb{Z})$ and $X_*(\mathbf{T})_k = \mathrm{Hom}(X^*(\mathbf{T})_k, \mathbb{Z})$. Similarly, $X_*(\mathbf{A})$ is equal to $X_*(\mathbf{A})_k$. We then have the standard homomorphisms

$$H_T : \ \mathbf{T}(k) \longrightarrow X_*(\mathbf{T})_k$$

by $q^{\langle H_T(t), \chi\rangle} = |\chi(t)|_k$ and

$$H_A : \ \mathbf{A}(k) \longrightarrow X_*(\mathbf{A})$$

The embedding $\theta : \mathbf{A} \subset \mathbf{T}$ induces an injection

$$0 \longrightarrow X_*(\mathbf{A}) \longrightarrow X_*(\mathbf{T})$$

Let ${}^0\mathbf{T}$ and ${}^0\mathbf{A}$ be the kernels of H_T and H_A, i.e. the largest compact subgroups of $\mathbf{T}(k)$ and $\mathbf{A}(k)$, we then have the following exact sequences with commuting vertical embeddings since ${}^0\mathbf{A}\backslash\mathbf{A} \hookrightarrow {}^0\mathbf{T}\backslash\mathbf{T}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^0\mathbf{A} & \longrightarrow & \mathbf{A} & \xrightarrow{H_A} & X_*(\mathbf{A}) & \longrightarrow & X_*(\mathbf{A}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \overline{\theta}} & & \downarrow{\scriptstyle \theta_*} & & \\ 0 & \longrightarrow & {}^0\mathbf{T} & \longrightarrow & \mathbf{T} & \xrightarrow{H_T} & H_T(\mathbf{T}(k)) & \longrightarrow & X_*(\mathbf{T})_k & & \end{array}$$

In general we have

$$X_*(\mathbf{A}) \subset H_T(\mathbf{T}(k)) \subset X_*(\mathbf{T})_k$$

and $H_T(\mathbf{T}(k))$ can coincide with either endpoint.

Proposition 2.1. *For an unramified torus $\boldsymbol{T}$, we have*

$$H_T(\boldsymbol{T}(k)) = X_*(\boldsymbol{A})$$

Proof. Assume that $\mathbf{T}$ splits over k'. Then we have

$$\mathbf{T}(k')/{}^0\mathbf{T}(k') \cong \mathrm{Hom}(X^*(\mathbf{T}), \mathbb{Z})$$

By Hilbert's Theorem 90 and the fact that k'/k is unramified, we have $H^1(\Gamma_{k'/k}, \mathcal{O}_{k'}^*) = 0$ which implies that $H^1(\Gamma_{k'/k}, {}^0T(k')) = 0$ as ${}^0\mathbf{T}'(k)$ is a product of copies of $\mathcal{O}_{k'}^*$.

Applying Galois cohomology to the sequence

$$0 \longrightarrow {}^0\mathbf{T}(k') \longrightarrow \mathbf{T}(k') \longrightarrow \mathbf{T}(k')/{}^0\mathbf{T}(k') \longrightarrow 0$$

and $H^0(\Gamma_{k'/k}, X) = X^{\Gamma_{k'/k}}$ where X is any group appearing in the sequence. We conclude that

$$(\mathbf{T}(k')/{}^0\mathbf{T}(k')^{\Gamma_{k'/k}} = \mathbf{T}(k)/{}^0\mathbf{T}(k)$$

where $({}^0\mathbf{T}(k'))^{\Gamma_{k'/k}} = {}^0\mathbf{T}(k') \cap \mathbf{T}(k)$ is in fact the largest compact subgroup ${}^0T$ of $T = \mathbf{T}(k)$. Consequently, H_T establishes an isomorphism $\mathbf{T}(k)/{}^0T \cong X_*(\mathbf{T})^{\Gamma_{k'/k}} = X_*(\mathbf{A})$. □

Set $\Lambda = \Lambda(T) = H_T(\mathbf{T}(k))$.

Definition 2.2. A character $\chi : \mathbf{T}(k) \longrightarrow \mathbb{C}^*$ is unramified if $\chi|^0T = 1$, i.e. χ factors through Λ. We denote the set of unramified characters of $\mathbf{T}(k)$ by X_{un}.

The embedding $\mathbf{A} \hookrightarrow \mathbf{T}$ gives

$$\mathbf{A}(k)/^0A \cong \mathbf{T}(k)/^0T \cong \Lambda$$

Hence $X_{un} = \mathrm{Hom}(\Lambda, \mathbb{C}^*) = \check{\mathbf{A}}$, where $\check{\mathbf{A}}$ is the dual torus of $\mathbf{A}$ defined via $X^*(\mathbf{A}) = X_*(\check{\mathbf{A}})$. Thus, we may identify X_{un} as the set of unramified characters for $\mathbf{A}$, which we will denote by A^*.

Now let $\mathbf{G}$ be a connected unramified group over k, and choose a Borel subgroup $\mathbf{B}$ with Levi decomposition $\mathbf{B} = \mathbf{TU}$. Let $\mathcal{H}(\mathbf{G}(k), K),\ \mathcal{H}(\mathbf{T}(k), {}^0T)$ be the corresponding smooth Hecke algebras. We can identify the Hecke algebra $\mathcal{H}(\mathbf{T}(k), {}^0T)$ with the group algebra $\mathbb{C}[\Lambda]$. We define the Satake transform

$$S :\ \mathcal{H}(\mathbf{G}(k), K) \to \mathcal{H}(\mathbf{T}(k), {}^0T)$$

by

$$\begin{aligned} Sf(t) &= \delta(t)^{1/2} \int_U f(tu)\ du \\ &= \delta(t)^{-1/2} \int_U f(ut)\ du \end{aligned}$$

The k-Weyl group $W_k := W(G, A)$ acts on $X^*(\check{\mathbf{A}}) = \Lambda$ and hence on $\mathbb{C}[\Lambda]$.

Theorem 2.3. *([Sha25, Theorem 2.2.1]) The Satake isomorphism S is an algebra isomorphism from $\mathcal{H}(\boldsymbol{G}(k), K)$ onto $\mathbb{C}[\Lambda]^{W_k}$.*

Let's denote $\mathfrak{o}$ the integer ring of k and ϖ its uniformizer. We have the following split exact sequence,

$$1 \longrightarrow \mathbf{T}(\mathfrak{o}) \longrightarrow \mathbf{T}(k) \xrightarrow{H_T} X_*(T)_k \longrightarrow 0$$

We generalize this to a closed subgroup $\mathbf{D} \subseteq \mathbf{T}$.

Proposition 2.4. *Assume $\boldsymbol{D} \subseteq \boldsymbol{T}$ is a closed k-subgroup and $\boldsymbol{T}$ is an unramified torus. Then there is a split exact sequence,*

$$1 \longrightarrow \boldsymbol{D}(k) \cap \boldsymbol{T}(\mathfrak{o}) \longrightarrow \boldsymbol{D}(k) \xrightarrow{-H_T|_{D(k)}} X_*(D^0)_k \longrightarrow 0$$

Proof. Let ℓ/k be a finite unramified extension which splits $\mathbf{T}$. Write $\sigma = \mathrm{Fr}$. For $t \in \mathbf{T}(k)$, then we have

$$\langle \chi, H_T(t) \rangle = -v_\ell(\chi(t))$$

$H_T(t) \in X_*(T)_k$ and for an unramified torus $\mathbf{T}$, $\ker H_T = \mathbf{T}(\mathfrak{o})$.

Let $i : D \hookrightarrow T$ be the inclusion, the restriction map on character groups is

$$i^* :\ X^*(T) \longrightarrow X^*(D)$$

Set

$$M_D = \{\chi \in X^*(T)\ :\ \chi|_{D^0} = 1\}$$

Take $\chi \in M_D$. Then $i^*\chi$ has finite order n. Hence for every $d \in \mathbf{D}(k)$, $\chi(d)^n = 1$.

Therefore $nv_\ell(\chi(d)) = v_\ell(\chi(d)^n) = v_\ell(1) = 0$. Hence $v_\ell(\chi(d)) = 0$ for every $\chi \in M_D$. The annihilator of M_D inside $X_*(T)_k$ is $X_*(D^0)_k$, we get $H_T(d) \in X_*(D^0)_k$.

Put $\lambda_d = -H_T(d)$, let $d_0 = \lambda_d(\varpi) \in \mathbf{D}^0(k)$, set $d_c = d_0^{-1}d$, then $H_T(d_c) = -H_T(d) + H_T(d_0) = 0$. We get $d_c \in \mathbf{T}(\mathfrak{o})$, hence $d_c \in \mathbf{T}(k) \cap \mathbf{T}(\mathfrak{o})$.

We get an exact sequence

$$1 \longrightarrow \mathbf{D}(k) \cap \mathbf{T}(\mathfrak{o}) \longrightarrow \mathbf{D}(k) \longrightarrow X_*(D^0)_k \longrightarrow 0$$

the splitting is given by $\lambda \in X_*(D)_k \mapsto \lambda(\varpi)$.

□

3. Knop action and Brion path over algebraic closure

In this section, $k = \overline{k}$ is an algebraically closed field of characteristic zero, $\mathbf{G}$ is a connected reductive group over $k = \overline{k}$. $\mathbf{X}$ is a $\mathbf{G}$-spherical variety. We denote $\Sigma_{L(X)}$ the simple roots of the Levi subgroup $\mathbf{L}(\mathbf{X})$ of $\mathbf{P}(\mathbf{X})$.

We describe a Weyl group action on the set of Borel orbits-the Knop action. For simple reflection ω_α corresponding to a simple root α it is defined as follows: Let $\mathbf{P}_\alpha$ denote the parabolic of semisimple split rank one associated to $\alpha \in \Sigma_k(G)$ for a fixed choice of maximal torus $\mathbf{T} \hookrightarrow \mathbf{B}$, and let $\mathbf{Y}$ be a $\mathbf{B}$-orbit. The simple reflection ω_α acts on the set of $\mathbf{B}$-orbits contained in the $\mathbf{P}_\alpha$-orbit $\mathbf{Y} \cdot \mathbf{P}_\alpha$, let $\mathbf{L}_\alpha$ denote the Levi subgroup of $\mathbf{P}_\alpha$, then $\mathbf{L}_\alpha / Z(\mathbf{L}_\alpha)$ is isomorphic to $\mathbf{PGL}_2$, we may view $\mathbf{Y} \cdot \mathbf{P}_\alpha / \mathbf{U}_{\mathbf{P}_\alpha} Z(\mathbf{L}_\alpha)$ as a spherical variety for $\mathbf{L}_\alpha / Z(\mathbf{L}_\alpha)$.

We define the action according to the type of spherical subgroup: If the spherical subgroup is of type G, ω_α will stabilize the unique $\mathbf{B}$-orbit in the given $\mathbf{P}_\alpha$-orbit. If the spherical subgroup is of type T, ω_α stabilizes the open orbit and interchanges the other two. If the spherical subgroup is of type N, ω_α stabilizes both orbits. Finally if the spherical subgroup is of type U, ω_α interchanges the two orbits.

For unramified computations, it is more convenient to consider the graph $\mathfrak{G}$, whose vertices are the $\mathbf{B}_k$-orbits on $\mathbf{X}_k$ of maximal rank and whose edges are labeled by the set Σ_G of simple roots, where an orbit $\mathbf{Y}$ is joined to an orbit $\mathbf{Z}$ by an edge labeled by α if ${}^{\omega_\alpha}\mathbf{Y} = \mathbf{Z}$ under Knop's action. We call this graph the *Knop graph*.

Let $\mathbf{Y}$ be a $\mathbf{B}$-orbit, for every oriented path γ from $\mathbf{Y}$ to $\mathring{\mathbf{X}}$ in the graph $\mathfrak{G}$, we denote by $\omega(\gamma)$ the Weyl group element corresponding to the path, that is $\omega_{\alpha_n} \cdots \omega_{\alpha_1}$, where $\alpha_1, \cdots \alpha_n$ are the consecutive labels for the edges in the oriented path. In particular ${}^{\omega(\gamma)}\mathbf{Y} = \mathring{\mathbf{X}}$ under the Knop action. We will denote $\mathfrak{G}(\mathbf{Y})$ the set of all paths from $\mathbf{Y}$ to $\mathring{\mathbf{X}}$ and $W(Y)$ the set of all $\omega(\gamma)$ for $\gamma \in \mathfrak{G}(Y)$.

Based on a description of the set $W(Y)$, Brion proves:

Theorem 3.1. *([Bri01, Theorem 4]) Denote by $stab_W(\mathring{\boldsymbol{X}})$, the stabilizer of $(\mathring{\boldsymbol{X}})$ under the Knop action, then a set of generators of $stab_W(\mathring{\boldsymbol{X}})$ consists of the elements ω_α, $\alpha \in \Sigma_{L(X)}$ and the element ω with the following property: there is a decomposition $\omega = \omega_1^{-1}\omega_2\omega_1$ such that*

- *${}^{\omega_1}\mathring{\boldsymbol{X}} = \boldsymbol{Y}$, with $codim(Y) = \ell(\omega_1)$, i.e. $\omega_1^{-1} \in W(Y)$.*
- *ω_2 is either equal to ω_α where $(\boldsymbol{Y}, \alpha)$ is of type T or N, or equal to $\omega_\alpha\omega_\beta$ where α, β are two orthogonal simple roots which raise the same orbit $\boldsymbol{Y}'$ of maximal rank to $\boldsymbol{Y}$.*

Definition 3.2. Consider the Knop graph $\mathfrak{G}$ and the paths $\mathfrak{g}$ corresponding to the $\mathrm{stab}_W(\mathring{\mathbf{X}})$-generators in Brion's theorem, that is the path consisting of a sequence of adjacent edges of the form $e_1 e_2 \cdots e_n E_1 E_2 e_n \cdots e_2 e_1$ where

- one vertex of the edge of e_1 is $\mathring{\mathbf{X}}$ and for all edges e_i if $\mathbf{Y}$ is a vertex and α is the label then $(\mathbf{Y}, \alpha)$ is of type U.
- if $\mathbf{Y}$ denotes the lower vertex of the edge e_n then one of the following is true: either $E_1 = E_2$ and it is labeled by a simple root α such that $(\mathbf{Y}, \alpha)$ is of type T, N, or E_1, E_2 are labeled by orthogonal simple roots α, β lowering $\mathbf{Y}$ to the same $\mathbf{B}$-orbit.

Let $\alpha_1, \cdots, \alpha_n$ denote the labels of $e_1, \cdots, e_n$, let $\omega_2 = \omega_\alpha$ in the first case and $\omega_2 = \omega_\alpha\omega_\beta$ in the second, if the path $\mathfrak{g}$ corresponds to a reduced decomposition of the corresponding Weyl group element $\omega = \omega_{\alpha_1} \cdots \omega_{\alpha_n}\omega_2\omega_{\alpha_n} \cdots \omega_{\alpha_1}$, then the path will be called a *Brion path* corresponding to ω.

Definition 3.3. We define $\mathbf{P}_\gamma, \mathbf{L}_\gamma$ to be the standard parabolic and Levi $\mathbf{P}_{\mathrm{supp}(\gamma)}, \mathbf{L}_{\mathrm{supp}(\gamma)}$, and $\mathbf{X}_\gamma$ the spherical variety $\mathring{\mathbf{X}}\mathbf{P}_\gamma/\mathbf{U}_{\mathbf{P}_\gamma}$, here $\mathrm{supp}(\gamma)$ denotes the minimal set of simple roots such that γ lies in their linear span.

Sakellaridis proves the following existence result for the Brion path:

Proposition 3.4. *([Sak13, Proposition 6.2.1]) For every spherical root γ, the corresponding element $\omega_\gamma \in W_X$ admits a Brion path.*

Sakellaridis proves the following rank bound on the spherical variety $\mathbf{X}_\gamma$:

Proposition 3.5. *([Sak13, Proposition 6.2.2]) A spherical variety for a group $\boldsymbol{G}$ with the property that there is a spherical root γ such that $\boldsymbol{P}_\gamma = \boldsymbol{G}$ is of rank at most two. Equivalently, for any spherical variety and any spherical root γ of it, the span of the support of γ contains at most two spherical roots.*

4. Knop action and Brion path over k

4.1. **Knop's action over algebraic closure.** In this section we will assume that k is an algebraically closed field of characteristic zero.

Let $\mathbf{X}$ be a $\mathbf{G}$-spherical variety, we fix $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$. We recall Knop's action of Weyl group on the set of Borel orbits. We denote W the absolute Weyl group of $\mathbf{G}$.

We denote $\mathfrak{B}(X)$ the set of all non-empty, closed, irreducible, $\mathbf{B}$-stable subsets of $\mathbf{X}$. For $\mathbf{X}$ a $\mathbf{G}$-spherical variety, we may identify this set as the set of Borel orbits in $\mathbf{X}$.

For a parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ and a $\mathbf{P}$-stable closed subvariety $\mathbf{Y} \subseteq \mathbf{X}$, let

$$\mathfrak{B}(Y, P) = \{\mathbf{Z} \in \mathfrak{B}(X) \mid \mathbf{PZ} = \mathbf{Y}\}$$

For every simple root α we construct an action of s_α on $\mathfrak{B}(X)$, we define this separately on each subset $\mathfrak{B}(Y, P_\alpha)$, there is an open $\mathbf{P}_\alpha$-stable subset $\mathbf{Y}^0 \subseteq \mathbf{Y}$ such that under the map $\Phi : P_\alpha \to \mathrm{Aut}(P_\alpha/B) = PGL_2$, for all $y \in \mathbf{Y}^0$ the group $\Phi(P_y)$ is either finite or conjugate to subgroups of $\mathbf{PGL}_2$ of types G, U, T or N, and we can define the s_α-action on $\mathfrak{B}(Y, P_\alpha)$ in each case. If it is type G, ω_α stabilizes the unique $\mathbf{B}$-orbit in the given $\mathbf{P}_\alpha$-orbit. In the case of type T, ω_α stabilizes the open orbit and interchanges the other two. In the case of type N, ω_α stabilizes both orbits. Finally in the case of type U, ω_α interchanges the two orbits.

Theorem 4.1. *([Kno95, Theorem 4.2]) Let $\boldsymbol{X}$ be a $\boldsymbol{G}$-spherical variety. Then the s_α-actions on $\mathfrak{B}(X)$ define an action of W.*

Let $\tilde{W}$ be the group generated by $\{s_\alpha \mid \alpha \in \Sigma_G\}$ with the relations $s_\alpha^2 = 1$, then there is a surjective homomorphism $\tilde{W} \to W$. The previous Theorem 4.1 tells us that the $\tilde{W}$-action on $\mathfrak{B}(X)$ factors through W.

We consider

$$\mathfrak{B}_0(X) = \{\mathbf{Z} \in \mathfrak{B}(X) \mid \mathrm{rk}\ Z = \mathrm{rk}\ X\}$$

The set is stable under the $\tilde{W}$-action.

Theorem 4.2. *([Kno95, Theorem 6.2]) The $\tilde{W}$-action on $\mathfrak{B}_0(X)$ factors through W, the isotropy group of (X) is $W_{(X)} = W_X \ltimes W_{P(X)}$.*

4.2. **Knop action over k.** In this section, k is a p-adic field of characteristic zero. We define the Knop action over k.

