ON LUNA'S CLASSIFICATION OF HOMOGENEOUS AFFINE SPHERICAL VARIETIES OVER p-ADIC FIELDS

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Date: November 2026.

1. Introduction

1.1. **Motivation.** Let k be an algebraically closed field of characteristic zero, and let G be a connected reductive group over k, we let B be a Borel subgroup of G. For a G-variety X, we say that X is a G-spherical variety if the B-action has an open orbit $\mathring{X}$ in X. If $X = H\backslash G$ is a G-spherical variety, we will say H is a spherical subgroup of G.

A fundamental starting of classification of spherical varieties is the study of wonderful varieties of lower rank. We recall the following classification result for rank-one wonderful varieties

Theorem 1.1. *Let G be a connected reductive group of adjoint type over k an algebraically closed field. Then there exists a bijection between the G-isomorphism classes of wonderful varieties of rank one to spherical systems of rank one for G.*

The classification of reductive spherical subgroups over algebraically closed fields was studied by Krämer [Krä79] and Brion [Bri87]. In particular, Brion classified all indecomposable spherical pairs (G, H), where G is semisimple and simply-connected, and H is connected and reductive. The classification consists of the spherical subgroups of simple groups appearing in Krämer's list, together with several additional families arising from products of simple groups (see Theorem 2.29). This classification provides an important starting point for our study of homogeneous affine spherical varieties over p-adic fields.

While Brion's classification gives an explicit description of connected reductive spherical subgroups in semisimple simplyconnected groups, Luna developed a more general combinatorial framework for the classification of homogeneous spherical varieties. In his framework, wonderful varieties are classified by spherical systems and the homogeneous spherical varieties are classified by augmentation of these systems.

Let's summarize Luna's classification of the homogeneous spherical varieties: To a G-wonderful variety X, we can associate to it a triple $\mathcal{S}_X = (S_X^p, \Sigma_X, \mathscr{A}_X)$, called the spherical system of X. Spherical systems are characterized by combinatorial axioms and there is a bijection between the G-spherical systems and G-isomorphism classes of wonderful G-varieties. If $H\backslash G$ is the open G-orbit of X, then H is called a *wonderful subgroup* of G. Moreover, every spherical subgroup H of G is contained in a wonderful subgroup, we denote it by $\overline{H}$, this is called the *spherical closure* of H. By introducing an augmentation of the spherical system, namely the homogeneous spherical datum $\mathscr{L}$, Luna obtained a bijection between the homogeneous spherical data and the isomorphism classes of homogeneous spherical varieties for groups of type A in [Lun01]. Later, Bravi and Pezzini completed Luna's classification for general reductive groups in [BP16] by introducing the classes of primitive wonderful varieties.

We can state Luna's classification theorem as follows

Theorem 1.2. *(cf. Theorem 2.38) For a connected reductive group G, the map φ*

$$X \mapsto (S_X^p, \Sigma_X, \mathscr{A}_X, \chi(X), \rho_X)$$

sending a homogeneous G-spherical variety X to the unique quintuple $(S_X^p, \Sigma_X, \mathscr{A}_X, \chi(X), \rho_X)$ is a bijection between the isomorphism classes of homogeneous G-spherical varieties and the homogeneous spherical data for G.

1.2. **Luna's classification over p-adic fields.** Having recalled Luna's classification of homogeneous affine spherical varieties over algebraically closed field, it is natural to ask whether an analogous classification can be established over non-algebraically closed field. In particular, motivated by question from harmonic analysis for spherical varieties, we are interested in the classification of homogeneous spherical varieties over p-adic fields. A fundamental difficulty is that Luna's homogeneous spherical data do not distinguish different k-forms of the same spherical varieties. Our

approach is to enrich Luna's homogeneous spherical data with suitable Galois characters, leading to the notion of extended homogeneous spherical datum.

In [Les24] Leslie introduced the *doubling automorphism group* $\mathrm{Aut}_d(X)$ of X. The finer arithmetic information of X is contained in a 1-cocycle $\mu_X^d \in H^1(k, \mathrm{Aut}_d(X))$. He also introduced the *outer data* out_X for X which comes from the 1-cocycle μ_X^d associated with X.