We have the following classification of adjoint unramified groups of split rank one: $\mathrm{Res}_{\ell/k}\mathbf{PGL}_2$, here ℓ is an unramified extension of k, $\mathrm{Res}_{\ell/k}\mathbf{PU}_3$, here ℓ is an unramified extension of k and $\mathbf{PU}_3$ is the projective unitary group over ℓ associated with a quadratic extension $\tilde{\ell}/\ell$.

Based on the study of transitive action on the spherical system for unramified adjoint groups of split rank one (see § **??**), we have the following classification of the spherical subgroups, we list the spherical subgroups for $\mathrm{Res}_{\ell/k}\mathbf{PGL}_2$ and $\mathrm{Res}_{\ell/k}\ \mathbf{PU}_3$ separately.

The following theorem is a classical theorem on the classification of spherical subgroups of $\mathbf{PGL}_2$:

Theorem 4.3. *The spherical subgroups $\boldsymbol{H}$ of $\boldsymbol{G} = \boldsymbol{PGL}_2$ are:*

Type G: $\boldsymbol{PGL}_2$, in this case there is a single $\boldsymbol{B}$-orbit on $\boldsymbol{H}\backslash\boldsymbol{G}$.
Type T split: a split maximal torus $\boldsymbol{T} \cong \mathbb{G}_m$, there are three $\boldsymbol{B}$-orbits: the open one, the two closed orbits of smaller rank.
Type T nonsplit: a nonsplit maximal torus $\boldsymbol{T} \cong U_1$, there are three $\boldsymbol{B}$-orbits: the open one, the two closed orbits of smaller rank, the closed orbits do not have rational k-points.
Type N: $N(\boldsymbol{T})$, there are two $\boldsymbol{B}$-orbits, the open one and a closed one of smaller rank.
Type U: $\boldsymbol{S}\cdot\boldsymbol{U}$, where $\boldsymbol{U}$ is a maximal unipotent subgroup and $\boldsymbol{S} \subset N(\boldsymbol{U})$, there are two $\boldsymbol{B}$-orbits, the open one and a closed one, both of the same rank.

Remark 4.4. The difference between the split type T and nonsplit type T cases is that they have different Galois action on the colors (the two closed Borel orbits).

Definition 4.5. For spherical subgroups $\mathbf{H}$, $\mathbf{H}'$ of an unramified adjoint group $\mathbf{G}$ of split rank one, we say that $\mathbf{H}$ and $\mathbf{H}'$ are of the same type if $(\mathbf{H}\backslash\mathbf{G})_{\overline{k}}$ and $(\mathbf{H}'\backslash\mathbf{G})_{\overline{k}}$ have the same spherical system over algebraic closure.

We have the following list of types of spherical subgroups for $\mathrm{Res}_{\ell/k}\mathbf{PGL}_2$:

Theorem 4.6. *We have the following list of types of spherical subgroups $\boldsymbol{H}$ of $\boldsymbol{G} = Res_{\ell/k}\boldsymbol{PGL}_2$:*

Type G: $Res_{\ell/k}\boldsymbol{PGL}_2$.
Type T: a maximal torus $Res_{\ell/k}\boldsymbol{T}$, here $\boldsymbol{T}$ is a maximal torus in $\boldsymbol{PGL}_2$.
Type N: $N(Res_{\ell/k}\boldsymbol{T})$.
Type U: $\boldsymbol{S}\cdot\boldsymbol{U}$, where $\boldsymbol{U}$ is a maximal unipotent subgroup of $\boldsymbol{G}$ and $\boldsymbol{S} \subset N(\boldsymbol{U})$.
Type TU:
- *$\boldsymbol{H} = Res_{k'/k}\boldsymbol{T}_1\boldsymbol{U}_1$, here $\boldsymbol{T}_1\boldsymbol{U}_1$ is a k'-spherical subgroup of $Res_{\ell/k'}PGL_2$, k'/k is an unramified extension.*

Group case: suppose ℓ/k' is an unramified quadratic extension, then $Res_{k'/k}\boldsymbol{PGL}_2$ is a spherical subgroup of $\boldsymbol{G}$.
Triple product: suppose ℓ/k' is an unramified cubic extension, then $Res_{\ell/k'}\boldsymbol{PGL}_2$ is a spherical subgroup of $\boldsymbol{G}$.

Proof. This follows from the classification in Section **??**. □

We have the following list of types of spherical subgroups for $\mathrm{Res}_{\ell/k}\mathbf{PU}_3$:

Theorem 4.7. *We have the following list of types of spherical subgroups $\boldsymbol{H}$ of $\boldsymbol{G} = Res_{\ell/k}\boldsymbol{PU}_3$:*

Type G: $Res_{\ell/k}\boldsymbol{PU}_3$.
Type N: $Res_{\ell/k}\boldsymbol{PO}_3$.
Type U: $\boldsymbol{S}\cdot\boldsymbol{U}$, where $\boldsymbol{U}$ is a maximal unipotent subgroup of $\boldsymbol{G}$ and $\boldsymbol{S} \subset N(\boldsymbol{U})$.
Type U_2: $\boldsymbol{H} = Res_{\ell/k}\boldsymbol{U}_2$, here $\boldsymbol{U}_2 \hookrightarrow \boldsymbol{PU}_3$ is a two-dimensional unitary subgroup.
Type TU:
- *$\boldsymbol{H} = Res_{k'/k}\boldsymbol{T}_1\boldsymbol{U}_1$, here $\boldsymbol{T}_1\boldsymbol{U}_1$ is a k'-spherical subgroup of $Res_{\ell/k'}\boldsymbol{PU}_3$, k'/k is an unramified extension.*
- *$\boldsymbol{H} = Res_{k'/k}\boldsymbol{T}_2\boldsymbol{U}_2$, here $\boldsymbol{T}_2\boldsymbol{U}_2$ is a k'-spherical subgroup of $Res_{\ell/k'}\boldsymbol{PU}_3$, k'/k is an unramified extension.*

Proof. This follows from the classification in Section **??**. □

We let $\mathfrak{B}_k(X)$ be the set of k-rational $\mathbf{B}$-orbits,

$$\mathfrak{B}_k(X) = \{x \cdot \mathbf{B} \mid x \in \mathbf{X}(k)\}$$

We extend the notions of the maximal split torus (Definition **??**) and maximal anisotropic torus to $\mathbf{Y} \in \mathfrak{B}_k(X)$.

Definition 4.8. For $\mathbf{Y} \in \mathfrak{B}_k(X)$, we define the split rank of $\mathbf{Y}$, denote by $\mathrm{rk}_k(Y)$ to be the dimension of $\mathbf{A_Y} := \mathbf{Y}/\mathbf{UM}$ viewed as an $\mathbf{A}$-variety. We define the anisotropic rank of $\mathbf{Y}$, denote as $\mathrm{s}_k(Y)$ to be the dimension of $\mathbf{M}_Y := \mathbf{Y}/\mathbf{UA}$ viewed as an $\mathbf{M}$-variety.

We have the following proposition,

Proposition 4.9. *([KZ20, Corollary 3.10]) In the lexicographic order, we have*

$$(rk_k(Y), s_k(Y)) \leq (rk_k(X), s_k(X))$$

Definition 4.10. Given $\mathbf{Y} \in \mathfrak{B}_k(X)$, $\mathbf{Y}$ is called principal if $(\mathrm{rk}_k(Y), s_k(Y))$ is maximal in the lexicographic order, and we denote the set of principal orbits by $\mathfrak{B}_0(X) \subseteq \mathfrak{B}_k(X)$.

We have the following result on the Borel orbits structure of semisimple unramified group $\mathbf{G}$ of split rank one.

Proposition 4.11. *Let $\boldsymbol{G}$ be an unramified semisimple group of k-rank 1 and let $\boldsymbol{H}$ be a k-subgroup, then for the $\boldsymbol{H}$-orbits on $\boldsymbol{B}\backslash\boldsymbol{G} = Fl$ one of the following holds:*

1. *$\boldsymbol{H}$ spherical and non-reductive*
   - *(Type U)*
   $$\mathfrak{B}(Fl/H) = \{Y > Z\},\ rk_k(Z) = rk_k(Y)$$
   - *(Type TU)*
   $$\mathfrak{B}(Fl/H) = \{Y > Z_1 > Z_2\},\ rk_k(Z_i) = rk_k(Y) - 1,\ s_k(Z_1) = s_k(Z_2)$$
2. *$\boldsymbol{H}$ is spherical, reductive*
   - *(Type T non-split, Type U_2 nonsplit)*
   $$\mathfrak{B}(Fl/H) = \{Y\}$$
   - *(Type T split)*
   $$\mathfrak{B}(Fl/H) = \{Y > Z_1,\ Z_2\},\ rk_k(Z_i) = rk_k(Y) - 1$$
   - *(Group case and triple product, Type U_2 split)*
   $$\mathfrak{B}(Fl/H) = \{Y > Z\},\ rk_k(Z) = rk_k(Y) - 1$$
   - *(Type N)*
   $$\mathfrak{B}(Fl/H) = \{Y > Z\},\ rk_k(Z) = rk_k(Y) - 1$$
3. *$\boldsymbol{H}$ is not spherical.*

Proof. This follows from [KZ20, Proposition 3.14] as we can identify the types of our spherical subgroups with types of spherical subgroups in [KZ20, Proposition 3.14]. □

We introduce the set of Borel orbits of $\mathbf{X}$ over algebraic closure:

$$\mathfrak{B}_{\overline{k}}(\mathbf{X}_{\overline{k}}) = \{x \cdot \mathbf{B}_{\overline{k}} \mid x \in \mathbf{X}(\overline{k})\}$$

Then we have an inclusion $\mathfrak{B}_k(\mathbf{X}) \subseteq \mathfrak{B}_{\overline{k}}(\mathbf{X}_{\overline{k}})$.

Lemma 4.12. *The Knop action is compatible with the Γ-action for $\boldsymbol{Y} \in \mathfrak{B}_{\overline{k}}(\boldsymbol{X}_{\overline{k}})$, $s \in W_{\overline{k}}$, we have*

$$\sigma(s * \boldsymbol{Y}) = \sigma(s) * \sigma(\boldsymbol{Y})$$

Proof. We prove the lemma case by case for the simple reflections.
Type G: We have $\mathbf{Y}\mathbf{P}_\alpha = \mathbf{Y}$, the Knop action fixes this

$$s_\alpha * \mathbf{Y} = \mathbf{Y}$$

applying σ, we get

$$\sigma(s_\alpha * \mathbf{Y}) = \sigma(\mathbf{Y})$$

note $\sigma(\mathbf{Y})$ is $\mathbf{P}_{\sigma(\alpha)}$-stable, so

$$s_{\sigma(\alpha)} * \sigma(\mathbf{Y}) = \sigma(\mathbf{Y})$$

hence $\sigma(s_\alpha * \mathbf{Y}) = \sigma(\mathbf{Y}) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y})$.
Type U: $\mathbf{Y}\mathbf{P}_\alpha$ contains two $\mathbf{B}$-orbits, $\mathbf{Y}_{op},\ \mathbf{Y}_{cl}$, here $\mathbf{Y}_{op}$ is open in $\mathbf{Y}_{op}\mathbf{P}_\alpha$ and $\mathbf{Y}_{cl}$ is closed. Knop's action exchanges them

$$s_\alpha * \mathbf{Y}_{op} = \mathbf{Y}_{cl}, \quad s_\alpha * \mathbf{Y}_{cl} = \mathbf{Y}_{op}$$

$\sigma(\mathbf{Y}_{op})$ is the open $\mathbf{B}$-orbit in $\mathbf{P}_{\sigma(\alpha)}\sigma(\mathbf{Y})$, and $\sigma(\mathbf{Y}_{cl})$ is the closed $\mathbf{B}$-orbit. If $\mathbf{Y} = \mathbf{Y}_{op}$, then

$$\sigma(s_\alpha * \mathbf{Y}) = \sigma(\mathbf{Y}_{cl}) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y}_{op}) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y})$$

If $\mathbf{Y} = \mathbf{Y}_{cl}$, then

$$\sigma(s_\alpha * \mathbf{Y}) = \sigma(\mathbf{Y}_{op}) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y}_{cl}) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y})$$

Type T: We denote $\mathbf{Y}_{op},\ \mathbf{Y}_0,\ \mathbf{Y}_\infty$ the open and two closed orbits in $\mathbf{Y}\mathbf{P}_\alpha$. Then if $\mathbf{Y}$ is the open orbit, $\mathbf{Y} = \mathbf{Y}_{op}$, we have

$$s_\alpha * \mathbf{Y} = \mathbf{Y}$$

thus

$$\sigma(s_\alpha * \mathbf{Y}) = \sigma(\mathbf{Y})$$

$\sigma(\mathbf{Y})$ is the open orbit in $\mathbf{P}_{\sigma(\alpha)}\sigma(\mathbf{Y})$, Knop's action fixes it

$$s_{\sigma(\alpha)} * \sigma(\mathbf{Y}) = \sigma(\mathbf{Y})$$

next we discuss the cases when $\mathbf{Y}$ is one of the closed orbits, suppose $\mathbf{Y} = \mathbf{Y}_0$, then

$$s_\alpha * \mathbf{Y}_0 = \mathbf{Y}_\infty$$

applying σ

$$\sigma(s_\alpha * \mathbf{Y}_0) = \sigma(\mathbf{Y}_\infty)$$

note $\sigma(\mathbf{Y}_0)$ and $\sigma(\mathbf{Y}_\infty)$ are precisely the two closed orbits, and $s_{\sigma(\alpha)}$ exchanges them, hence

$$s_{\sigma(\alpha)} * \sigma(\mathbf{Y}_0) = \sigma(\mathbf{Y}_\infty)$$

therefore

$$\sigma(s_\alpha * \mathbf{Y}_0) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y}_0)$$

the same argument works with $\mathbf{Y}_\infty$.
Type N: The $\mathbf{P}_\alpha$ orbit contains two $\mathbf{B}$-orbits: $\mathbf{Y}_{op}$ and $\mathbf{Y}_{cl}$. Knop's action fixes both:

$$s_\alpha * \mathbf{Y}_{op} = \mathbf{Y}_{op}, \quad s_\alpha * \mathbf{Y}_{cl} = \mathbf{Y}_{cl}$$

If $\mathbf{Y} = \mathbf{Y}_{op}$, then $\sigma(\mathbf{Y}_{op})$ is the open orbit which is fixed by $s_{\sigma(\alpha)}$, thus

$$\sigma(s_\alpha * \mathbf{Y}) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y})$$

If $\mathbf{Y} = \mathbf{Y}_{cl}$, the same argument gives

$$\sigma(s_\alpha * \mathbf{Y}_{cl}) = \sigma(\mathbf{Y}_{cl}) = s_{\sigma(\alpha)} * \sigma(\mathbf{Y}_{cl})$$

For a general Weyl element $s = s_{\alpha_1} \cdots s_{\alpha_r}$, induction gives

$$\sigma(s * \mathbf{Y}) = \sigma(s) * \sigma(\mathbf{Y})$$

□

Proposition 4.13. *For $\boldsymbol{X} = \boldsymbol{H}\backslash Res_{\ell/k}\ \boldsymbol{SL}_2$ a spherical variety with a-A-2-n type spherical systems over algebraic closure, there are three $\boldsymbol{B}$-orbits in $\boldsymbol{X}$: $\mathring{\boldsymbol{X}}$, $\boldsymbol{Y}_+$, $\boldsymbol{Y}_0$, here $\boldsymbol{Y}_+$ corresponds to the color D_+ and $\boldsymbol{Y}_0$ the closed Borel orbit $\boldsymbol{H}\backslash\boldsymbol{HB}$. We have*

$$s * \boldsymbol{Y}_0 = \boldsymbol{Y}_+, \quad s * \boldsymbol{Y}_+ = \boldsymbol{Y}_0$$

for $s \in W_k \hookrightarrow W_{\overline{k}}$, and $*$ is the $\overline{k}$-Knop action on the set of Borel orbits.

Proof. We have $\mathbf{Y}_0 = \mathbf{H}\backslash\mathbf{B}$ is a closed Borel orbit over k, we have $\mathbf{Y}_+ = \mathbf{H}\tilde{\omega}_0\mathbf{B}$ is a Borel orbit over k, from Proposition 4.11 we get the set of k-rational Borel orbits for $\mathbf{X}$ as $\{\mathring{\mathbf{X}}, \mathbf{Y}_+, \mathbf{Y}_0\}$.

Denote $\alpha_1, \cdots, \alpha_n$ the simple roots for $(\mathbf{SL}_2)^n = (\mathrm{Res}_{\ell/k}\ \mathbf{SL}_2)_{\overline{k}}$ with α_i the simple root for the i-th $\mathbf{SL}_2$ copy. We can choose the representative for $s \in W_k$ as $s = s_1 \cdots s_n$.

For $s = s_1 s_2 \cdots s_n$, since $(\alpha_i, \mathbf{Y}_0)$ is of type U for all i, for each i choose $\tilde{s}_i$ the representatives of s_i in $\mathbf{SL}_2(\overline{k})$ and denote $\tilde{s} = \tilde{s}_1 \cdots \tilde{s}_n$. We have

$$s * \mathbf{Y}_0 = \mathbf{H}\tilde{s}\mathbf{B}$$

we deduce $\dim \mathbf{Y}_0 < \dim (s * \mathbf{Y}_0) < \dim \mathring{\mathbf{X}}$ and hence $s * \mathbf{Y}_0 = \mathbf{Y}_+$. Since the $\overline{k}$-Knop action satisfies the braid relation, we get $s * \mathbf{Y}_+ = \mathbf{Y}_0$.

□

Proposition 4.14. *We denote ω_0 the longest Weyl group element of $(Res_{\ell/k}\ \boldsymbol{SU}_3)_{\overline{k}}$ with its representative $\tilde{\omega}_0 \in Res_{\ell/k}\ \boldsymbol{SU}_3(k)$.*

For $\boldsymbol{X} = \boldsymbol{H}\backslash Res_{\ell/k}\ \boldsymbol{SU}_3$ a spherical variety with a-A-4-n type spherical systems over algebraic closure, there are three $\boldsymbol{B}$-orbits in $\boldsymbol{X}$: $\mathring{\boldsymbol{X}}$, $\boldsymbol{Y}_+$, $\boldsymbol{Y}_0$, here $\boldsymbol{Y}_+ = \boldsymbol{H}\tilde{\omega}_0\boldsymbol{B}$ and $\boldsymbol{Y}_0$ the closed Borel orbit $\boldsymbol{H}\backslash\boldsymbol{HB}$. We have

$$s * \boldsymbol{Y}_0 = \boldsymbol{Y}_+, \quad s * \boldsymbol{Y}_+ = \boldsymbol{Y}_0$$

for $s \in W_k \hookrightarrow W_{\overline{k}}$, and $*$ is the $\overline{k}$-Knop action on the set of Borel orbits.

Proof. We have $\mathbf{Y}_0 = \mathbf{H}\backslash\mathbf{B}$ is a closed Borel orbit over k, $\mathbf{Y}_+ = \mathbf{H}\tilde{\omega}_0\mathbf{B}$ is also a Borel orbit over k, from Proposition 4.11 we get the set of k-rational Borel orbits for $\mathbf{X}$ as $\{\mathring{\mathbf{X}}, \mathbf{Y}_+, \mathbf{Y}_0\}$.

We have $s \in W_k$ lifts to the longest Weyl group element ω_0 of $W_{\overline{k}}$. Denote α_i^1, α_i^2 the simple roots of the i-th copy of $(\mathbf{SL}_3)^n = (\mathrm{Res}_{\ell/k}\ \mathbf{SU}_3)_{\overline{k}}$, then $(\alpha_i^1, \mathbf{Y}_0)$ and $(\alpha_i^2, \mathbf{Y}_0)$ are both of type U. We have

$$s * \mathbf{Y}_0 = \mathbf{H}\tilde{\omega}_0\mathbf{B} = \mathbf{Y}_+$$

Since the $\overline{k}$-Knop action satisfies the braid relation, we get $s * \mathbf{Y}_+ = \mathbf{Y}_0$.

□

Proposition 4.15. *We denote ω_0 the longest Weyl group element of $(Res_{\ell/k}\ \boldsymbol{SU}_3)_{\overline{k}}$ with its representative $\tilde{\omega}_0 \in Res_{\ell/k}\ \boldsymbol{SU}_3(k)$.*

For $\boldsymbol{X} = \boldsymbol{H}\backslash Res_{\ell/k}\ \boldsymbol{SU}_3$ a spherical variety with a-A-5-n type spherical systems over algebraic closure, there are three $\boldsymbol{B}$-orbits in $\boldsymbol{X}$: $\mathring{\boldsymbol{X}}$, $\boldsymbol{Y}_+$, $\boldsymbol{Y}_0$, here $\boldsymbol{Y}_+ = \boldsymbol{H}\tilde{\omega}_0\boldsymbol{B}$ and $\boldsymbol{Y}_0$ is the closed Borel orbit $\boldsymbol{H}\backslash\boldsymbol{HB}$. We have

$$s * \boldsymbol{Y}_0 = \boldsymbol{Y}_+, \quad s * \boldsymbol{Y}_+ = \boldsymbol{Y}_0$$

for $s \in W_k \hookrightarrow W_{\overline{k}}$, and $*$ is the $\overline{k}$-Knop action on the set of Borel orbits.

Proof. We have $\mathbf{Y}_0 = \mathbf{H}\backslash\mathbf{B}$ is a closed Borel orbit over k, $\mathbf{Y}_+ = \mathbf{H}\tilde{\omega}_0\mathbf{B}$ is also a Borel orbit over k, from Proposition 4.11 we get the set of k-rational Borel orbits for $\mathbf{X}$ as $\{\mathring{\mathbf{X}}, \mathbf{Y}_+, \mathbf{Y}_0\}$.

We have $s \in W_k$ lifts to the longest Weyl group element ω_0 of $W_{\overline{k}}$. Denote α_i^1, α_i^2 the simple roots of the i-th copy of $(\mathbf{SL}_3)^n = (\mathrm{Res}_{\ell/k}\ \mathbf{SU}_3)_{\overline{k}}$, then $(\alpha_i^1, \mathbf{Y}_0)$ and $(\alpha_i^2, \mathbf{Y}_0)$ are both of type U. We have

$$s * \mathbf{Y}_0 = \mathbf{H}\tilde{\omega}_0\mathbf{B} = \mathbf{Y}_+$$

Since the $\overline{k}$-Knop action satisfies the braid relation, we get $s * \mathbf{Y}_+ = \mathbf{Y}_0$.

□

Proposition 4.16. *Let $\boldsymbol{G}$ be an unramified semisimple group of split rank one and let $\boldsymbol{H}$ be a k-spherical subgroup. Under the identification $W_k \cong W_{\overline{k}}^{\Gamma}$, we can describe the W_k-action on $\mathfrak{B}_k(\boldsymbol{X}) \subseteq \mathfrak{B}_{\overline{k}}(\boldsymbol{X}_{\overline{k}})$ as follows: we denote $s \in W_k$ the nontrivial reflection and $\boldsymbol{Y} = \mathring{\boldsymbol{X}}$,*

1. $\boldsymbol{H}$ spherical and non-reductive

- *(Type U)*
$$\mathfrak{B}_k(\boldsymbol{X}) = \{Y > Z\}$$
we have $s * Z = Y$ and $s * Y = Z$.
- *(Type TU)*
$$\mathfrak{B}_k(\boldsymbol{X}) = \{Y > Z_1 > Z_2\}$$
we have $s * Y = Y$, and $s * Z_1 = Z_2$.

2. $\boldsymbol{H}$ is spherical, reductive

- *(Type T non-split, Type U_2 nonsplit)*
$$\mathfrak{B}_k(\boldsymbol{X}) = \{Y\}$$
we have $s * Y = Y$.
- *(Type T split)*
$$\mathfrak{B}_k(\boldsymbol{X}) = \{Y > Z_1,\ Z_2\}$$
we have $s * Y = Y$ and $s * Z_1 = Z_2$.
- *(Group case and triple product, Type U_2 split)*
$$\mathfrak{B}_k(\boldsymbol{X}) = \{Y > Z\}$$
we have $s * Y = Y$ and $s * Z = Z$.
- *(Type N)*
$$\mathfrak{B}_k(\boldsymbol{X}) = \{Y > Z\}$$
we have $s * Y = Y$ and $s * Z = Z$.

Proof. Replacing G by its simply-connected cover, we may assume G is simply-connected.

We first consider the cases when X is of split rank zero, these are the type U cases, we only need to prove our proposition for $U_{SL_2}\backslash SL_2$ and $U_{SU_3}\backslash SU_3$. The $U_{SL_2}\backslash SL_2$ case follows from definition for Knop action over algebraic closure. We denote α_1, α_2 the simple roots of $(SU_3)_{\overline{k}}$ and we may choose the lift of $s \in W_k$ in $W_{\overline{k}}$ as $s_1 s_2 s_1$ for s_i the simple reflections associated with α_i. By a computation over algebraic closure, we have $s_1 s_2 s_1 * Y = Z$ and $s_1 s_2 s_1 * Z = Y$.

We now consider the cases when X is of split rank one. From the equality $W_k \cap W_{\overline{k}}(X) = W_k(X) = \mathbb{Z}/2\mathbb{Z}$ proved in Proposition 4.25, we get $s * Y = Y$ in all the cases. We only need to show
$$s * Z_1 = Z_2$$
in the type T split case and type TU case. The type T split case follows from definition for Knop action over algebraic closure. The relations
$$s * Z_1 = Z_2$$
for type TU cases are proved case by case in Proposition 4.13, Proposition 4.14 and Proposition 4.15.

□

Proposition 4.17. *Under the identification $W_k \cong W_{\overline{k}}^{\Gamma}$, we have the subset $\mathfrak{B}_k(\boldsymbol{X})$ is W_k-stable.*

Proof. This is because the Knop action is compatible with the Γ-action: $\sigma(s*\mathbf{Y}) = \sigma(s)*\mathbf{Y}^\sigma = s*\mathbf{Y}$.

We want to show if $\mathbf{Y}(k) \neq \emptyset$, for $s \in W_k$ and $\tilde{s} \in W_k^\Gamma$ the chosen lift, then $\tilde{s} * \mathbf{Y}(k) \neq \emptyset$. To prove this, it is enough to prove it for a simple reflection $s = s_\alpha$, since they generate W_k.

For $\alpha \in \Sigma_k(G)$ with $s_\alpha \in W_k$ the associated simple reflection, we denote $\tilde{s}_\alpha \in W_{\overline{k}}(G)$ the lift of s_α. Let $\mathbf{P}_\alpha$ be the corresponding k-parabolic. For $\mathbf{Y} \in \mathfrak{B}_k(X)$, choose $y \in \mathbf{Y}(k)$ so that $\mathbf{Y} = y\mathbf{B}$ and put $\mathbf{H}_y = \mathrm{stab}_{\mathbf{P}_\alpha}(y)$. After quotienting by $R_u(\mathbf{P}_\alpha)$ and the center of a Levi subgroup this is the rank-one situation for the adjoint group $\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$. We can use the result in Proposition 4.16 to show that the $\tilde{s}_\alpha$-action preserves the orbits which contain k-points.

The $\mathbf{B}$-orbits in $\mathbf{YP}_\alpha$ are in bijection with the $\mathbf{H}_y$-orbits on $\mathbf{P}_\alpha/\mathbf{B}$

$$H_y pB \mapsto ypB$$

Let $\mathscr{O}'$ be the $\mathbf{H}_y$-orbit corresponding to $\tilde{s}_\alpha * \mathbf{Y}$, then we have $\mathscr{O}'(k) \neq \emptyset$, choose $z \in \mathscr{O}'(k)$. The map $\mathbf{P}_\alpha(k) \to (\mathbf{P}_\alpha/\mathbf{B})(k)$ is surjective, hence $z = pB$ for some $p \in \mathbf{P}_\alpha(k)$. The point $yp \in \mathbf{X}(k)$ then lies in the $\mathbf{B}$-orbit $\tilde{s}_\alpha * \mathbf{Y}$. Therefore $\tilde{s}_\alpha * \mathbf{Y} \in \mathfrak{B}_k(X)$. □

Definition 4.18. We define the k-Knop action of W_k on $\mathfrak{B}_k(\mathbf{X})$ as the action induced by the inclusions $W_k \hookrightarrow W_{\overline{k}}$ and $\mathfrak{B}_k(\mathbf{X}) \hookrightarrow \mathfrak{B}_{\overline{k}}(\mathbf{X}_{\overline{k}})$.

Definition 4.19. We define the Knop-Zhgoon action according to the type of spherical subgroup: If it is type G, ω_α stabilizes the unique $\mathbf{B}$-orbit in the given $\mathbf{P}_\alpha$-orbit. In the case of type T, ω_α stabilizes the open orbit and interchanges the other two. In the case of type TU, ω_α stabilizes the open orbit and interchanges the closed orbits. In the case of type N, ω_α stabilizes both orbits. Finally in the case of type U, ω_α interchanges the two orbits. We will view the Knop-Zhgoon action as an $\tilde{W}_k = \langle \omega_\alpha \rangle$, $\alpha \in \Sigma_k(G)$ action on $\mathfrak{B}_k(X)$.

The following conjecture is due to [KZ20]:

Conjecture 4.20. *For a field k such that char $k \neq 2$, the Knop-Zhgoon action of $\tilde{W}_k$ on the set $\mathfrak{B}_k(X)$ factors through the restricted Weyl group W_k.*

This conjecture is known for algebraically closed fields (Theorem 4.1), Knop and Zhgoon proved this conjecture on the subset $\mathfrak{B}_0(X) \subset \mathfrak{B}(X)$ of principal families of $\mathbf{B}$-orbits for characteristic zero fields by means of equivariant geometry of cotangent bundle.

Proposition 4.21. *For $\boldsymbol{G}$ an unramified group, the Knop-Zhgoon action (Definition 4.19) and the k-Knop action (Definition 4.18) of an element $\omega_\alpha \in W_k$ for $\alpha \in \Sigma_k(G)$ on $\mathfrak{B}_k(\boldsymbol{X})$ are the same.*

Proof. For $\omega_\alpha \in W_k$ a simple reflection, we will use $*_K$ and $*_{KZ}$ to denote the k-Knop action and Knop-Zhgoon action of ω_α on $\mathbf{Y} \in \mathfrak{B}_k(\mathbf{X})$. We denote $\mathbf{P}_\alpha$ the k-parabolic subgroup corresponding to α and $\mathbf{L}_\alpha$ the Levi subgroup of $\mathbf{P}_\alpha$.

Let's denote $\mathfrak{B}_k^{\mathbf{L}_\alpha}(\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha})$ (resp. $\mathfrak{B}_k^{\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)}(\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}Z(\mathbf{L}_\alpha))$) denote the set of k-rational $\mathbf{B}_\alpha$-orbits in $\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}$ as an $\mathbf{L}_\alpha$ spherical variety (resp. the $\overline{\mathbf{B}}_\alpha = \mathbf{B}_\alpha/Z(\mathbf{L}_\alpha)$-orbits in $\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}Z(\mathbf{L}_\alpha)$ as an $\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$-variety).

Now to compare $\omega_\alpha *_K \mathbf{Y}$ and $\omega_\alpha *_{KZ} \mathbf{Y}$, note that there are bijections:

$$\mathfrak{B}_k^{\mathbf{G}}(\mathbf{YP}_\alpha) \leftrightarrow \mathfrak{B}_k^{\mathbf{L}_\alpha}(\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}) \leftrightarrow \mathfrak{B}_k^{\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)}(\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}Z(\mathbf{L}_\alpha))$$

Furthermore, we notice that these bijections are compatible with the $*_K$-action and the $*_{KZ}$ action: for $\mathbf{Y}_1 \in \mathfrak{B}_k(\mathbf{YP}_\alpha)$, if $\omega_\alpha *_K \mathbf{Y} = \mathbf{Y}_1$, let $\overline{\mathbf{Y}}, \overline{\mathbf{Y}}_1 \in \mathfrak{B}_k^{\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)}(\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha})$ be the corresponding $\overline{\mathbf{B}}_\alpha$-orbits, then we have $\omega_\alpha *_K \overline{\mathbf{Y}} = \overline{\mathbf{Y}}_1$, as such a property holds over algebraic closure, here we view ω_α as an element in $W_k(\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha))$, a similar discussion applies to the $*_{KZ}$-action.

Finally, note that $\mathbf{YP}_\alpha/\mathbf{U}_{\mathbf{P}_\alpha}Z(\mathbf{L}_\alpha)$ is a spherical variety for the adjoint split rank one group $\mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$ appearing in Proposition 4.11, we checked that the k-Knop and Knop-Zhgoon action are the same in Proposition 4.16. □

Corollary 4.22. *Conjecture 4.20 holds for unramified groups.*

Proof. This is because $W_{\overline{k}}$-action on $\mathfrak{B}_{\overline{k}}(X)$ satisfies the braid relation (Theorem 4.1). □

4.3. **Identification of the k-little Weyl group.** In this section, we let k be a characteristic zero field, G a quasisplit group over k, X a spherical variety for G, we fix B a Borel subgroup of G, $T \subset B$ a maximal torus, $A \subset T$ a maximal split torus. We identify the k-little Weyl group of X as a subgroup of the k-Weyl group. We denote $\mathfrak{t}$ the Lie algebra of T. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+_{\overline{k}}(G)} \alpha$$

be the half sum of the positive roots and ρ_k the image of ρ under the restriction map

$$\mathrm{Res}: \ X^*(T) \longrightarrow X^*(A)$$

For any subset $\mathfrak{s}$ of $\mathfrak{t}^*$ let

$$N_{W_{\overline{k}}}(\mathfrak{s}) = \{\omega \in W_{\overline{k}} \mid \omega\mathfrak{s} = \mathfrak{s}\}$$
$$C_{W_{\overline{k}}}(\mathfrak{s}) = \{\omega \in W_{\overline{k}} \mid \omega|_{\mathfrak{s}} = \mathrm{id}_{\mathfrak{s}}\}$$
$$W(\mathfrak{s}) = N_{W_{\overline{k}}}(\mathfrak{s})/C_{W_{\overline{k}}}(\mathfrak{s})$$

we have $C_{W_{\overline{k}}}(\rho+\mathfrak{t}^*_X) = N_{W_{\overline{k}}}(\rho+\mathfrak{t}^*_X) \cap C_{W_{\overline{k}}}(\mathfrak{t}^*_X)$, the isotropy group of ρ is trivial, hence $C_{W_{\overline{k}}}(\rho+\mathfrak{t}^*_X) = 1$, this gives $N_{W_{\overline{k}}}(\rho + \mathfrak{t}^*_X) \hookrightarrow W(\mathfrak{t}^*_X)$.

First, recall that we have the following result from Knop:

Proposition 4.23. *[Kno94, Theorem 6.5] Let X be a G-spherical variety over $k = \overline{k}$, then there is a canonical lifting $W_{\overline{k}}(X) \hookrightarrow N_{W_{\overline{k}}}(\rho + \mathfrak{t}^*_X)$ coming from the inclusion $W_X \subset W(\mathfrak{t}^*_X)$.*

Using the lifting over the algebraic closure, we prove the following canonical lifting of the k-little Weyl group:

Proposition 4.24. *We have*

$$C_{W_k(G)}(\rho_k + \mathfrak{a}^*_X) = 1$$

and there exists a canonical lifting

$$W_k(X) \hookrightarrow N_{W_k(G)}(\rho_k + \mathfrak{a}^*_X)$$

here by canonical, we mean this lifting descends from the canonical lifting of Knop in Proposition 4.23.

Proof. Knop's theorem gives a canonical embedding

$$W_{\overline{k}}(X) \hookrightarrow N_{W_{\overline{k}}(G)}(\rho + \mathfrak{t}^*_X)$$

Now over k, the k-little Weyl group is defined from the absolute Weyl group by

$$N(\mathfrak{a}^*_X) := \{\omega \in W_{\overline{k}}(X) \mid \omega\mathfrak{a}^*_X = \mathfrak{a}^*_X\}$$
$$C(\mathfrak{a}^*_X) := \{\omega \in W_{\overline{k}}(X) \mid \omega|_{\mathfrak{a}^*_X} = \mathrm{id}\}$$

and $W_k(X) = N(\mathfrak{a}^*_X)/C(\mathfrak{a}^*_X)$ (Definition **??**). The claim is that Knop's canonical lift descends to a canonical embedding

$$W_k(X) \hookrightarrow N_{W_k(G)}(\rho_k + \mathfrak{a}^*_X)$$

Take $\overline{\omega} \in W_k(X) = N(\mathfrak{a}^*_X)/C(\mathfrak{a}^*_X)$, and a representative $\omega \in N(\mathfrak{a}^*_X) \subset W_{\overline{k}}(X)$. By Knop's theorem over $\overline{k}$, ω has a canonical absolute lift $\tilde{\omega} \in W_{\overline{k}}(G)$, now since ω normalizes $\mathfrak{a}^*_X$, we have $\tilde{\omega}$ induces an element of the restricted Weyl group $W_k(G)$, and this induced element preserves the shifted affine subspace,

$$\tilde{\omega}(\rho_k + \mathfrak{a}^*_X) = \rho_k + \mathfrak{a}^*_X$$

we obtain a candidate map

$$W_k(X) \longrightarrow N_{W_k(G)}(\rho_k + \mathfrak{a}_X^*)$$

To show that this is independent of the lift, replace ω by ωc with $c \in C(\mathfrak{a}_X^*)$, then c acts trivially on $\mathfrak{a}_X^*$, its Knop lift $\tilde{c}$ preserves $\rho + \mathfrak{t}_X^*$, so after restriction it preserves $\rho_k + \mathfrak{a}_X^*$, note that $\tilde{c}_k$ acts as a finite order affine translation on the affine space, therefore $\tilde{c}_k|_{\rho_k + \mathfrak{a}_X^*} = \mathrm{id}$. Note that

$$C_{W_k(G)}(\rho_k + \mathfrak{a}_X^*) = 1$$

this is because ρ_k is regular dominant for the Weyl group. So $\tilde{c}_k = 1$, and the image of $\overline{\omega}$ is independent of the chosen representative ω.