Motivated by Leslie's theory of geometric cocycles, we introduce an extended version of Luna's homogeneous spherical datum-extended homogeneous spherical datum and denote it by $\overline{\mathscr{L}}$, which can be viewed as a homogeneous spherical datum $\mathscr{L}$ together with a set of homomorphisms μ_X^{Gal} coming from the Galois action on colors of even sphere types.

We state the first main theorem of our paper which can be viewed as a k-version of the classification of wonderful varieties of rank one

Theorem 1.3. *(cf. Theorem 6.20) Let G be a connected quasisplit group of adjoint type over k a p-adic local field of characteristic zero. Then there exists a surjection from the G-isomorphism classes of wonderful varieties of split rank one to extended spherical systems with transitive Galois group action on the Luna diagram which is compatible with the Galois action on the Dynkin diagram for G.*

We can now state the second main theorem of our paper which can be viewed as a k-version of Brion's classification of reductive wonderful subgroups over algebraic closure

Theorem 1.4. *(cf. Theorem 7.24) Let G be a connected quasisplit group of adjoint type over k a characteristic zero field, then we have a bijection between the G-isomorphism classes of wonderful varieties X over k associated with reductive wonderful subgroups and the extended spherical systems of reductive type.*

For the injectivity property, we prove it based on the equality $\mathrm{out}_X = \mu_X^{\mathrm{Gal}}$. We can now state the main theorem of our paper which can be viewed as Luna's classification over k for homogeneous affine spherical varieties

Theorem 1.5. *(cf. Theorem 7.2) Let G be a quasisplit group over k a p-adic local field. Then there exists a surjection from the G-isomorphism classes of homogeneous affine spherical varieties over k to extended homogeneous spherical data associated with homogeneous spherical data of reductive type.*

1.3. **Future direction.** Motivated by Luna's classification (Theorem 1.2) over algebraic closure and our classification result over k a p-adic field (Theorem 7.2), we formulate the following conjecture

Conjecture 1.6. *Let G be a quasisplit group over k a p-adic local field, given $(\mathscr{L}, (\mu_i^{Gal})_{i\in I})$ an extended homogeneous spherical datum, then there exists a homogeneous G-spherical variety X over k with $(\mathscr{L}_X, (\mu_i^{Gal})_{i\in I_X}) = (\mathscr{L}, (\mu_i^{Gal})_{i\in I})$.*

2. Classification of spherical varieties over algebraic closure

2.1. **Spherical varieties.** In this section, we assume that k is an algebraically closed field of characteristic zero. We introduce a class of homogeneous varieties, namely spherical varieties and follow the treatment of [How95].

Let G be a connected reductive group over k, X a G-variety, in particular, X could be a vector space in which case X is a G-representation, we denote by $\mathcal{R}(X)$ the ring of regular functions on X which has a natural action of the group G.

Definition 2.1. We will say that the action of G on $\mathcal{R}(X)$ is *multiplicity-free* if the multiplicity of each irreducible representation of G is at most one

$$\dim \mathrm{Hom}_G(\sigma, \mathcal{R}(X)) \leq 1$$

for σ an irreducible representation of G.

Proposition 2.2. *The action of G on the affine variety X is multiplicity-free if and only if the Borel subgroup has an open dense orbit.*

Definition 2.3. For a G-variety X, we say X is a G-spherical variety if the B-action has an open orbit $\overset{\circ}{X}$ in X.

Affine varieties X with multiplicity-free G-action play an important role in the invariant theory of classical groups, we refer to [How95] for more details on this point of view.

Classifications of homogeneous affine spherical varieties can be found in [Bri87], classifications of multiplicity-free representations can be found in [Kno06].

2.2. **Wonderful varieties.** In this section, we review the theory of wonderful varieties following the presentation of [BP05, Section 2]. We assume that k is an algebraically closed field of characteristic zero, G is a connected reductive group over k, B is a fixed Borel subgroup of G, Σ_G the corresponding set of simple roots.

Definition 2.4. An algebraic G-variety X is *wonderful* of rank r if the following conditions hold:

(1) X is smooth and complete.
(2) G has a dense orbit in X whose complement is the union of r smooth prime divisors $D_i,\ i = 1, \cdots r$ with normal crossings.
(3) the intersection of the divisors D_i is nonempty and for any subset $I \subseteq \{1, \cdots, r\}$

$$(\cap_{i\in I} D_i) \backslash (\cup_{i\notin I} D_i)$$

is a G-orbit.