Finally, the map is injective. If $\overline{\omega}$ maps to the identity in $W_k(G)$, then its action on $\mathfrak{a}_X^*$ is trivial, hence its representative lies in $C(\mathfrak{a}_X^*)$, so $\overline{\omega} = 1 \in W_k(X)$. Therefore there is a canonical embedding

$$W_k(X) \hookrightarrow W_k(G)$$

with image in $N_{W_k(G)}(\rho_k + \mathfrak{a}_X^*)$.

□

Proposition 4.25. *We have an equality,*

$$W_k(G) \cap W_{\overline{k}}(X) = W_k(X)$$

where $W_{\overline{k}}(X) \subset W_{\overline{k}}(G)$ is Knop's ρ-shift lifting, and $W_k(X)$ is identified with its image inside $W_k(G)$.

Proof. $W_k(X)$ is originally defined as the quotient $N_{W_k(G)}(\mathfrak{a}_X^*)/C_{W_k(G)}(\mathfrak{a}_X^*)$. Set $H = W_k(G) \cap W_{\overline{k}}(X) \subset W_{\overline{k}}(G)$, then H acts on $\mathfrak{a}_X^*$. This is because elements of $W_k(G)$ preserve the split torus A, while elements of $W_{\overline{k}}(X)$ preserve $\mathfrak{t}_X^*$. Hence H preserves $\mathfrak{a}_X^* = \mathrm{Im}(\mathfrak{a}^* \to \mathfrak{t}_X^*)$. Hence there is a natural map

$$\varphi : \ H \longrightarrow W_k(X) = N(\mathfrak{a}_X^*)/C(\mathfrak{a}_X^*)$$

Let's prove injectivity of the map φ. If $h \in H$ acts trivially on $\mathfrak{a}_X^*$, then the induced action on $\rho_k + \mathfrak{a}_X^*$ has trivial linear part, since h is of finite order, this action is trivial. So h fixes ρ_k, but ρ_k is regular dominant, its stabilizer in $W_k(G)$ is trivial, therefore $h = 1$.

Surjectivity follows from Corollary **??**.

Therefore we have

$$W_k(G) \cap W_{\overline{k}}(X) = W_k(X)$$

□

4.4. **Brion path over k.** In this section, $\mathbf{G}$ will be a quasisplit connected reductive group over a characteristic zero field k and $\mathbf{X}$ a $\mathbf{G}$-spherical variety, we want to introduce the Brion path over k.

Definition 4.26. We say that an orbit $\mathbf{Y}$ is raised to an orbit $\mathbf{Z}$ by an edge labeled by α if $\mathbf{Z}$ is the open $\mathbf{B}$-orbit in $\mathbf{YP}_\alpha$ for $\alpha \in \Sigma_k$.

We consider the following graph $\mathfrak{G}_k$, whose vertices are $\mathbf{Y} \in \mathfrak{B}_k(X)$ and whose edges are labeled by the set Σ_k of k-simple roots, where an orbit $\mathbf{Y}$ is joined to an orbit $\mathbf{Z}$ by an edge labeled by α if ${}^{\omega_\alpha}\mathbf{Y} = \mathbf{Z}$ under Knop's action. We will call the graph $\mathfrak{G}_k$ the k-Knop graph.

Let $\mathbf{Y} \in \mathfrak{B}_k(X)$, for every oriented path γ from $\mathbf{Y}$ to $\mathring{\mathbf{X}}$ in the graph $\mathfrak{G}_k$, we denote by $\omega(\gamma) \in W_k$ the Weyl group element corresponding to the path, that is $\omega_{\alpha_n} \cdots \omega_{\alpha_1}$, where $\alpha_1, \cdots \alpha_n$ are the consecutive labels for the edges in the oriented path. We will denote by $\mathfrak{G}(Y)$ the set of all paths from $\mathbf{Y}$ to $\mathring{\mathbf{X}}$ and $W(Y)$ the set of all $\omega(\gamma)$ for $\gamma \in \mathfrak{G}(Y)$.

We make the following definition of the Levi variety as in the split case:

Definition 4.27. For $\gamma \in \Sigma_k(X)$, we define $\mathbf{P}_\gamma$ and $\mathbf{L}_\gamma$ to be the standard parabolic and Levi $\mathbf{P}_{\mathrm{supp}(\gamma)}$, $\mathbf{L}_{\mathrm{supp}(\gamma)}$ and X_γ the spherical variety $\mathring{\mathbf{X}}\mathbf{P}_\gamma/\mathbf{U}_{\mathbf{P}_\gamma}$, here $\mathrm{supp}(\gamma)$ denotes the minimal set of simple roots such that γ lies in their linear span.

Based on the Brion's Theorem 3.1, we expect the following result over k: Denote by $\mathrm{stab}_{W_k}(\mathring{\mathbf{X}})$, the stabilizer of $(\mathring{\mathbf{X}})$ under the k-Knop action, then a set of generators of $\mathrm{stab}_{W_k}(\mathring{\mathbf{X}})$ consists of the elements ω_α, $\alpha \in \Sigma_k(L(X))$ and the element ω with the following property: there is a decomposition $\omega = \omega_1^{-1}\omega_2\omega_1$ such that

- ${}^{\omega_1}\mathring{\mathbf{X}} = \mathbf{Y}$, i.e. $\omega_1^{-1} \in W(Y)$.
- ω_2 is either equal to ω_α where $(\mathbf{Y}, \alpha)$ is not of type G or type U, or equal to $\omega_\alpha\omega_\beta$ where α, β are two orthogonal simple roots which raise the same orbit $\mathbf{Y}'$ to $\mathbf{Y}$.

We plan to check this assumption for all homogeneous affine spherical varieties using the reduction to Levi varieties in our future work. For every homogeneous affine spherical varieties of split rank one this assumption can be checked directly. Based on this expected result, we can make a definition on k-Brion path.

5. The main theorem on the unramified spectrum

Following the strategy of Sakellaridis, we study the unramified spectrum in this section, we first introduce the intertwining operators S_χ^Y associated with a B-orbit, then we prove a necessary condition for distinction and we study the basic theorem which describes the action of the standard intertwining operators on S_χ^Y.

5.1. **Distribution on a single orbit.** In this section, we let k be a p-adic field of characteristic zero. $\mathbf{G}$ an unramified group over k, $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$, $\mathbf{T} \subset \mathbf{B}$ a maximal torus and $\mathbf{A}$ a maximal split torus of $\mathbf{T}$. Let $\mathbf{X}$ be a $\mathbf{G}$-spherical variety and $\mathbf{Y} \in \mathfrak{B}_k(X)$.

In this section, we will study the B-equivariant distributions on $Y = \mathbf{Y}(k)$. We begin with the case $Y = \mathbf{Y}(k)$ is a single B-orbit.

We examine the question of $(B, \chi^{-1}\delta^{-\frac{1}{2}})$-equivariant distributions on $C_c^\infty(Y)$ where $Y = y \cdot B$. We will use the natural projection $C_c^\infty(B) \to C_c^\infty(yB)$, given by integration on the left over B_y with respect to right Haar measure

$$p(f)(x) = \int_{B_y} f(bx)\ d_r b$$

If we pull back a $(B, \chi^{-1}\delta^{-\frac{1}{2}})$-equivariant distribution to B, we get a $(B, \chi^{-1}\delta^{-\frac{1}{2}})$-equivariant distribution on B, which has to be $\chi^{-1}\delta^{-\frac{1}{2}}$ times right Haar measure, hence equal to $\chi^{-1}\delta^{\frac{1}{2}} d_l b$. In other words, the distribution is given by

$$S_\chi^Y :\ \phi \mapsto \int_B f(b)\chi^{-1}\delta^{\frac{1}{2}}(b)\ d_l b$$

where $f \in C_c^\infty(B)$ such that $p(f) = \phi$. This distribution is well defined if and only if

$$\chi^{-1}\delta^{\frac{1}{2}}|_{B_y} = \delta_{B_y} \tag{5.1}$$

where δ_{B_y} is the modular character of B_y. The definition of the admissible characters doesn't depend on the choice of $y \in Y$: Set $\eta_\chi = \chi^{-1}\delta^{1/2}$, then

$$\eta_\chi|_{B_y} = \delta_{B_y}$$

If $y' = yb_0$ for $b_0 \in B$, then $B_{y'} = b_0^{-1}B_y b_0$ and $\eta_\chi(b_0^{-1}hb_0) = \eta_\chi(h)$. While $\delta_{B_{y'}}(b_0^{-1}hb_0) = \delta_{B_y}(h)$, hence

$$\eta_\chi|_{B_y} = \delta_{B_y} \leftrightarrow \eta_\chi|_{B_{y'}} = \delta_{B_{y'}}$$

The definition of S^Y_χ depends on the choice of the base point y, if $y' = yb$, then $S^Y_{\chi,y'} = \eta(b_0)^{-1}S^Y_{\chi,y}$. It also depends on the Haar measure on Y which we will fix later.

Definition 5.1. Given a B-orbit Y, the unramified characters satisfying (5.1) will be called Y-admissible, they form a complex subvariety of A^* which will be denoted by Adm_Y.

For $\mathbf{Y}$ a $\mathbf{B}$-orbit closure, we will use the notation $\mathrm{Adm}_Y = \mathrm{Adm}_{\mathring{Y}}$. Note that two Y-admissible characters differ by an unramified character of T_Y which is that same as an unramified character of A_Y, hence we may view Adm_Y as a translate of A^*_Y, in particular $\dim \mathrm{Adm}_Y = \mathrm{rk}(Y)$.

We can summarize the above discussion as follows:

Lemma 5.2. *For each B-orbit Y we have a unique up to scalar B-morphism $C^\infty_c(Y) \to \mathbb{C}_{\chi\delta^{\frac{1}{2}}}$ if and only if $\chi \in Adm_Y$. These morphisms are given as specializations of a regular family S^Y_χ.*

Next let's discuss the case of non-transitive action of B, we assume that $(\mathbf{Y}, \alpha)$ is not of type N. Let $\mathbf{Y}$ be a k-rational $\mathbf{B}$-orbit and denote $\underline{S}^Y_\chi$ the vector space of B-morphisms $C^\infty_c(Y) \to \mathbb{C}_{\chi\delta^{\frac{1}{2}}}$. It has a basis consisting of $S^{Y_i}_\chi$, where Y_i ranges over the distinct B-orbits, however it turns out that this basis is not the correct basis to use.

Lemma 5.3. *Given a $\boldsymbol{B}$-orbit $\boldsymbol{Y}$, the number of B-orbits in $\boldsymbol{Y}(k)$ is $ker(H^1(k, \boldsymbol{T}^Y) \longrightarrow H^1(k, \boldsymbol{T}))$ which is the same as the number of complex characters of T_Y which coincide with χ on the image of T.*

Proof. This is because the number of B-orbits is given by the cardinality of $\ker(H^1(k, \mathbf{B}_y) \to H^1(k, \mathbf{B}))$ and $H^1(k, \mathbf{U}) = 1$ for an unipotent group $\mathbf{U}$ over a characteristic zero p-adic local field, hence $\ker(H^1(k, \mathbf{B}_y) \to H^1(k, \mathbf{B})) = \ker(H^1(k, \mathbf{T}^Y) \to H^1(k, \mathbf{T}))$. From the short exact sequence

$$1 \longrightarrow \mathbf{T}^Y \longrightarrow \mathbf{T} \longrightarrow \mathbf{T}_Y \longrightarrow 1$$

we get $\mathbf{T}_Y(k)/\mathrm{Im}(\mathbf{T}(k)) \cong \ker(H^1(k, \mathbf{T}^Y) \to H^1(k, \mathbf{T}))$, hence we can identify $\ker(H^1(k, \mathbf{T}^Y) \to H^1(k, \mathbf{T}))$ as the set of complex characters of T_Y which coincide with χ on the image of T. □

We introduce a basis for $\underline{S}^Y_\chi$:

Definition 5.4. Fix a point $y \in Y$, obtaining an identification $\mathbf{Y}/\mathbf{U} \cong \mathbf{T_Y} \cong \mathbf{T}/\mathbf{T^Y}$. We may form a basis for $\underline{S}^Y_\chi$ indexed by the set of complex characters $\tilde{\chi}$ of T_Y which coincide with χ on the image of T. $S^Y_{\tilde{\chi}}$ can be described as the composition of two morphisms: the first is integration over the horocycles on Y

$$C^\infty_c(Y) \longrightarrow C^\infty_c(T_Y, \mathcal{L}_{\delta\delta^{-1}_{B_y}}) : \ f(\bullet) = \int_{U_y\backslash U} \phi(y \cdot u\bullet)\, du$$

followed by the integration over the torus

$$\int_{T_Y} f(t)\tilde{\chi}^{-1}\delta^{-\frac{1}{2}}(t)\, dt$$

Here $\mathcal{L}_{\delta\delta^{-1}_{B_y}}$ is the complex T-line bundle over T_Y associated with the character $\delta\delta^{-1}_{B_y}$ of T^Y.

We prove a useful lemma on the set of admissible characters.

Lemma 5.5. *For $\boldsymbol{H} \subseteq \boldsymbol{B}$ a k-subgroup, we have $Adm_H = Adm_{H^0}$ where*

$$Adm_H := \{\chi \in T^* \mid \chi^{-1}\delta^{1/2}|_H\delta_H^{-1} = 1\}$$
$$Adm_{H^0} := \{\chi \in T^* \mid \chi^{-1}\delta^{1/2}|_{H^0}\delta_{H^0}^{-1} = 1\}$$

Here H^0 is the connected component of H.

Proof. We have $\mathbf{H} \subseteq \mathbf{B} = \mathbf{TU}$, the disconnectedness of $\mathbf{H}$ comes from its image in $\mathbf{T}$ as algebraic unipotent subgroup in characteristic 0 is connected.

Denote $\mathbf{H}_T = \mathrm{im}(\mathbf{H} \longrightarrow \mathbf{T})$, then $\pi_0(H) \cong \pi_0(H_T)$. From the Proposition 2.4, every $\mathbf{H}_T(k)$ can be written as $h = h_0 h_c$ with $h_0 \in \mathbf{H}_T^0(k)$ and $h_c \in \mathbf{H}_T(k) \cap \mathbf{T}(\mathfrak{o})$.

If $\chi \in \mathrm{Adm}_{H^0}$, then

$$\chi(h) = \chi(h_0)\chi(h_c) = \chi(h_c) = 1$$

as χ is an unramified character of T. □

5.2. **Convergence and rationality.** Following Sakellaridis, we study the convergence and rationality for the intertwining operators we constructed in the previous section. In this section k will be a p-adic local field of characteristic zero.

The discussion in this section does not depend on whether $\mathbf{Y}(k)$ consists of a single B-orbit or not, so we will assume $\mathbf{Y}(k)$ is a single B-orbit.

The distribution S_χ^Y can be expressed as

$$S_\chi^Y : \ \phi \mapsto \int_Y \phi(y) d\mu_\chi(y) \tag{5.2}$$

here $d\mu_\chi$ is an $\chi^{-1}\delta^{-1/2}$-eigenmeasure.

By Hironaka's embedded resolution of singularities [Hir64a], [Hir64b], given a k-rational $\mathbf{B}$-orbit $\mathbf{Y}$ with closure $\overline{\mathbf{Y}}$, there exists a canonical regular k-scheme $\tilde{\mathbf{Y}}$ and a proper k-morphism $p : \tilde{\mathbf{Y}} \to \overline{\mathbf{Y}}$, which is an isomorphism on $\mathbf{Y}$ and the inverse image $\mathbf{E}$ of $\overline{\mathbf{Y}} \backslash \mathbf{Y}$ is an effective divisor. Moreover, for every point $y_0 \in \tilde{\mathbf{Y}}(k)$ the geometrically irreducible components of $\mathbf{E}$ which contain y_0 are defined over k and have normal crossings. Notice that the map $\tilde{Y} \to \overline{Y}$ is surjective on k-points, this is because the proper algebraic morphism induces a proper map in the Hausdorff topology.

To understand the behavior of S_χ^Y applied to any $\phi \in C_c^\infty(\overline{Y})$, we will write it as $\int_Y \phi(y)\ d\mu(y)$ where $d\mu$ is a B-eigenmeasure on Y with weight $\chi^{-1}\delta^{-\frac{1}{2}}$. In turn $d\mu$ can be written as $|\omega||f_1|^{s_1} \cdots |f_m|^{s_m}$ where ω is a top-degree $\mathbf{B}$-eigenform and f_i are $\mathbf{B}$-semi-invariants on $\mathbf{Y}$.

We can pull back ϕ, the functions f_i and ω to $\tilde{\mathbf{Y}}$ in order to express the integral as an integral on $\tilde{Y}$, the corresponding measure will be denoted by $p^* d\mu$. A divisor $\mathbf{D}$ on $\tilde{\mathbf{Y}}$ defines a valuation v_D on rational sections of line bundles, and we can extend the exponential of this valuation to $d\mu$:

$$q^{v_D(\chi^{-1}\delta^{-\frac{1}{2}})} = q^{v_D(\omega)+s_1 v_D(f_1)+\cdots+s_m v_D(f_m)}$$

Let $y_0 \in \tilde{Y} \cap \mathrm{supp}(p^*\phi)$. Let $\mathbf{D}_1, \cdots, \mathbf{D}_k$ be the irreducible components of $\mathbf{E}$ which contain y_0, then there are local coordinates $x_1, \cdots, x_n$ identifying a neighborhood of y_0 in the Hausdorff topology with a neighborhood of 0 in k^n such that $D_i = \{x_i = 0\}$ for $1 \leq i \leq k$. Moreover $|p^* d\mu| = \prod_{i=1}^k |x_i|^{r_i}\ dx_1 \cdots dx_n$, where $q^{r_i} = q^{v_{D_i}(\chi^{-1}\delta^{-\frac{1}{2}})}$.

Therefore in a neighborhood of y_0, the integral (5.2) is equal to

$$\int p^*\phi(x) \prod_{i=1}^k |x_i|^{r_i}\ dx_1 \cdots dx_n \tag{5.3}$$

Proposition 5.6. *The integral (5.2) representing $S_\chi^Y(\phi)$ converges for all $\phi \in C_c^\infty(\overline{Y})$ for χ in an open subregion of Adm_Y. It is rational in χ and its poles are products of the form*

$$\frac{1}{1 - q^{-v_D(\chi^{-1}\delta^{-\frac{1}{2}})-1}}$$

where D denotes an irreducible component of the exceptional divisor E of $\tilde{Y}$. We will still denote the resulting functional by S_χ^Y.

Proof. Using (5.3), in a neighborhood V of y_0, the integral $S_\chi^Y(\phi)$ is equal to

$$I_V(\chi) = \int_V p^*\phi(x) \prod_{i=1}^{k} |x_i|^{r_i} \, dx_1 \cdots dx_n$$

We denote $p^*\phi$ by Φ. Since Φ is locally constant and compactly supported, there exists N such that Φ is constant on sufficiently small cosets of $\varpi^N \mathfrak{o}^n$. Consequently, Φ can be written as a finite linear combination

$$\Phi(x) = \sum_{\nu=1}^{M} c_\nu 1_{C_{\nu,1}\times\cdots\times C_{\nu,n}}(x)$$

where each $C_{\nu,i} \subset k$ is an open compact subset.

Hence we obtain

$$I_V(\chi) = \sum_{\nu=1}^{M} c_\nu \int_{C_{\nu,1}\times\cdots\times C_{\nu,n}} \prod_{i=1}^{k} |x_i|^{r_i} \, dx_1 \cdots dx_n$$

By Fubini's theorem, we get

$$I_V(\chi) = \sum_{\nu=1}^{M} c_\nu \prod_{i=1}^{k} \Big(\int_{C_{\nu,i}} |x_i|^{r_i} dx_i \Big) \prod_{i=k+1}^{n} \mathrm{vol}(C_{\nu,i})$$

Now for each local factor

$$\int_{C_{\nu,i}} |x_i|^{r_i} \, dx_i$$

This factor converges when $\mathrm{Re}\ r_i(\chi) > -1$, this local integral is rational and its pole is of the form $\frac{1}{1-q^{-r_i-1}}$.

We define

$$\Omega_Y = \{\chi \in \mathrm{Adm}_Y \ : \ \mathrm{Re}\ r_D(\chi) > -1, \ \text{for every irreducible component } D \subset E\}$$

where $r_D(\chi) = v_D(\chi^{-1}\delta^{-\frac{1}{2}})$, then if $\Omega_Y \neq \emptyset$, we have Ω_Y is an open subregion of Adm_Y.

Next we want to show $\Omega_Y \neq \emptyset$. We can find $f \in k[Y]^{(B)}$, f vanishes on $\overline{Y}\backslash Y$, we denote ψ^{-1} the weight of f

$$f(yb) = \psi^{-1}(b) f(y)$$

Fix $\chi_0 \in \mathrm{Adm}_Y$. Up to a nonzero scalar, the eigenmeasure corresponding to $\chi_0\psi^k$ is

$$d\mu_{\chi_0\psi^k} = |f|^k d\mu_{\chi_0}$$

For every irreducible component $D \subset E$, the function p^*f vanishes along D, because $p(D) \subset \partial Y$. Therefore $v_D(f) > 0$. We have

$$r_D(\chi_0\psi^k) = r_D(\chi_0) + k v_D(f)$$

Since E has only finitely many irreducible components and $v_D(f)$ is positive for every $D \subset E$. We can find sufficiently large k, such that

$$\mathrm{Re}\ r_D(\chi_0\psi^k) > -1$$

for every $D \subset E$. Thus $\chi_0\psi^k \in \Omega_Y$. We conclude $\Omega_Y \neq \emptyset$.

Above all, we see the integral representing $S_\chi^Y(\phi)$ converges for all $\phi \in C_c^\infty(\overline{Y})$, for χ in an open subregion of Adm_Y. It is rational in χ and its poles are products of factors of the form:

$$\frac{1}{1 - q^{-v_D(\chi^{-1}\delta^{-\frac{1}{2}})-1}}$$

where D denotes an irreducible component of the exceptional divisor E of $\widetilde{Y}$. □

5.3. **Comparison between admissible characters for different orbits.** In this section we describe the relation between admissible characters for different Borel orbits.

For $\mathbf{G}$ a quasisplit group of semisimple split rank one, let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a k-spherical homogeneous space with $\Sigma_k(X) \neq \emptyset$, we fix a choice of Borel subgroup $\mathbf{B}$ so that $\mathbf{HB}$ is open in $\mathbf{G}$. Let $\mathbf{Y} = \mathbf{HB}$ be the open $\mathbf{B}$-orbit and $\mathbf{Z}$ a closed $\mathbf{B}$-orbit.

We denote the maximal split tori of $\mathbf{Y}$ and $\mathbf{Z}$ by $\mathbf{A_Y} = I_Y\backslash\mathbf{A}$ and $\mathbf{A_Z} = I_Z\backslash\mathbf{A}$. We denote I_Y^0 and I_Z^0 the identity components of I_Y and I_Z. We introduce $\mathbf{A}_\mathbf{Y}^0 = I_Y^0\backslash\mathbf{A}$ and $\mathbf{A}_\mathbf{Z}^0 = I_Z^0\backslash\mathbf{A}$. We decompose the split torus as

$$\mathbf{A} = \mathbf{A}_G^{cent} \cdot \mathbf{A}_{ss}$$

Here $\mathbf{A}_G^{cent} = \mathbf{A} \cap Z(G)$ and $\mathbf{A}_{ss}$ is a split rank one torus such that $\mathbf{A}_G^{cent} \cap \mathbf{A}_{ss}$ is a finite group.

Lemma 5.7. *For $\boldsymbol{G}$ a quasisplit group of semisimple split rank one, let $\boldsymbol{X}$ be a homogeneous spherical variety with $\Sigma_k(X) \neq \emptyset$. For $\boldsymbol{Y}$ the open Borel orbit of $\boldsymbol{X}$, if we denote $\boldsymbol{A_Y} = I_Y\backslash\boldsymbol{A}$, and denote by I_Y^0 the connected component of I_Y. We have $I_Y^0 \subseteq \boldsymbol{A}_G^{cent}$.*

Proof. For the decomposition $\mathbf{A} = \mathbf{A}_G^{cent} \cdot \mathbf{A}_{ss}$, we have $\dim \mathbf{A}_{ss} = 1$ by our assumption. We denote

$$\mathrm{pr}: \ \mathbf{A} \longrightarrow \mathbf{A}_{ss}$$

the projection map. We have $\dim \mathbf{A}_\mathbf{Y}^{ss} := \mathrm{pr}(I_Y)\backslash\mathbf{A}_{ss} = 1$ by $\Sigma_k(X) \neq \emptyset$.

We must have $I_Y^0 \subseteq \ker \mathrm{pr} = \mathbf{A}_G^{cent}$, this is because otherwise we will have $\mathrm{pr}(I_Y)$ is a dimension one subtorus of $\mathbf{A}_{ss}$, then we get $\dim \mathbf{A}_\mathbf{Y}^{ss} = 0$, a contradiction. Hence $I_Y^0 \subseteq \mathbf{A}_G^{cent}$. □

Lemma 5.8. *We have*

$$I_Y \cap \boldsymbol{A}_G^{cent} = I_Z \cap \boldsymbol{A}_G^{cent}$$

Proof. We have $\mathbf{Y} = \mathbf{HB}$, and $\mathbf{Z} = \mathbf{H}g\mathbf{B} = x \cdot \mathbf{B}$. For an element $c \in \mathbf{A}_G^{cent}$, we have

$$c \in \mathbf{B}_x \leftrightarrow c \in g^{-1}\mathbf{H}g \leftrightarrow gcg^{-1} \in \mathbf{H}$$

Because c is central, we get

$$c \in \mathbf{B}_x \leftrightarrow c \in \mathbf{H}$$

hence we have

$$I_Z \cap \mathbf{A}_G^{cent} = \mathbf{H} \cap \mathbf{A}_G^{cent} = I_Y \cap \mathbf{A}_G^{cent}$$

□

Proposition 5.9. *For $\boldsymbol{G}$ a quasisplit group of semisimple split rank one, let $\boldsymbol{X} = \boldsymbol{H}\backslash\boldsymbol{G}$ be a k-spherical homogeneous variety with $\Sigma_k(X) \neq \emptyset$. Let $\boldsymbol{Y}$ be the open $\boldsymbol{B}$-orbit and $\boldsymbol{Z}$ a closed $\boldsymbol{B}$-orbit, we have a surjection $\boldsymbol{A}_\boldsymbol{Y}^0 \to \boldsymbol{A}_\boldsymbol{Z}^0$.*

Proof. Note that from Lemma 5.7, we have

$$I_Y^0 \subseteq I_Y \cap \mathbf{A}_G^{cent}$$

hence $I_Y^0 \subseteq (I_Y \cap \mathbf{A}_G^{cent})^0$.

Then from Lemma 5.8

$$I_Y^0 \subseteq (I_Y \cap \mathbf{A}_G^{cent})^0 = (I_Z \cap \mathbf{A}_G^{cent})^0 \subseteq I_Z^0 \cap (\mathbf{A}_G^{cent})^0 \subseteq I_Z^0$$

□

Let $(\mathbf{Y}, \alpha)$ be a parabolic pair, then we have a bijection

$$\mathfrak{B}(\mathbf{YP}_\alpha) \leftrightarrow \mathfrak{B}(\mathbf{Y}_\alpha\mathbf{L}_\alpha)$$

here the left-hand side is the Borel orbits inside $\mathbf{YP}_\alpha$ and the right hand side is the $\mathbf{B}_\alpha$ orbits inside $\mathbf{Y}_\alpha\mathbf{L}_\alpha$, $\mathbf{Y}_\alpha$ is the $\mathbf{B}_\alpha$ orbit corresponding to $\mathbf{Y}$ under the quotient map by $\mathbf{U}_{\mathbf{P}_\alpha}$

$$(\mathbf{P}_\alpha)_y\backslash\mathbf{P}_\alpha \longrightarrow \mathbf{H}_\alpha\backslash\mathbf{L}_\alpha$$

$\mathbf{H}_\alpha = ((\mathbf{P}_\alpha)_y \bmod \mathbf{U}_{\mathbf{P}_\alpha}) \subset \mathbf{L}_\alpha$, we denote $\mathbf{Y} = \mathbf{B}_y\backslash\mathbf{B}$ and $\mathbf{Y}_\alpha = (\mathbf{B}_\alpha)_{\overline{y}}\backslash\mathbf{B}_\alpha$.

Lemma 5.10. *Let $(\boldsymbol{Y}, \alpha)$ be a parabolic pair not of type G or type U. If we denote $\boldsymbol{A}_{\boldsymbol{Y}_\alpha} = I_{Y_\alpha}\backslash\boldsymbol{A}$, $\boldsymbol{A}_{\boldsymbol{Z}_\alpha} = I_{Z_\alpha}\backslash\boldsymbol{A}$, then there is a surjection*

$$\boldsymbol{A}^0_{\boldsymbol{Y}_\alpha} \twoheadrightarrow \boldsymbol{A}^0_{\boldsymbol{Z}_\alpha}$$

Proof. This follows from Proposition 5.9. □

Recall that we have listed the types of spherical subgroups in Theorem 4.6 and Theorem 4.7 whose proof is based on the classification in Section **??**. We have the following proposition as a consequence of Lemma 5.10:

Proposition 5.11. *We let ℓ/k be an unramified extension of p-adic fields. Given $(\boldsymbol{Y}, \alpha)$ a parabolic pair, $\boldsymbol{P}_\alpha$ the parabolic subgroup associated with α with Levi decomposition $\boldsymbol{P}_\alpha = \boldsymbol{L}_\alpha \boldsymbol{U}_{\boldsymbol{P}_\alpha}$. Let $\boldsymbol{Y}$ denote the open orbit in $\boldsymbol{Y}\boldsymbol{P}_\alpha$ and let $\boldsymbol{Z}_*$ denote the closed orbits, the following relations hold between their character groups:*
If $\boldsymbol{L}_\alpha/Z(\boldsymbol{L}_\alpha) \cong Res_{\ell/k}\boldsymbol{PGL}_2$, then we have:

Type G: We have $\chi(\check{\alpha}(\varpi)) = q_\ell$.
Type U: the stabilizer B_z of the closed orbit differs from the stabilizer B_y by the root subgroup U_α, and we have $Adm_Z = Adm_{\omega_\alpha Y} = {}^{\omega_\alpha}Adm_Y$.
Type T: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}_*$ the closed ones, then we have $Adm_{Z_*} \subset Adm_Y$.
Type N: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}$ the closed one, then we have $Adm_Z \subset Adm_Y$.
Type TU: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}_*$ the closed ones, then we have $Adm_{Z_*} \subset Adm_Y$.
Group: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}$ the closed one, then we have $Adm_Z \subset Adm_Y$.
Cubic: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}_*$ the closed ones, then we have $Adm_{Z_*} \subset Adm_Y$.

If $\boldsymbol{L}_\alpha/Z(\boldsymbol{L}_\alpha) \cong Res_{\ell/k}\boldsymbol{PU}_3$, then we have:

Type G: We have $\chi(\check{\alpha}(\varpi)) = q_\ell^2$.
Type U: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}$ the closed one, then we have $Adm_Z = {}^{\omega_\alpha}Adm_Y$.
Type U_2: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}$ the closed one, then we have $Adm_Z \subset Adm_Y$.
Type N: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}$ the closed one, then we have $Adm_Z \subset Adm_Y$.
Type TU: If $\boldsymbol{Y}$ denotes the open orbit and $\boldsymbol{Z}_*$ the closed ones, then we have $Adm_{Z_*} \subset Adm_Y$.

Proof. In the type G case, this proposition follows from calculation for the admissible characters for the spherical varieties $\mathrm{Res}_{\ell/k}\mathbf{PGL}_2\backslash\mathrm{Res}_{\ell/k}\mathbf{PGL}_2$ and $\mathrm{Res}_{\ell/k}\mathbf{PU}_3\backslash\mathrm{Res}_{\ell/k}\mathbf{PU}_3$.

If $(\mathbf{Y}, \alpha)$ is of type U. Choose a representative $\omega \in N_G(A)(k)$ for the simple reflection associated with α. We may choose $z \in \mathbf{Z}(k)$ and $y = z\omega \in \mathbf{Y}(k)$. Put $\overline{\mathbf{L}_\alpha} = \mathbf{L}_\alpha/Z(\mathbf{L}_\alpha)$ and let $\overline{\mathbf{U}}_\alpha$ be the unipotent radical of its standard Borel subgroup $\overline{\mathbf{B}}_\alpha$. A spherical subgroup of $\overline{\mathbf{L}}_\alpha$ of type U has the form

$$S_\alpha\overline{\mathbf{U}}_\alpha,\ S_\alpha \subset N_{\overline{\mathbf{L}}_\alpha}(\overline{\mathbf{U}}_\alpha)$$

Denote $\overline{\mathbf{Y}}$ and $\overline{\mathbf{Z}}$ the $\overline{\mathbf{B}}_\alpha$-orbits corresponding to $\mathbf{Y}$ and $\mathbf{Z}$. The stabilizers are

$$(\overline{\mathbf{B}}_\alpha)_z = S_\alpha\overline{\mathbf{U}}_\alpha, \quad (\overline{\mathbf{B}}_\alpha)_y = \omega^{-1}S_\alpha\omega$$

Set

$$\Delta_\alpha(a) = |\det(\mathrm{Ad}(a); \mathrm{Lie}\ \overline{\mathbf{U}}_\alpha)|_k$$

then we have

$$\delta_{B_z} = \Delta_\alpha \cdot {}^\omega\delta_{B_y}$$

from $(\overline{\mathbf{B}}_\alpha)_y = \omega^{-1}S_\alpha\omega$ and $(\overline{\mathbf{B}}_\alpha)_z = S_\alpha\overline{\mathbf{U}}_\alpha$.

The definition of Δ_α and $(\overline{\mathbf{B}}_\alpha)_y = \omega^{-1}S_\alpha\omega$ gives

$$\delta^{1/2}|_{B_z} = \Delta_\alpha \cdot {}^\omega\delta^{1/2}|_{B_y}$$

Hence for unramified character χ,

$$\begin{aligned} {}^{\omega}\chi \in \mathrm{Adm}_Z &\Leftrightarrow {}^{\omega}\chi^{-1}\delta^{1/2}|_{B_z} = \delta_{B_z} \\ &\Leftrightarrow {}^{\omega}\chi^{-1}\Delta_\alpha{}^{\omega}\delta^{1/2}|_{B_y} = \Delta_\alpha{}^{\omega}\delta_{B_y} \\ &\Leftrightarrow \chi^{-1}\delta^{1/2}|_{B_y} = \delta_{B_y} \\ &\Leftrightarrow \chi \in \mathrm{Adm}_Y \end{aligned}$$

Consequently $\mathrm{Adm}_Z = {}^{\omega}\mathrm{Adm}_Y$.

We put

$$\mathbf{P} = \mathbf{P}_\alpha = \mathbf{LN}$$

here $\mathbf{L} = \mathbf{L}_\alpha$, $\mathbf{N} = \mathbf{U}_{P_\alpha}$ and let $\pi : \mathbf{P} \to \mathbf{L}$ be the quotient map.

For a pair that is neither type G or type U, Lemma 5.7 says that the split torus in its open stabilizer is central in $\mathbf{L}$. Consequently, we can choose a maximal k-split torus C of $\mathbf{B}_y^0$ which satisfies

$$C \subset Z(L) \cap \mathbf{B}_y$$

we write $z = y\ell$ for $\ell \in L$. Since C is central in $\mathbf{L}$, we get

$$C \subset \mathbf{B}_y \cap \mathbf{B}_z$$

now define $K_y = \mathbf{B}_y \cap \mathbf{N}$ and $K_z = \mathbf{B}_y \cap \mathbf{N}$.

For $x = y, z$ consider

$$0 \longrightarrow \mathrm{Lie}\ K_x \longrightarrow \mathrm{Lie}\ \mathbf{B}_x \longrightarrow \mathrm{Lie}\ \pi(\mathbf{B}_x) \longrightarrow 0$$

Hence for

$$\delta_{B_x}(c) = |\det(\mathrm{Ad}(c); \mathrm{Lie}\ K_x)|$$

since $K_z = \ell^{-1}K_y\ell$, we get

$$\delta_{B_z}|_{C(k)} = \delta_{B_y}|_{C(k)}$$

From

$$\chi(c)^{-1}\delta(c)^{1/2} = \delta_{B_z}(c) = \delta_{B_y}(c)$$

we get the character $\eta_{y,\chi} := \chi^{-1}\delta^{1/2}\delta_{B_y}^{-1}$ is trivial on $C(k)$.

We put $\mathbf{B}_y^0 = S \ltimes \mathbf{U}_y$ with S an unramified torus and C its maximal split torus. For the unramified torus S, we have $S(k) = C(k)S(\mathfrak{o})$. Now since $\eta_{y,\chi}$ is trivial on U_y, and $S(\mathfrak{o})$, $C(k)$, we conclude $\eta_{y,\chi}$ is trivial on B_y^0.

Using the connected-component reduction of Lemma 5.5, we get

$$\chi^{-1}\delta^{1/2}|_{B_y} = \delta_{B_y}$$

and hence

$$\mathrm{Adm}_Z \subseteq \mathrm{Adm}_Y$$

□

5.4. **Necessary condition for the distinction.** In this section $\mathbf{X}$ will be a homogeneous spherical variety for an unramified p-adic group $\mathbf{G}$.

Definition 5.12. We will say $\alpha \in \Sigma_k(G)$ raises a $\mathbf{B}$-orbit $\mathbf{Y}$ to a $\mathbf{B}$-orbit $\mathbf{Z}$ if $\mathbf{Z} \neq \mathbf{Y}$ is the open orbit in $\mathbf{YP}_\alpha$.

We have the following lemma.