A subgroup H of G is *spherical* if $H\backslash G$ is a spherical G-variety, and it is *wonderful* if $H\backslash G$ admits an embedding which is a wonderful variety.

A wonderful G-variety is always projective and spherical. This implies that any B-proper rational function on X is uniquely determined (up to a scalar factor) by its B-weight. We denote the lattice of such weights by $\chi(X)$. We let $\Lambda_X = \mathrm{Hom}_{\mathbb{Z}}(\chi(X), \mathbb{Z})$ and $\mathcal{V}_X \subset \Lambda_X \otimes_{\mathbb{Z}} \mathbb{Q}$ be the cone generated by the images of G-invariant valuations, we call $\mathcal{V}_X$ the valuation cone of X.

Definition 2.5. Consider the cone negative-dual to $\mathcal{V}_X$: $\mathcal{V}_X^{\perp} = \{\chi \in \chi(X) \otimes \mathbb{R} \mid \langle \chi, v\rangle \leq 0 \text{ for every } v \in \mathcal{V}_X\}$. The generators of the intersection of its extremal rays with $\chi(X)$ are called the *spherical roots* of X, we denote this set by Σ_X

If a prime divisor of X is B-stable but not G-stable, then it is called a *color*. The set of all colors is denoted by Δ_X.

The set of spherical roots of all wonderful G-varieties of rank 1 is denoted by $\Sigma(G)$. For G of adjoint type the elements of $\Sigma(G)$ are always linear combinations of simple roots Σ_G with nonnegative integer coefficients.

Now let's recall some lemmas on colors: let X be a wonderful G-variety, let P_α be the standard parabolic subgroup associated to $\alpha \in \Sigma_G$. The set of colors of X is denoted by Δ_X and let $\Delta_X(\alpha)$ denote the set of non-P_α-stable colors. We say that α moves the colors in $\Delta_X(\alpha)$, that is P_α is not contained in the stabilizer of $D \in \Delta_X(\alpha)$.

Lemma 2.6. *([BP05, Lemma 2.3]) For any $\alpha \in \Sigma_G$, $\Delta_X(\alpha)$ has at most two elements and only one of the following four cases can appear:*

(1) *$\Delta_X(\alpha) = \emptyset$, which happens when the open Borel orbit $\mathring{X}$ is stable under P_α, and the set of all such α will be denoted by S_X^p.*
(2) *$\Delta_X(\alpha)$ has two elements, which happens exactly when $\alpha \in \Sigma_X$, the two colors in $\Delta_X(\alpha)$ will be denoted by D_α^+, D_α^- and we have*

$$\langle \rho(D_\alpha^+), \gamma\rangle + \langle \rho(D_\alpha^-), \gamma\rangle = \langle \alpha^\vee, \gamma\rangle$$

for every $\gamma \in \Sigma_X$. We will denote by $\mathscr{A}_X$ the union of all $\Delta_X(\alpha)$ for every $\alpha \in \Sigma_G \cap \Sigma_X$.
(3) *$\Delta_X(\alpha)$ has one element and $2\alpha \in \Sigma_X$, the color in $\Delta_X(\alpha)$ is denoted by D'_α and we have:*

$$\langle \rho(D'_\alpha), \gamma\rangle = \frac{1}{2}\langle \alpha^\vee, \gamma\rangle$$

(4) *The remaining case, i.e. $\Delta_X(\alpha)$ has one element but $2\alpha \notin \Sigma_X$. In this case, the color in $\Delta_X(\alpha)$ is denoted by D_α and*

$$\langle \rho(D_\alpha), \gamma\rangle = \langle \alpha^\vee, \gamma\rangle$$

for every $\gamma \in \Sigma_X$.

Lemma 2.7. *([BP05, Lemma 2.4]) For all $\alpha, \beta \in \Sigma_G$, the condition $\Delta_X(\alpha) \cap \Delta_X(\beta) \neq \emptyset$ occurs only in the following two cases:*

- *if $\alpha, \beta \in \Sigma_G \cap \Sigma_X$ then we can have $|\Delta_X(\alpha) \cup \Delta_X(\beta)| = 3$.*
- *if α and β are orthogonal and $\alpha + \beta$ (or $\frac{1}{2}(\alpha+\beta)$) belongs to Σ_X, then $D_\alpha = D_\beta$.*

The relations in these two lemmas come from the analysis of the rank one and rank two cases, they will appear in the next section as the axioms for *spherical systems*.