Lemma 5.13. *Suppose $\boldsymbol{Y}$ is a $\boldsymbol{B}$-orbit such that $\boldsymbol{Y} \neq \mathring{\boldsymbol{X}}$, then we can find $\alpha \in \Sigma_k$ such that α raises $\boldsymbol{Y}$.*

Proof. We argue by contradiction. If this is not the case, then we have

$$\mathbf{YP}_\alpha = \mathbf{Y}$$

for all $\alpha \in \Sigma_k(G)$, note $\mathbf{YB} = \mathbf{Y}$, and $\mathbf{G}$ is generated by $\mathbf{B}$ and $\mathbf{P}_\alpha$ for $\alpha \in \Sigma_k(G)$, we deduce $\mathbf{Y} = \mathbf{GY}$, this can not happen, since this would imply $\mathbf{Y} = \mathbf{X}$. Hence there must exist $\alpha \in \Sigma_k(G)$ with $\mathbf{YP}_\alpha \neq \mathbf{Y}$. □

Corollary 5.14. *For a $\boldsymbol{B}$-orbit $\boldsymbol{Y} \neq \mathring{\boldsymbol{X}}$, we can find a sequence of simple roots $\alpha_1, \cdots, \alpha_r$, such that $\alpha_1, \cdots, \alpha_r$ raise $\boldsymbol{Y}$ to $\mathring{\boldsymbol{X}}$.*

Proof. Suppose $\mathbf{Y} \neq \mathring{\mathbf{X}}$. From the previous lemma, we can find $\alpha \in \Sigma_k(G)$, such that α raises $\mathbf{Y}$ to $\mathbf{Y}'$, then $\dim(\mathbf{Y}') > \dim(\mathbf{Y})$. We can repeat this argument, since $\mathring{\mathbf{X}}$ is the unique open Borel orbit, eventually we obtain α_r raises $\mathbf{Y}^{(r-1)}$ to $\mathring{\mathbf{X}}$. □

Lemma 5.15. *Denote $\boldsymbol{Y}$ the open Borel orbit in $\boldsymbol{YP}_\alpha$ such that $(\boldsymbol{Y}, \alpha)$ is not of type G or type U, then we have*

$$Adm_Y = {}^{\omega_\alpha} Adm_Y$$

Proof. This follows from

$${}^{\omega_\alpha}\chi(\mathbf{A_Y}) = \chi(\mathbf{A_Y})$$

which is proved in [KZ20, Section 3.1] and [KZ20, Section 3.2], and the definition of the set of admissible characters. □

Lemma 5.16. *Suppose α raises $\boldsymbol{Y}$ to $\boldsymbol{Z}$, then we have*

$$Adm_Y \subset {}^{\omega_\alpha} Adm_Z$$

Proof. If $(\mathbf{Y}, \alpha)$ is of type U, we have $\mathrm{Adm}_Z = {}^{\omega_\alpha}\mathrm{Adm}_Y$.

For other types, this follows from

$$\mathrm{Adm}_Y \subset \mathrm{Adm}_Z = {}^{\omega_\alpha}\mathrm{Adm}_Z$$

□

Proposition 5.17. *For every $\boldsymbol{B}$-orbit $\boldsymbol{Y}$, we have $Adm_Y \subset {}^{\omega} Adm_{\mathring{X}}$ for some $\omega \in W_k$.*

Proof. We denote the simple roots $\alpha_1, \cdots, \alpha_r$ such that $\alpha_1, \cdots, \alpha_r$ raise $\mathbf{Y}$ to $\mathring{\mathbf{X}}$, now this proposition follows from Lemma 5.16 applying to $\alpha_1, \cdots, \alpha_r$. □

As a preparation of our main theorem in this section, we discuss the poles of the intertwining operators. Let $S^Y_\chi : C_c^\infty(X) \to I(\chi)$ be an intertwining operator. By the "poles" of S^Y_χ we mean the smallest divisor $M \subset \mathrm{Adm}_Y$ which contains the polar divisors of $S^Y_\chi(\phi)$ for all $\phi \in C_c^\infty(X)$.

Proposition 5.18. *For M to be a polar divisor of S^Y_χ, it has to be contained in the variety of admissible characters of a smaller orbit in the closure of Y.*

Proof. We can repeat the proof in [Sak08, § 4.6.1]. □

Our main theorem on the necessary condition of distinction is the following.

Theorem 5.19. *Let $\boldsymbol{X}$ be a homogeneous spherical variety for an unramified group $\boldsymbol{G}$. Assume that $\mathring{X} = \mathring{\boldsymbol{X}}(k)$ carries a B-invariant measure. A necessary condition for the existence of a non-zero morphism $C_c^\infty(X) \to I(\chi)$ is that*

$$\chi \in {}^{\omega}(\delta^{-\frac{1}{2}} A_X^*) \tag{5.4}$$

for some $\omega \in W_k$. For every B-orbit Y on X and $\chi \in Adm_Y$, there exists a natural (up to a non-zero regular function on Adm_Y) family $S^Y_\chi : C_c^\infty(X) \to I(\chi)$, rational in χ. For almost all χ satisfying

the condition (5.4), the space of morphisms $C_c^\infty(X) \to I(\chi)$ admits a basis consisting of all S_χ^Y with $\chi \in Adm_Y$. Here "for almost all" means Zariski-generically: for $\chi \in Adm_Y$ after removing the proper lower-dimensional algebraic subsets-the polar divisors M of S_χ^Y (Proposition 5.18).

Proof. The necessary condition follows from Proposition 5.17. The statement that the space of morphisms $C_c^\infty(X) \to I(\chi)$ admits a basis follows from the analytic continuation of S_χ^Y as discussed in Proposition 5.6 and a Mackey theoretic argument as in [Sak08, Section 4.3]. □

Remark 5.20. Denote $\delta_{(X)}$ the modular character of the Borel subgroup of $\delta_{L(X)}$, then we have

$$\delta = \delta_{(X)}\delta_{P(X)},\ \delta^{-1/2}A_X^* = \delta_{(X)}^{-1/2}A_X^*, \quad \omega_{L(X)}(\delta^{-1/2}A_X^*) = \delta_{(X)}^{1/2}A_X^*$$

here $\omega_{L(X)}$ is the longest Weyl group element of $L(X)$.

5.5. **Basic theorem.** In this section, let $\mathbf{G}$ be an unramified group over k and $\mathbf{X}$ a homogeneous $\mathbf{G}$-variety, we will assume that k is a characteristic zero p-adic local field. We fix $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ and $\mathbf{B} = \mathbf{TU}$, $\mathbf{T}$ a maximal torus and $\mathbf{A} \subset \mathbf{T}$ a maximal split torus.

We recall the definition of the standard intertwining operators for unramified principal series:

$$T_\omega :\ I(\chi) \longrightarrow I({}^\omega\chi)$$

which are rational for $\chi \in T^*$ and are given by rational continuation of the integral:

$$T_\omega(f)(g) = \int_{U \cap \omega^{-1}U\omega \backslash U} \phi(\omega u g)\ du$$

This integral expression depends on the choice of a representative for $\omega \in N_{\mathbf{G}}(\mathbf{A})(k)$. We fix the Chevalley normalization for T_ω:

$$\mathrm{vol}(\mathbf{U} \cap \omega^{-1}\mathbf{U}\omega(\mathfrak{o})\backslash \mathbf{U}(\mathfrak{o})) = 1 \tag{5.5}$$

Recall that we have defined $S_{\tilde\chi}^Y$ in Definition 5.4. Next we want to investigate what happens when one composes the morphism $S_{\tilde\chi}^Y$ with the intertwining operators on principal series T_ω.

Theorem 5.21. *Let $\boldsymbol{Y}$ be a k-rational $\boldsymbol{B}$-orbit on $\boldsymbol{X}$ and α be a simple root such that $\boldsymbol{Y}$ is open in $\boldsymbol{Y}\cdot\boldsymbol{P}_\alpha$. The following describes the composition of T_{ω_α} with elements of $\underline{S}_\chi^Y$ for $\chi \in Adm_Y$:*

- *if $(\boldsymbol{Y}, \alpha)$ is of type G, then $T_{\omega_\alpha} \circ S_1 = 0$ for every $S_1 \in \underline{S}_\chi^Y$.*
- *if $(\boldsymbol{Y}, \alpha)$ is of type U, then $T_{\omega_\alpha} \circ S_{\tilde\chi}^Y \sim S_{\omega_\alpha\tilde\chi}^{\omega_\alpha Y}$.*
- *if $(\boldsymbol{Y}, \alpha)$ is of strongly tempered type (Definition **??**), then $T_{\omega_\alpha} \circ S_{\tilde\chi}^Y \sim S_{\omega_\alpha\tilde\chi}^{\omega_\alpha Y}$.*

Here we use $\sim$ to mean that the functionals on the two sides are equal to each other up to a meromorphic function of $\tilde\chi$.

Our theorem covers all non affine cases of the types of rank one spherical pairs in Proposition 4.11 (we exclude the type N cases), we can only prove these three types of weak local functional equations as these are the only cases we can adapt the proof strategy for split groups. For the basic homogeneous affine spherical pairs, we can provide direct computational argument to weak local functional equations: The Galois case $\mathbf{SL}_2\backslash\mathbf{Res}_{\ell/k}\mathbf{SL}_2$ for ℓ/k a quadratic extension is proved in Proposition 7.9 and $\mathbf{SU}_2\backslash\mathbf{SU}_3$ for $\mathbf{SU}_3$ an unramified unitary group has multiplicity one property for unramified principal series (Proposition 6.8).

Similar to the proof for split groups, the proof of Theorem 5.21 consists of two steps. First, we show it for functions whose support on $(\mathbf{Y}\cdot\mathbf{P}_\alpha)(k)$ is compact and then we will extend it to all $\phi \in C_c^\infty(X)$.

Proposition 5.22. *The statement of Theorem 5.21 is true when S_*^*'s, $T_{\omega_\alpha} \circ S_*^*$ are viewed as functionals and restricted to $\phi \in C_c^\infty(X)$ with $\phi|_{\boldsymbol{Y}\boldsymbol{P}_\alpha(k)} \in C_c^\infty((\boldsymbol{Y}\boldsymbol{P}_\alpha)(k))$.*

Proof. We let $S_1 \in \underline{S}^Y_\chi$ be one of the basis elements. If $(\mathbf{Y}, \alpha)$ is of type G, then we have $YP_\alpha = Y$ and the image of every $S_1 \in \underline{S}^Y_\chi$ is contained in $\mathrm{Ind}_{P_\alpha}^G(\chi\delta^{\frac{1}{2}}) \subset I(\chi)$ and T_{ω_α} annihilates that subspace.

Let $\mathbf{H}_\alpha = (\mathbf{G}_y \cap \mathbf{P}_\alpha \bmod \mathbf{U}_{P_\alpha}) \subset \mathbf{L}_\alpha$. We can write the functional S_1 as (the rational continuation of):

$$S_1(\phi) = \int_{U_y \backslash U \times T_Y} \phi(y \cdot ua)\tilde{\chi}^{-1}\delta^{\frac{1}{2}}(a)\ duda$$

If $(\mathbf{Y}, \alpha)$ is of strongly tempered type, we have $\mathbf{H}_\alpha \cap \mathbf{B} \subset \mathbf{Z}_\alpha$ as shown in the previous Proposition **??**, and hence we can define $\widetilde{\mathbf{L}}_\alpha$ as $\mathbf{L}_\alpha/\mathbf{H}_\alpha \cap \mathbf{B}$, we denote $\widetilde{\mathbf{B}}_\alpha$ the Borel subgroup of $\widetilde{\mathbf{L}}_\alpha$. The open Borel orbit in $\mathbf{H}_\alpha \backslash \mathbf{L}_\alpha$ has trivial stabilizer under the $\widetilde{\mathbf{B}}_\alpha$-action. Proposition **??** tells us that $\widetilde{\mathbf{B}}_\alpha$ acts transitively on the open Borel orbit and hence $\widetilde{\mathbf{L}}_\alpha$ acts transitively on $\widetilde{\mathbf{H}}_\alpha \backslash \widetilde{\mathbf{L}_\alpha}(k)$. We may view $\tilde{\chi}$ as a character of $\widetilde{\mathbf{B}}_\alpha$.

Now S_1 can be considered as a morphism $C_c^\infty((\mathbf{YP}_\alpha)(k)) \to \mathrm{Ind}_{\widetilde{B}_\alpha}^{\widetilde{L_\alpha}}(\tilde{\chi}\delta^{\frac{1}{2}})$, we can analyze it as the composition of two morphisms: first integration over $(U_{P_\alpha})_y \backslash U_{P_\alpha}$

$$f \mapsto \int_{(U_{P_\alpha})_y \backslash U_{P_\alpha}} f(y \cdot u\bullet)\ du$$

this gives us a morphism

$$\iota_\alpha :\ C_c^\infty(\mathbf{YP}_\alpha(k)) \longrightarrow C_c^\infty(\widetilde{H_\alpha}\backslash\widetilde{L_\alpha}, \mathcal{L}_{\delta_{P_\alpha}\delta^{-1}_{P_y \cap U_{P_\alpha}}})$$

this is followed by integration over $\tilde{y}\widetilde{B_\alpha}$:

$$\widetilde{S^Y_{\tilde{\chi}}} :\ C_c^\infty(\widetilde{H_\alpha}\backslash\widetilde{L_\alpha}, \mathcal{L}_{\delta_{P_\alpha}\delta^{-1}_{P_y \cap U_{P_\alpha}}}) \to \mathrm{Ind}_{\widetilde{B_\alpha}}^{\widetilde{L_\alpha}}(\tilde{\chi}\delta^{\frac{1}{2}})$$

Now by the fact that $\widetilde{B_\alpha}$ has a unique open orbit, it follows from a Mackey theoretic argument that for a generic $\chi \in \mathrm{Adm}_Y$, there is a unique morphism (the untwisted case is proved in Proposition 5.24)

$$C_c^\infty(\widetilde{H_\alpha}\backslash\widetilde{L_\alpha}, \mathcal{L}_{\delta_{P_\alpha}\delta^{-1}_{P_y \cap U_{P_\alpha}}}) \to \mathrm{Ind}_{\widetilde{B_\alpha}}^{\widetilde{L_\alpha}}({}^{\omega_\alpha}\tilde{\chi}\delta^{\frac{1}{2}})$$

note that $T_{\omega_\alpha} \circ \widetilde{S^Y_{\tilde{\chi}}}$ has image in $\mathrm{Ind}_{\widetilde{B_\alpha}}^{\widetilde{L_\alpha}}({}^{\omega_\alpha}\tilde{\chi}\delta^{\frac{1}{2}})$, we conclude $T_{\omega_\alpha} \circ S^Y_{\tilde{\chi}} \sim S^{{}^{\omega}Y}_{\tilde{\chi}}$ for $(\mathbf{Y}, \alpha)$ of strongly tempered type.

If $(\mathbf{Y}, \alpha)$ is of type U, since $T_{\omega_\alpha} \circ T_{\omega_\alpha} \sim \mathrm{id}$, let $\mathbf{Y}$ be the closed orbit in $\mathbf{YP}_\alpha$, then $\widetilde{S^Y_{\tilde{\chi}}}$ is given by the functional

$$\phi \mapsto \int_{\widetilde{T_Y}\backslash\widetilde{T}} \phi(\overline{y} \cdot a)\tilde{\chi}^{-1}\delta^{-1/2}(a)\ da$$

this converges absolutely for every $\tilde{\chi}$ if $\phi \in C_c^\infty(\widetilde{H_\alpha}\backslash\widetilde{L_\alpha}, \mathcal{L}_{\delta_{P_\alpha}\delta^{-1}_{P_y \cap U_{P_\alpha}}})$ and in the domain of convergence of T_{ω_α} we get

$$T_{\omega_\alpha} \circ \widetilde{S^Y_{\tilde{\chi}}}(\phi) = \int_{U_2}\int_{\widetilde{T_Y}\backslash\widetilde{T}} \phi(\overline{y} \cdot a\omega u)\tilde{\chi}^{-1}\delta^{-1/2}(a)da\ du$$

which is precisely $\widetilde{S^{{}^{\omega_\alpha}Y}_{{}^{\omega_\alpha}\tilde{\chi}}}$. □

To conclude the proof of Theorem 5.21, we repeat the reduction argument of Sakellaridis for split groups, we must show that the orbits in the closure do not contribute, that is we need to show what we proved for ϕ compactly supported on $(\mathbf{Y} \cdot \mathbf{P}_\alpha)(k)$ continues to hold for ϕ supported in its closure. We use S^Y_χ to denote any of the basis elements in the theorem, according to the type of

$(\mathbf{Y}, \alpha)$ that we are considering, the idea is to split the intersection of the support of ϕ with P_α into infinitely many compact pieces, let ϕ_i denote the restriction of ϕ to the i-th piece and use the fact that $S^Y_\chi = \sum_i S^Y_\chi(\phi_i)$ when χ is such that the integral expression for S^Y_χ converges. The problem is that T_{ω_α} and S^Y_χ will not converge simultaneously so we can not use their integral expressions to prove directly that $T_{\omega_\alpha} \sum_i S^Y_\chi(\phi_i) = \sum_i T_{\omega_\alpha} S^Y_\chi(\phi_i)$. We have the following lemma:

Lemma 5.23. *Let K_1 be an open compact subgroup of P_α, let $g_1, \cdots, g_m$ be representatives for orbits of K_1 on $B\backslash P_\alpha$. Then there are rational functions $r_1, \cdots, r_m$ of χ such that for $\phi \in I(\chi)^{K_1}$ we have*

$$(T_{\omega_\alpha}\phi)(1) = \sum_{j=1}^{m} r_j(\chi)\phi(g_j)$$

This lemma is a consequence of the rationality of T_ω and the fact that $\mathrm{Ind}_B^{P_\alpha}(\chi\delta^{\frac{1}{2}})^{K_1}$ is finite dimensional.

Now given $\phi \in C_c^\infty(\overline{\mathbf{YP}_\alpha}(k))$, fix an open-compact $K_1 \subset P_\alpha$ such that ϕ is K_1-invariant and representatives $g_1, \cdots, g_m$ as above and enumerate the K_1-orbits on $(\mathbf{YP}_\alpha)(k) : \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \cdots$, let $\phi_i = \phi \cdot 1_{\mathcal{O}_i} \in C_c^\infty((\mathbf{Y}\cdot\mathbf{P}_\alpha)(k))$, for $g \in P_\alpha$, the sets $\mathcal{O}_i g$ define a partition of $(\mathbf{Y}\cdot\mathbf{P}_\alpha)(k)$ into $g^{-1}K_1 g$-orbits. For χ in the region of convergence of the integral expression for S^Y_χ we have: $S^Y_\chi(R(g_j)\phi) = \sum_i S^Y_\chi(R(g_j)\phi_i)$ for every j. Using the previous lemma

$$T_{\omega_\alpha}\sum_i S^Y_\chi(\phi_i) = \sum_{j=1}^m r_j(\chi)\sum_i R(g_j)S^Y_\chi(\phi_i) =$$
$$= \sum_i\sum_{j=1}^m r_j(\chi)R(g_j)S^Y_\chi(\phi_i) = \sum_i T_{\omega_\alpha}S^Y_\chi(\phi_i)$$

Using Proposition 5.22, we have $T_{\omega_\alpha} \circ S^Y_\chi(\phi_i) \sim S^{\omega_\alpha Y}_{\omega_\alpha\chi}(\phi_i)$, hence $T_{\omega_\alpha}S^Y_\chi \sim S^{\omega_\alpha Y}_{\omega_\alpha\chi}$.

We prove a proposition concerning the generic multiplicity-free property for spherical varieties with a single Borel orbit.

Proposition 5.24. *For $\boldsymbol{G}$ an unramified group of semisimple split rank one, let $\boldsymbol{X} = \boldsymbol{H}\backslash\boldsymbol{G}$ be a k-spherical homogeneous variety with $\Sigma_k(X) \neq \emptyset$. Assume $\mathring{\boldsymbol{X}}(k)$ consists of a single B-orbit, then, generically*

$$dim\ Hom_G(C_c^\infty(X), I(\chi)) \leq 1$$

for $I(\chi)$ an irreducible unramified principal series.

Proof. This follows because, by a Mackey theory argument ([Sak08, §4.3]) we have $\mathrm{Hom}_G(C_c^\infty(X), I(\chi))$ is spanned by the intertwining morphisms associated with the closed and the open Borel orbits.

We have $\dim \mathrm{Hom}_G(C_c^\infty(\mathring{X}), I(\chi)) \leq 1$ as $\mathring{X}$ is a single B-orbit. Note the closed Borel orbits Y are of semisimple split rank zero for $\Sigma_k(X) \neq \emptyset$, we can check this case by case using Proposition 4.11. We conclude

$$\dim \mathrm{Hom}_G(C_c^\infty(X), I(\chi)) \leq 1$$

except for those $I(\chi)$ that admit non-zero intertwining morphism in $\mathrm{Hom}_G(C_c^\infty(Y), I(\chi))$, and such χ's form a dimension zero subset in Adm_X. □

Remark 5.25. The same proof can be applied to $C_c^\infty(X, \mathcal{L})$ the twisted case.

6. Some multiplicity formulas

We verify the multiplicity formulas for the basic spherical varieties whose local functional equations we want to study. This is based on the calculation of the maximal split torus A_X.

Given G a quasisplit group over k and X a G-spherical variety, we denote the maximal k-torus of X by T_X and the maximal k-split torus by A_X.

We have the following commutative diagram (Diagram (**??**))

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^X & \longrightarrow & T & \longrightarrow & T_X & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & A^X & \longrightarrow & A & \longrightarrow & A_X & \longrightarrow & 1 \end{array}$$

Example 6.1. We want to calculate the above diagram for $X = GL_2\backslash \mathrm{Res}_{\ell/k} GL_2$, where $T_X \cong \mathrm{Res}_{\ell/k}\mathbb{G}_m$ and $A_X \cong \mathbb{G}_m$, we have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & (\mathrm{Res}_{\ell/k}\mathbb{G}_m)^2 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & (\mathbb{G}_m)^2 & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

the map $\mathrm{Res}_{\ell/k}\mathbb{G}_m \to (\mathrm{Res}_{\ell/k}\mathbb{G}_m)^2$ is given by $a \mapsto (a, \overline{a})$ at the level of k-points, the induced map on the second row is $a \mapsto (a, a)$, namely the diagonal embedding.

We have the following diagram for $X = SL_2\backslash \mathrm{Res}_{\ell/k} SL_2$ where ℓ/k is a quadratic extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_1 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

We have $H^1(k, U_1) \cong \mathbb{Z}/2\mathbb{Z} = \ker(H^1(k, T^X) \to H^1(k, T))$.

Example 6.2. We want to calculate the previous diagram for $X = U_2\backslash \mathrm{Res}_{\ell/k} GL_2$ where ℓ/k is a quadratic extension and U_2 is the unitary group associated with a Hermitian space over ℓ, note that $T_X \cong \mathbb{G}_m^2 \cong A_X$, hence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (U_1)^2 & \longrightarrow & (\mathrm{Res}_{\ell/k}\mathbb{G}_m)^2 & \longrightarrow & \mathbb{G}_m^2 & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m^2 & \longrightarrow & 1 \end{array}$$

Example 6.3. We want to calculate the previous diagram for $X = U_2\backslash U_3$ where U_3 is the unitary group associated with a Hermitian space over ℓ, we have $T \cong \mathrm{Res}_{\ell/k}\mathbb{G}_m \times U_1$, and $T_X \cong \mathrm{Res}_{\ell/k}\mathbb{G}_m$, $A_X \cong \mathbb{G}_m$; hence we have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_1 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m \times U_1 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

Note that the maximal torus of SU_3 is isomorphic to $\mathrm{Res}_{\ell/k}\mathbb{G}_m$, hence we have the following diagram for $SU_2\backslash SU_3$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

Example 6.4. We want to calculate the previous diagram for $X = GL_2\backslash \mathrm{Res}_{\ell/k}GL_2$ for ℓ/k a cubic extension, we have $T \cong (\mathrm{Res}_{\ell/k}\mathbb{G}_m)^2$ and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & (\mathrm{Res}_{\ell/k}\mathbb{G}_m)^2 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

hence we have the following diagram for $SL_2\backslash \mathrm{Res}_{\ell/k}SL_2$ where ℓ/k is a cubic extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Res}_{\ell/k}\mathbb{G}_m & \longrightarrow & T_X & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbb{G}_m & \longrightarrow & A_X & \longrightarrow & 1 \end{array}$$

Note the first Galois cohomology $H^1(k, \mu_2) \cong k^\times/k^{\times 2}$. For a characteristic zero p-adic local field k with residue characteristic not equal to 2, we can prove that $k^\times/k^{\times 2} \cong (\mathbb{Z}/2\mathbb{Z})^2$, the three nontrivial square classes correspond to the set of quadratic extensions of k.

6.1. **Some multiplicity formulas.** In this section we prove several multiplicity formulas. Let k be a p-adic field of characteristic zero with odd residue characteristic. Let $\mathbf{G}$ be a connected unramified group over k and $\mathbf{H}$ a k-spherical subgroup of $\mathbf{G}$. In the following, for ℓ/k an extension of p-adic local fields and L/ℓ a quadratic extension, we denote $\omega_{L/\ell} : \ell^\times/\mathrm{Nm}_{L/\ell}(L^\times) \to \mathbb{Z}/2\mathbb{Z}$, the quadratic characters associated with the quadratic extension L/ℓ.

For the next proposition, note that given an unramified representation $\pi = \mathrm{Ind}(\chi^2)$ of $SL_2(\ell)$ with $\chi \in (\ell^\times)^*$. For $\pi_1 = \mathrm{Ind}(\chi, \chi^{-1})$ and $\pi_2 = \mathrm{Ind}(\chi\omega_{L/\ell}, \chi^{-1}\omega_{L/\ell}^{-1})$, we have $\pi_1|_{SL_2(\ell)} = \pi,\ \pi_2|_{SL_2(\ell)} = \pi$ where $\omega_{L/\ell}$ is the unramified quadratic character associated with the unramified quadratic extension L of ℓ.

Definition 6.5. For a split rank one spherical pair $(\mathbf{G}(k), \mathbf{H}(k)) = (G, H)$, we will say that an irreducible H-distinguished unramified representation π is of *general position*, if the space of invariant functionals $\mathrm{Hom}_H(\pi, \mathbb{C})$ does not contain the contribution from the closed Borel orbits inside $\mathbf{H}\backslash\mathbf{G}$.

Proposition 6.6. *Given an irreducible unramified $SL_2(k)$-distinguished representation π for $SL_2(\ell)$ of general position, where ℓ/k is an unramified quadratic extension, we have*

$$dim\ Hom_{SL_2(k)}(\pi, \mathbb{C}) = 2$$

Proof. Denote $X = SL_2(k)\backslash SL_2(\ell)$ and $\pi = \mathrm{Ind}(\chi)$, then we have

$$\mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C}) = \mathrm{Hom}_{SL_2(\ell)}(\pi, C^\infty(X)) = \mathrm{Hom}_{SL_2(\ell)}(C_c^\infty(X), \tilde{\pi})$$

Since π is of general position, we conclude that $\mathrm{Hom}_{SL_2(\ell)}(C_c^\infty(X), \tilde{\pi}) = \mathrm{Hom}_{B_2(\ell)}(C_c^\infty(X), \mathbb{C}_{\chi^{-1}\delta^{1/2}})$ has a basis given by intertwining operators indexed by the number of open Borel orbits inside $\mathring{X}$. $\square$

For the next proposition, we denote the three quadratic extensions of ℓ by $\ell_1,\ \ell_2,\ \ell_3$, given an unramified representation $\pi = \mathrm{Ind}(\chi^2)$ of $SL_2(\ell)$, denote the irreducible representations $\pi_1 = \mathrm{Ind}(\chi, \chi^{-1}),\ \pi_2 = \mathrm{Ind}(\chi\omega_{\ell_1/\ell}, \chi^{-1}\omega_{\ell_1/\ell}^{-1}),\ \pi_3 = \mathrm{Ind}(\chi\omega_{\ell_2/\ell}, \chi^{-1}\omega_{\ell_2/\ell}^{-1}),\ \pi_4 = \mathrm{Ind}(\chi\omega_{\ell_3/\ell}, \chi^{-1}\omega_{\ell_3/\ell}^{-1})$, here $\omega_{\ell_i/\ell}$ are the quadratic characters associated with the extensions ℓ_i.

The following proposition can be proved in the same way as Proposition 6.6.

Proposition 6.7. *Given an irreducible unramified $SL_2(k)$-distinguished representation π for $SL_2(\ell)$ of general position for ℓ/k a cubic unramified extension, we have*

$$dim\ Hom_{SL_2(k)}(\pi, \mathbb{C}) = 4$$

Let SU_3 denote the unramified unitary group associated with an ℓ-Hermitian space for ℓ/k an unrmaified quadratic extension, and let SU_2 denote its subgroup with embedding $\begin{pmatrix} * & * \\ * & * \end{pmatrix} \hookrightarrow \begin{pmatrix} * & & * \\ & 1 & \\ * & & * \end{pmatrix}$ over algebraic closure. The following proposition can be proved in the same way as Proposition 6.6.

Proposition 6.8. *Given an irreducible unramified $SU_2(k)$-distinguished representation π for $SU_3(k)$ of general position, we have*

$$dim\ Hom_{SU_2(k)}(\pi, \mathbb{C}) = 1$$

Moreover, a basis of this space can be constructed via the restriction of the basis $\ell \in Hom_{U_2(k)}(\tilde{\pi}, \mathbb{C})$, where $\tilde{\pi}$ is an unramified representation of $U_3(k)$ such that $\tilde{\pi}|_{SU_3(k)} = \pi$.

7. Weak local functional equations for $\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\ell)$, ℓ/k quadratic

In this section, we let $G = SL_2(\ell)$, $H = SL_2(k)$, $B = B_2(\ell)$, ℓ/k is an unramified quadratic extension of p-adic fields. Choose $\tau \in \ell^\times,\ \overline{\tau} = -\tau$, then we have

$$\mathring{X} = \bigsqcup_{c \in k^\times / N_{\ell/k}(\ell^\times)} Hg_cB$$

here $g_c = \begin{pmatrix} c\tau & -1 \\ 1 & 0 \end{pmatrix}$. We introduce

$$H_c = SL_2(k) \cap g_c B_2(\ell) g_c^{-1}$$

We introduce the following coordinates on $SL_2(\ell)$:

$$B = AU,\ a(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},\ u(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

here $t \in \ell^\times,\ n \in \ell$.

Then we have the following expression for the intertwining operator:

Proposition 7.1. *For χ an unramified character of $\ell^\times$, $\tilde{\chi}$ an unramified character of $\boldsymbol{T}_X(k) \cong k^\times$, such that*

$$\tilde{\chi}(N_{\ell/k}(t)) = \chi(t)$$

then we have

$$S_{\tilde{\chi}} = S_1^\chi + \tilde{\chi}^{-1}(\varpi)|\varpi| S_\varpi^\chi$$

here for $\phi \in C_c^\infty(X)$,

$$S_c^\chi(\phi) := \int_{B_c \backslash B} \phi(Hg_cb)\chi^{-1}(b)\delta^{-\frac{1}{2}}(b)\ db$$

with db the right Haar measure on B.

Proof. By definition $S_{\tilde{\chi}}$ is an integration over the horocycles and then followed by an integration over T_X with the factor $\tilde{\chi}^{-1}\delta^{-\frac{1}{2}}$.

We get,

$$S_{\tilde{\chi}}(\phi) = \int_{U^1\backslash \ell^\times} \int_\ell \phi(Hg_1 a(t)u(n))\chi(t)^{-1}\delta(a(t))^{1/2}\, dn\, d^\times t$$
$$+ \tilde{\chi}(\varpi)^{-1}|\varpi| \int_{U^1\backslash \ell^\times} \int_\ell \phi(Hg_c a(t)u(n))\chi(t)^{-1}\delta(a(t))^{1/2} dn\, d^\times t$$

□

We denote $\langle \cdot, \cdot \rangle :\ I(\chi) \times I(\chi^{-1}) \longrightarrow \mathbb{C}$ the standard G-invariant pairing,

$$\langle F, \phi \rangle = \int_K F(k)\phi(k)\, dk$$

Here $F \in I(\chi)$ and $\phi \in I(\chi^{-1})$.

Lemma 7.2. *Under the identifications,*

$$Hom_B(C_c^\infty(X), \mathbb{C}_{\chi\delta^{1/2}}) = Hom_G(C_c^\infty(X), I(\chi)) = Hom_G(I(\chi^{-1}), C^\infty(X))$$

For $S \in Hom_B(C_c^\infty(X), \mathbb{C}_{\chi\delta^{1/2}})$, the corresponding map T_S in $Hom_G(I(\chi^{-1}), C^\infty(X))$ is characterized by

$$\int_X f(x)T_S(\phi)(x)\, dx = \langle A_S(f), \phi \rangle$$

where $A_S(f)(g) = S(R(g)f)$.

Proof. Fix a G-invariant measure dx on X, we get a duality

$$C_c^\infty(X) \times C^\infty(X) \longrightarrow \mathbb{C},\ \phi \times f \mapsto \int_X \phi(x)f(x)\, dx$$

From this we deduce the existence of $T_S :\ I(\chi^{-1}) \to C^\infty(X)$: Given $\phi \in I(\chi^{-1})$, we define $T_S(\phi) \in C^\infty(X)$ to be the function corresponds to the linear functional $\langle A_S(f), \phi \rangle$ on $C_c^\infty(X)$.

We check the map T_S is G-equivariant. Indeed, for $g_0 \in G$,

$$\begin{aligned}\int_X f(x)T_S(R(g_0)\phi)(x)\, dx &= \langle A_S(f), R(g_0)\phi \rangle \\ &= \langle R(g_0^{-1})A_S(f), \phi \rangle \\ &= \langle A_S(R(g_0^{-1})f), \phi \rangle \\ &= \int_X f(x)\, R(g_0)T_S(\phi)(x)\, dx.\end{aligned}$$

The nondegeneracy of the pairing on X implies,

$$T_S(R(g_0)\phi) = R(g_0)T_S(\phi).$$

□

Lemma 7.3. *For $S = S_c^\chi$, we have*

$$\langle A_S(f), \phi \rangle = \int_{B_c\backslash G} f(Hg_c g)\phi(g)\, dg$$

Proof. Use the normalized induction convention, for $F \in I(\chi) = \mathrm{Ind}_B^G(\chi\delta^{1/2})$, we have

$$F(bg) = \chi(b)\delta^{1/2}(b)F(g)$$

then for $\phi \in I(\chi^{-1})$,

$$\phi(bg) = \chi^{-1}(b)\delta^{1/2}(b)\phi(g)$$

Now let $S = S_c^\chi$, as a B-eigenfunctional, it is given by

$$S_c^\chi(f) = \int_{B_c\backslash B} f(Hg_cb)\chi^{-1}(b)\delta^{-1/2}(b)\ db$$

Therefore

$$A_S(f)(g) = \int_{B_c\backslash B} f(Hg_cbg)\chi^{-1}\delta^{-1/2}(b)db$$

Plugging into the definition for $\langle\cdot,\cdot\rangle$, we get

$$\langle A_S(f), \phi\rangle = \int_K \int_{B_c\backslash B} f(Hg_cbk)\chi^{-1}(b)\delta^{-1/2}(b)\phi(k)\ dbdk$$

Now use the B-equivariance of ϕ, we get

$$\chi^{-1}(b)\delta^{-1/2}(b)\phi(k) = \delta^{-1}(b)\phi(bk)$$

so the integral becomes

$$\langle A_S(f), \phi\rangle = \int_K \int_{B_c\backslash B} f(Hg_cbk)\phi(bk)\delta^{-1}(b)\ dbdk$$

finally the Iwasawa decomposition $G = BK$ gives us

$$dg = \delta^{-1}(b)d_rb\ dk$$

hence we get,

$$\langle A_S(f), \phi\rangle = \int_{B_c\backslash G} f(Hg_cg)\phi(g)\ dg$$

□

Lemma 7.4. *We denote the quotient map,*

$$\begin{aligned} q_c :\ B_c\backslash G &\longrightarrow H\backslash G \\ B_cg &\longmapsto Hg_cg \end{aligned}$$

then we have

$$T_{S_c^\chi}(\phi)(H\cdot 1) = \int_{H_c\backslash H} \phi(g_c^{-1}h)\ dh$$

Proof. Denote $S = S_c^\chi$. From

$$\langle A_S(f), \phi\rangle = \int_{B_c\backslash G} f(Hg_cg)\phi(g)\ dg$$

we get $T_S(\phi)$ is a fiber integral,

$$T_S(\phi)(x) = \int_{q_c^{-1}(x)} \phi(g)\ d\mu_x(g)$$

where $d\mu_x$ is the induced measure on the fiber of q_c. For the base point $x = H$, we have

$$q_c^{-1}(H) = B_c\backslash g_c^{-1}H \cong H_c\backslash H$$

hence

$$T_S(\phi)(H\cdot 1)=\int_{H_c\backslash H} f(g_c^{-1}h)\ dh$$

□

For $c\in k^\times/N_{\ell/k}(\ell^\times)$, $f\in I(\chi^{-1})$, we define

$$\Lambda_c(f):=\int_{H_c\backslash H} f(g_c^{-1}h)\ dh$$

viewed as a functional in $\mathrm{Hom}_H(I(\chi^{-1}),1)$.

Proposition 7.5. *Under the identifications,*

$$Hom_B(C_c^\infty(X),\mathbb{C}_{\chi\delta^{1/2}})=Hom_G(I(\chi^{-1}),C^\infty(X))\cong Hom_H(I(\chi^{-1}),1)$$

we have $S_c^\chi\leftrightarrow\Lambda_c$.

Proof. Note that the isomorphism,

$$\mathrm{Hom}_G(I(\chi^{-1}),C^\infty(X))\cong\mathrm{Hom}_H(I(\chi^{-1}),1)$$

is given by the evaluation at $H\cdot 1$, hence the bijection $S_c^\chi\leftrightarrow\Lambda_c$ follows from the Lemma 7.4. □

We will call the map

$$\begin{aligned}\mathcal{F}:\ \mathrm{Hom}_B(C_c^\infty(X),\mathbb{C}_{\chi\delta^{1/2}})&\longrightarrow\mathrm{Hom}_H(I(\chi^{-1}),1)\\ S&\longmapsto\mathcal{F}(S)\end{aligned}$$

the Frobenius map.