Definition 2.8. The spherical system $\mathcal{S}_X$ for a wonderful variety X consists of a triple $(S_X^p, \Sigma_X, \mathscr{A}_X)$, where S_X^p is the set of the simple roots moving no colors, Σ_X is the set of spherical roots, and $\mathscr{A}_X$ is the subset of colors defined in Lemma 2.6.

We define the rank $\mathrm{rk}(X)$ of a wonderful variety X to be $\mathrm{rk}_{\mathbb{Z}}\ \chi(X)$. Note that since Σ_X spans $\chi(X)$, we have $\mathrm{rk}(X) = \#\Sigma_X$.

2.3. **Spherical systems and homogeneous spherical data.** We want to axiomize the results of the classification of wonderful varieties of rank less than or equal to two and some geometric properties of colors studied by Luna (2.6, 2.7). I will follow the presentation of [BP16, Definition 1.2.2] on spherical systems.

Definition 2.9. Given a root datum $\mathcal{R} = (X^*(T), \Phi, X_*(T), \Phi^\vee)$ of a connected reductive algebraic group G with Borel subgroup B and T a maximal torus of B, and a set of positive roots Σ_G, we denote by $\Sigma(G)$ the spherical roots of all wonderful G-varieties of rank 1, a triple $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ such that $S^p \subseteq \Sigma_G$, $\Sigma \subset \Sigma(G)$, $\mathscr{A}$ is a finite set endowed with a map $\rho : \mathscr{A} \longrightarrow \chi^*$ for $\chi = \mathbb{Z}\Sigma \subset X^*(T)$, $\mathcal{S}$ will be called a *spherical system* if the following axioms are satisfied:

(A1) $\forall D \in \mathscr{A}$, $\rho(D)(\alpha) \leq 1$ for all $\alpha \in \Sigma$, equality holds only if $\alpha \in \Sigma_G \cap \Sigma$.

(A2) $\forall \alpha \in \Sigma_G \cap \Sigma$, $\mathscr{A}(\alpha) := \{D \in \mathscr{A} |\ \rho(D)(\alpha) = 1\} = \{D_\alpha^+,\ D_\alpha^-\}$, and $\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee$.
(A3) $\mathscr{A} = \bigcup_{\alpha \in \Sigma_G \cap \Sigma} \mathscr{A}(\alpha)$.
(Σ1) If $2\alpha \in \Sigma \cap 2\Sigma_G$, then $\frac{1}{2}\langle \alpha^\vee, \beta \rangle$ is a non-positive integer, $\forall \beta \in \Sigma \backslash \{2\alpha\}$, furthermore $\alpha \notin \chi$ and $\frac{1}{2}\langle \alpha^\vee, \beta \rangle$ is an integer for all $\beta \in \chi$.
(Σ2) If $\alpha, \beta \in \Sigma_G$ are orthogonal and $\alpha + \beta$ belongs to Σ or 2Σ, then $\langle \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle$, $\forall \gamma \in \chi$.
(S) For all $\alpha \in \Sigma$, there is a wonderful G-variety X of rank 1 with $S_X^p = S^p$, and $\Sigma_X = \{\alpha\}$.

The cardinality of Σ will be called the *rank* of the spherical system. The axiom of the spherical system come from the following lemma:

Lemma 2.10. *For every wonderful G-variety X the triple $(S_X^p, \Sigma_X, \mathscr{A}_X)$ is a spherical system.*

Let's sketch the proof for this lemma: axioms (A2), (A3) correspond to Lemma 2.6 (2), axiom (Σ1) corresponds to Lemma 2.6 (3), axiom (Σ2) corresponds to Lemma 2.7 (2) and axiom (S) follows from the definition of Σ_X and $\Sigma(G)$.

Lemma 2.11. *The map $X \mapsto (S_X^p, \Sigma_X, \mathscr{A}_X)$ is a bijection between rank one (resp. rank two) wonderful varieties (up to G-isomorphism) and rank one (resp. rank two) spherical systems.*

This lemma is a reformulation of the result of Wasserman [Was96].