Corollary 7.6. *If we denote* $\Lambda_{\tilde\chi}$ *the Frobenius dual of* $S_{\tilde\chi}$*, then we have*

$$\Lambda_{\tilde\chi}=\Lambda_1+\tilde\chi^{-1}(\varpi)|\varpi|\Lambda_\varpi\in Hom_H(I(\chi^{-1}),1)$$

Next we study the restriction of the invariant functionals. For representations $\Pi_1=I(\chi,\chi^{-1})$ and $\Pi_2=I(\chi\omega_{L/\ell},\chi^{-1}\omega_{L/\ell}^{-1})$ of $GL_2(\ell)$, where χ is an unramified character of $\ell^\times$ and $\omega_{L/\ell}$ the quadratic character associated with the unramified extension L/ℓ. Denote $\pi=I(\chi^2)$, then we have natural restriction maps:

$$\mathrm{Hom}_{GL_2(k)}(\Pi_1,\mathbb{C})\to\mathrm{Hom}_{SL_2(k)}(\pi,\mathbb{C}),\ \mathrm{Hom}_{GL_2(k)}(\Pi_2,\mathbb{C})\to\mathrm{Hom}_{SL_2(k)}(\pi,\mathbb{C})$$

note that $\mathrm{Hom}_{GL_2(k)}(\Pi_1,\mathbb{C})$ and $\mathrm{Hom}_{GL_2(k)}(\Pi_2,\mathbb{C})$ are both one-dimensional.

$L_1\in\mathrm{Hom}_{GL_2(k)}(\Pi_1,\mathbb{C})$ can be constructed as:

$$L_1(F)=\int_{H_{g_1}\backslash GL_2(k)} F(g_1^{-1}h)\ dh$$

here $H_{g_1}=GL_2(k)\cap g_1B_{GL_2}(\ell)g_1^{-1}\cong\ell^\times$.

Proposition 7.7. L_1 *is mapped to the basis element* $\Lambda_1+\chi(\varpi)|\varpi|\Lambda_2\in Hom_{SL_2(k)}(\pi,\mathbb{C})$.

Proof. We let $f=F|_{SL_2(\ell)}$, note the following decomposition

$$\ell^\times\backslash GL_2(k)=\bigsqcup_{c\in k^\times/\mathrm{N}_{\ell/k}(\ell^\times)}\ell^\times m_cSL_2(k)$$

where $m_c=\begin{pmatrix}1&0\\0&c\end{pmatrix}$, $c\in\{1,\varpi\}$.

We get

$$L_1(F) = \int_{H_c \backslash H} F(g_1^{-1} m_c h)\ dh$$

here $H_c = SL_2(k) \cap m_c^{-1} \ell^\times m_c = SL_2(\ell) \cap g_c B_2(\ell) g_c^{-1}$.

Now since $g_1^{-1} m_c = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} g_c^{-1}$. We get

$$F(\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} g_c^{-1} h) = \chi(c)|c|_\ell^{1/2} F(g_c^{-1} h)$$

note $g_c^{-1} h \in SL_2(\ell)$, we get $F(g_c^{-1} h) = f(g_c^{-1} h)$.

Hence

$$L_1(F) = \Lambda_1(f) + \chi(\varpi)|\varpi|\Lambda_2(f)$$

□

Similarly, we have the following proposition,

Proposition 7.8. *Denote $L_2 \in Hom_{GL_2(k)}(\Pi_2, \mathbb{C})$ the basis element,*

$$L_2(F) = \int_{H_{g_1} \backslash GL_2(k)} F(g_1^{-1} h)\ dh$$

then we have L_2 is mapped to the element $\Lambda_1 - \chi(\varpi)|\varpi|\Lambda_2 \in Hom_{SL_2(k)}(\pi, \mathbb{C})$.

Given χ an unramified character of $\ell^\times$, we introduce the following two characters $\tilde{\chi}_i$ of T_X as extensions of χ^{-2}:

- $\tilde{\chi}_1(N_{\ell/k}(t)) = \chi^{-2}(t)$ for $t \in \ell^\times$, $\tilde{\chi}_1(\varpi) = \chi^{-1}(\varpi)$.
- $\tilde{\chi}_2(N_{\ell/k}(t)) = \chi^{-2}(t)$ for $t \in \ell^\times$, $\tilde{\chi}_2(\varpi) = -\chi^{-1}(\varpi)$.

then we get $\Lambda_{\tilde{\chi}_1} \in \mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C})$ and $\Lambda_{\tilde{\chi}_2} \in \mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C})$. From the Corollary 7.6, we get

- $\Lambda_{\tilde{\chi}_1} = \Lambda_1 + \chi(\varpi)|\varpi|\Lambda_2$.
- $\Lambda_{\tilde{\chi}_2} = \Lambda_1 - \chi(\varpi)|\varpi|\Lambda_2$.

Proposition 7.9. *The intertwining functionals $S_{\tilde{\chi}_i}$ satisfy the weak local functional equations,*

$$T_\omega \circ S_{\tilde{\chi}_i} \sim S_{\omega_{\tilde{\chi}_i}}$$

Proof. This is because we have $\ell_1 = \mathcal{F}(S_{\tilde{\chi}_1}) \in \mathrm{Hom}_{SL_2(k)}(\pi, 1)$ and $\ell_2 = \mathcal{F}(S_{\tilde{\chi}_2}) \in \mathrm{Hom}_{SL_2(k)}(\pi, 1)$, the Frobenius duals of ℓ_1 and ℓ_2 satisfy weak local functional equations as they are restrictions of the GL_2-invariant functionals. □

8. Weak local functional equations for $\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\ell)$, ℓ/k cubic

In this section, we let $G = SL_2(\ell),\ H = SL_2(k),\ B = B_2(\ell)$, ℓ/k is an unramified cubic extension of p-adic fields with $p \neq 2$. Choose $\alpha \in \ell^\times, \alpha \notin k^\times$, then we have

$$\mathring{X} = \bigsqcup_{c \in k^\times / k^{\times 2}} H g_c B$$

here $g_c = \begin{pmatrix} c\alpha & -1 \\ 1 & 0 \end{pmatrix}$.

We have the following sequence,

$$\mu_2 \longrightarrow \mathbf{T}(k) \longrightarrow \mathbf{T}_X(k) \longrightarrow H^1(k, \mu_2) \cong k^\times / k^{\times 2}$$

we denote the quotient map,

$$q : \mathbf{T}(k) = \ell^\times \longrightarrow \mathbf{T}_X(k) = T_X$$

We choose $\sqrt{c} \in \overline{k}$, and denote $\xi_c = q(\sqrt{c}) \in T_X$ the representative for $c \in k^\times/k^{\times 2}$.

From the exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Res}_{\ell/k}\ \mathbb{G}_m \longrightarrow \mathbf{T}_X \longrightarrow 1$$

we get

$$1 \longrightarrow \mu_2(k) \longrightarrow \ell^\times \xrightarrow{q} T_X \xrightarrow{\delta} H^1(k, \mu_2) \longrightarrow H^1(k, \mathrm{Res}_{\ell/k}\mathbb{G}_m)$$

we have $\delta(\xi_c) = [c]$.

Proposition 8.1. *For χ an unramified character of $\ell^\times$, $\tilde{\chi}$ an unramified character of T_X, such that $\tilde{\chi}(q(t)) = \chi(t)$ for $t \in \ell^\times$. Then we have*

$$S_{\tilde{\chi}} = \sum_{c \in k^\times/k^{\times 2}} \tilde{\chi}(\xi_c)^{-1} |c|_\ell^{1/2} S_c^\chi$$

here for $\phi \in C_c^\infty(X)$,

$$S_c^\chi(\phi) := \int_{B_c \backslash B} \phi(H g_c b) \chi^{-1}(b) \delta^{-\frac{1}{2}}(b)\ db$$

with db the right Haar measure on B.

For $c \in k^\times/k^{\times 2}$, we denote

$$\Lambda_c(f) = \int_{H_c \backslash H} f(g_c^{-1} h)\ dh$$

here $H_c = SL_2(\ell) \cap g_c B_2(\ell) g_c^{-1}$.

The proof of the following proposition is similar to the quadratic case:

Proposition 8.2. *If we denote $\Lambda_{\tilde{\chi}}$ the Frobenius dual of $S_{\tilde{\chi}}$, then we have*

$$\Lambda_{\tilde{\chi}} = \sum_{c \in k^\times/k^{\times 2}} \tilde{\chi}(\xi_c)^{-1} |c|_\ell^{1/2} \Lambda_c$$

here $\Lambda_{\tilde{\chi}} \in Hom_{SL_2(k)}(I(\chi^{-1}), 1)$.

Next we study the restriction of the invariant functionals. For representations $\Pi_1 = I(\chi, \chi^{-1})$ and $\Pi_2 = I(\chi\omega_{L/\ell}, \chi^{-1}\omega_{L/\ell}^{-1})$ with L/ℓ a quadratic extension, where χ is an unramified character of $\ell^\times$ and $\omega_{L/\ell}$ the quadratic character associated with the quadratic extension L/ℓ. For $\pi = I(\chi^2)$, we have the restriction map

$$\mathrm{Hom}_{GL_2(k)}(\Pi_1, \mathbb{C}) \longrightarrow \mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C}) \tag{8.1}$$

$L_1 \in \mathrm{Hom}_{GL_2(k)}(\Pi_1, \mathbb{C})$ can be constructed as:

$$L_1(F) = \int_{k^\times \backslash GL_2(k)} F(g_1^{-1} h)\ dh$$

Lemma 8.3. *We have a bijection of sets:*

$$k^\times \backslash GL_2(k)/SL_2(k) \cong k^\times/k^{\times 2}$$

Proof. Define a map

$$\psi :\ k^\times \backslash GL_2(k)/SL_2(k) \longrightarrow k^\times/k^{\times 2}$$

by $\psi([g]) = [\det g]$. Map ψ is well defined because if we replace g by ags with $a \in k^\times,\ s \in SL_2(k)$. Then

$$\det(ags) = \det(aI_2)\det(g)\det(s) = a^2 \det(g)$$

It is surjective because for $c \in k^\times$, take $m_c = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$, then $\psi([m_c]) = [c]$.

It is injective because if $[\det\ g_1] = [\det\ g_2]$, then $\det\ g_2 = a^2 \det\ g_1$, take $s = g_1^{-1} a^{-1} g_2$, we get $\det\ s = \det(g_1)^{-1} a^{-2} \det(g_2) = 1$, so $s \in SL_2(k)$, and $k^\times g_2 SL_2(k) = k^\times g_1 SL_2(k)$. □

Proposition 8.4. *L_1 is mapped to the basis element $\ell_1 = \sum_{c \in k^\times/k^{\times 2}} \chi(c)|c|_\ell^{1/2} \Lambda_c$ under the restriction map (8.4).*

Proof. We let $f = F|_{SL_2(\ell)}$, note the following decomposition,

$$k^\times \backslash GL_2(k) = \bigsqcup_{c \in k^\times/k^{\times 2}} k^\times m_c SL_2(k)$$

we have

$$L_1(F) = \sum_{c \in k^\times/k^{\times 2}} \int_{\{\pm I\}\backslash SL_2(k)} F(g_1^{-1} m_c h)\ dh$$

using the matrix identity,

$$g_1^{-1} m_c = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} g_c^{-1}$$

we get

$$F(g_1^{-1} m_c h) = \chi(c)|c|_\ell^{1/2} F(g_c^{-1} h)$$

therefore

$$\int_{\{\pm I\}\backslash SL_2(k)} F(g_1^{-1} m_c h)\ dh = \chi(c)|c|_\ell^{1/2} \int_{\{\pm I\}\backslash SL_2(k)} f(g_c^{-1} h)\ dh$$

Hence

$$L_1(F) = \sum_{c \in k^\times/k^{\times 2}} \chi(c)|c|_\ell^{1/2} \Lambda_c(f)$$

□

Proposition 8.5. *Given χ an unramified character of $\ell^\times$, we introduce the following character χ_1 of T_X as extensions of χ^{-2}: $\chi_1(q(t)) = \chi^{-2}(t)$ for $t \in \ell^\times$ and $\chi_1(\xi_c) = \chi^{-1}(c)$ for $c \in k^\times/k^{\times 2}$. Then we have*

$$\Lambda_{\chi_1} = \ell_1 \in Hom_{SL_2(k)}(\pi, \mathbb{C})$$

Proof. This is because from Proposition 8.2, we have

$$\Lambda_{\chi_1} = \sum \chi_1^{-1}(\xi_c)|c|_\ell^{1/2} \Lambda_c$$

From Proposition 8.4, we have

$$\ell_1 = \sum_{c \in k^\times/k^{\times 2}} \chi(c)|c|_\ell^{1/2} \Lambda_c$$

Since $\chi_1(\xi_c) = \chi^{-1}(c)$, we get

$$\Lambda_{\chi_1} = \ell_1$$

□

If we denote $L_2 \in \mathrm{Hom}_{GL_2(k)}(\Pi_2, \mathbb{C})$ as:

$$L_2(F) = \int_{k^\times \backslash GL_2(k)} F(g_1^{-1} h)\ dh$$

for $F \in \Pi_2$ and $\ell_2 \in \mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C})$, the image of L_2 under the restriction map

$$\mathrm{Hom}_{GL_2(k)}(\Pi_2, \mathbb{C}) \longrightarrow \mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C})$$

We have the following proposition:

Proposition 8.6. *Given χ an unramified character of $\ell^\times$, we introduce the following character χ_2 of T_X as an extension of $(\chi\omega_{L/\ell})^{-2}$: $\chi_2(q(t)) = (\chi\omega_{L/\ell})^{-2}(t)$ for $t \in \ell^\times$ and $\chi_2(\xi_c) = (\chi\omega_{L/\ell})^{-1}(c)$ for $c \in k^\times/k^{\times 2}$. Then we have*

$$\Lambda_{\chi_2} = \ell_2 \in Hom_{SL_2(k)}(\pi, \mathbb{C})$$

Proof. We can repeat the proof for Proposition 8.5. □

Now we consider the weak local functional equation, we fix χ an unramified character of $\ell^\times$. For the irreducible unramified representation $\pi = I(\chi^2)$ of $SL_2(\ell)$, note that π is the restriction of $\Pi_1 = I(\chi, \chi^{-1})$, $\Pi_2 = I(\chi\omega_{\ell_1/\ell}, \chi^{-1}\omega_{\ell_1/\ell}^{-1})$, $\Pi_3 = I(\chi\omega_{\ell_2/\ell}, \chi^{-1}\omega_{\ell_2/\ell}^{-1})$, $\Pi_4 = I(\chi\omega_{\ell_3/\ell}, \chi^{-1}\omega_{\ell_3/\ell}^{-1})$ of $GL_2(\ell)$.

From the exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Res}_{\ell/k}\, \mathbb{G}_m \longrightarrow \mathbf{T}_X \longrightarrow 1$$

we get

$$1 \longrightarrow \mu_2(k) \longrightarrow \ell^\times \xrightarrow{q} T_X \xrightarrow{\delta} H^1(k, \mu_2) \longrightarrow H^1(k, \mathrm{Res}_{\ell/k}\mathbb{G}_m)$$

Here

$$q : \ell^\times \longrightarrow T_X$$

is the quotient map. We also get

$$T_X/q(\ell^\times) \cong H^1(k, \mu_2) \cong k^\times/k^{\times 2}$$

Since k has odd residue characteristic, we have

$$k^\times/k^{\times 2} \cong \{1, \epsilon, \varpi, \epsilon\varpi\}$$

where $\epsilon \in \mathfrak{o}_k^\times$ is a nonsquare unit.

Let χ be an unramified character of $\ell^\times$, trivial on $\mu_2(k)$, the four characters of $T(k)$ restricting to the same χ on $q(\ell^\times)$ can be described as

$$\chi_\eta(t) = \chi(t)\eta(\delta(t)), \quad \eta \in \mathrm{Hom}(k^\times/k^{\times 2}, \mathbb{C}^\times)$$

here

$$\delta : \ T_X \longrightarrow k^\times/k^{\times 2}$$

is the connecting homomorphism. Denote η_ϵ, η_ϖ, $\eta_{\epsilon\varpi}$ the three nontrivial quadratic characters corresponding to the three nontrivial square classes ϵ, ϖ, $\epsilon\varpi$ in $k^\times/k^{\times 2}$. We denote the four extensions of χ to T_X by $\tilde{\chi}_i$, $i = 1, 2, 3, 4$.

We introduce the following four characters χ_i, $i = 1, 2, 3, 4$:

- $\chi_1(q(t)) = \chi^{-2}(t)$ for $t \in \ell^\times$ and $\chi_1(\xi_c) = \chi^{-1}(c)$ for $c \in k^\times/k^{\times 2}$.
- $\chi_2(q(t)) = \chi^{-2}(t)$ for $t \in \ell^\times$ and $\chi_2(\xi_c) = \chi^{-1}\omega_{\ell_1/\ell}(c)$ for $c \in k^\times/k^{\times 2}$.
- $\chi_3(q(t)) = \chi^{-2}(t)$ for $t \in \ell^\times$ and $\chi_3(\xi_c) = \chi^{-1}\omega_{\ell_2/\ell}(c)$ for $c \in k^\times/k^{\times 2}$.
- $\chi_4(q(t)) = \chi^{-2}(t)$ for $t \in \ell^\times$ and $\chi_4(\xi_c) = \chi^{-1}\omega_{\ell_3/\ell}(c)$ for $c \in k^\times/k^{\times 2}$.

We denote S_{χ_i} the intertwining functionals associated with χ_i.

Proposition 8.7. *The intertwining functionals S_{χ_i} satisfy the weak local functional equations,*

$$T_\omega \circ S_{\chi_i} \sim S_{\omega_{\chi_i}}$$

Proof. Denote the GL_2-invariant functionals associated with Π_1, Π_2, Π_3 by $\overline{L}_2$, $\overline{L}_3$, $\overline{L}_4$.

For the adjoint $\Lambda_i \in \mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C})$ of S_{χ_i}, from the Propositions 8.5 and 8.6 we have

$$\Lambda_i = \ell_i \in \mathrm{Hom}_{SL_2(k)}(\pi, \mathbb{C})$$

where ℓ_1 is the restriction of the GL_2-invariant functional L_1, ℓ_i are restrictions of $\overline{L}_i$ for $2 \leq i \leq 4$. Since L_1 and $\overline{L}_i,\ 2 \leq i \leq 4$ satisfy the weak local functional equations, we conclude ℓ_i satisfy the weak local functional equations, hence S_{χ_i} satisfy the weak local functional equations.

□

Declaration of generative AI use

The first draft of this paper is part of my thesis and it is purely human written. We used ChatGPT- 6 Astra and a verification framework OSVerify [NYL+26] with ChatGPT-6 Astra to polish the draft and improve writing. The author verified all the output. The author is responsible for the correctness of this paper.

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