Definition 2.12. A spherical system $\mathcal{S}$ is called *geometrically realizable* if it is of the form $\mathcal{S} = \mathcal{S}_X$ for a wonderful variety X.

Theorem 2.13. *([BP16, Theorem 1.2.3]) There is a bijection $X \mapsto (S_X^p, \Sigma_X, \mathscr{A}_X)$ from G-isomorphism classes of wonderful G-varieties to spherical G-systems.*

As we will see, the injectivity of the map is a uniqueness result for homogeneous spherical varieties proved by Losev (Theorem 2.30), and I will review Luna and Bravi-Pezzini's reduction steps in the next section.

Let $(S^p, \Sigma, \mathscr{A})$ be a spherical system, we consider the couple (χ', ρ') with

- χ' is a subgroup of $X^*(T)$ containing Σ.
- a map $\rho' :\ \mathscr{A} \to (\chi')^*$.

For X a homogeneous spherical variety, we can associate $(\chi(X), \rho_X)$ to it and we want to extend the axioms for spherical systems from wonderful varieties to an extended version of spherical systems associated with homogeneous spherical varieties, and this motivates the following definition of augmentation of spherical systems:

Definition 2.14. We say that (χ', ρ') is an augmentation of $(S^p, \Sigma, \mathscr{A})$ if it satisfies

- If we denote $\chi = \langle \Sigma \rangle$, under the natural map $(\chi')^* \to (\chi)^*$, ρ' is equal to ρ.
- For all $\alpha \in \Sigma \cap \Sigma_G$, $\mathscr{A}(\alpha) = \{D_\alpha^+,\ D_\alpha^-\}$, and $\langle \rho'(D_\alpha^+), \gamma \rangle + \langle \rho'(D_\alpha^-), \gamma \rangle = \langle \alpha^\vee, \gamma \rangle$ for $\gamma \in \chi'$.
- If $2\alpha \in \Sigma \cap 2\Sigma_G$, then $\alpha \notin \chi'$ and $\langle \alpha^\vee, \gamma \rangle$ is an even integer for all $\gamma \in \chi'$.
- If $\alpha + \beta \in \Sigma$ or $\frac{1}{2}(\alpha + \beta) \in \Sigma$, $\alpha,\ \beta$ are orthogonal, then $\langle \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle$ for $\gamma \in \chi'$.
- For all $\alpha \in S^p$, $\alpha^\vee$ annihilates χ'.

We say that $(S^p, \Sigma, \mathscr{A}, \chi', \rho')$ is an augmented spherical system, and an augmented spherical system such that every element of Σ is primitive in χ' will be called a *homogeneous spherical datum*, we denote it by $\mathscr{L}$.

We have the following proposition:

Proposition 2.15. *Given X a homogeneous spherical variety, then*

$$\mathscr{L}_X = (S_X^p, \Sigma_X, \mathscr{A}_X, \chi(X), \rho_X)$$

is a homogeneous spherical datum, we call this the homogeneous spherical datum associated with X.

The primitivity of Σ_X in $\chi(X)$ follows from our definition for the set of spherical roots.

Definition 2.16. We let H be a spherical subgroup for G and denote $X = H\backslash G$. We let

$$\mathscr{L}_X = (S_X^p, \Sigma_X, \mathcal{A}_X, \chi(X), \rho_X)$$

be the homogeneous spherical datum of $X = H\backslash G$. We denote $\mathcal{S}$ the spherical system $(S_X^p, \Sigma_X, \mathcal{A}_X)$ and we say that the spherical system of the spherical subgroup H is of type $\mathcal{S}$.

2.4. **Overview of classification of wonderful reductive subgroups.** In this section, we recall Luna's parabolic induction, Bravi and Pezzini's classification of reductive wonderful subgroups.

2.4.1. *Colors and quotients.* Let $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ be a spherical system for G, we will define its set of colors, such that when $\mathcal{S} = \mathcal{S}_X$ for X a wonderful G-variety, this will recover the set of colors Δ_X.

Definition 2.17. The set of colors of $\mathcal{S}$ is the finite set Δ obtained as disjoint union $\Delta = \Delta^a \cup \Delta^{2a} \cup \Delta^b$ where:

- $\Delta^a = \mathscr{A}$.
- $\Delta_{2a} = \{D_\alpha \mid \alpha \in \Sigma_G \cap \frac{1}{2}\Sigma\}$.
- $\Delta^b = \{D_\alpha \mid \alpha \in \Sigma_G\backslash(S^p \cup \Sigma \cup \frac{1}{2}\Sigma)\}/\sim$ where $D_\alpha \sim D_\beta$ if α and β are orthogonal and $\alpha + \beta \in \Sigma$ or $\frac{1}{2}(\alpha + \beta) \in \Sigma$.

For all $\alpha \in \Sigma_G$ the set of colors *moved by* α is:

$$\Delta(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in S^p \\ \mathscr{A}(\alpha) & \text{if } \alpha \in \Sigma \\ \{D_\alpha\} & \text{otherwise} \end{cases}$$

We extend the map ρ from $\mathscr{A}$ to Δ as

$$\langle \rho(D), \gamma \rangle = \begin{cases} \langle \rho(D), \gamma \rangle & \text{if } D \in \Delta^a \\ \frac{1}{2}\langle \alpha^\vee, \gamma \rangle & \text{if } D = D_\alpha \in \Delta^{2a} \\ \langle \alpha^\vee, \gamma \rangle & \text{if } D = D_\alpha \in \Delta^b \end{cases}$$

We are going to define quotients of spherical systems, the combinatorial counterpart of a certain class of morphisms between wonderful varieties.

Definition 2.18. A subset of colors $\Delta' \subset \Delta$ is distinguished if for all $D \in \Delta'$ there exists a positive integer a_D such that $\sum_{D \in \Delta'} a_D \langle \rho(D), \gamma \rangle \geq 0$ for all $\gamma \in \Sigma$.

We have the following definition from [BP16]:

Definition 2.19. If $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ is a spherical system and Δ' is a distinguished subset of Δ, we define the quotient of the spherical system $\mathcal{S}$ as: $\mathcal{S}/\Delta' = (S^p, \Sigma, \mathscr{A})/\Delta' = (S^p/\Delta', \Sigma/\Delta', \mathscr{A}/\Delta')$ where

- $S^p/\Delta' = \{\alpha \in \Sigma_G, \Delta(\alpha) \subset \Delta'\}$.
- Σ/Δ' is the basis of the monoid

$$\{\sigma \in \mathbb{N}\Sigma \mid \langle \rho(D), \sigma \rangle = 0 \text{ for all } D \in \Delta'\}$$

- We define $\mathscr{A}/\Delta' = \cup_{\alpha \in \Sigma_G \cap \Sigma/\Delta'} \mathscr{A}(\alpha)$, and we define $\rho/\Delta' : \mathscr{A}/\Delta' \to (\chi/\Delta')^*$ as a restriction of ρ to $\mathscr{A}/\Delta'$ under the natural map $\chi^* \to (\chi/\Delta')^*$.

The following proposition is a result of Bravi,

Proposition 2.20. *([Bra13, Theorem 3.1]) If $\Delta' \subset \Delta$ is distinguished then:*

- *the monoid $\{\sigma \in \mathbb{N}\Sigma \mid \langle \rho(D), \sigma \rangle = 0 \text{ for all } D \in \Delta'\}$ is free.*
- *setting $S^p/\Delta' = \{\alpha \mid \Delta(\alpha) \subset \Delta'\}$, then Σ/Δ' is equal to the basis of the above monoid and $\mathscr{A}/\Delta' = \cup_{\alpha \in \Sigma_G \cap \Sigma/\Delta'} \mathscr{A}(\alpha)$, the triple $(S^p/\Delta', \Sigma/\Delta', \mathscr{A}/\Delta')$ is a spherical G-system.*

Proposition 2.20 shows that $\mathcal{S}/\Delta'$ is indeed a spherical system.

Let $f : X \to Y$ be a surjective G-morphism with connected fibers between wonderful G-varieties, then the subset $\Delta_f := \{D \in \Delta_X \mid f(D) = Y\}$ is distinguished, and $\mathcal{S}_Y = \mathcal{S}_X/\Delta_f$. The following holds:

Proposition 2.21. *([Lun01, § 3.3]) Let X be a wonderful G-variety, then the map $f \mapsto \Delta_f$ induces a bijection between distinguished subsets of Δ_X and G-isomorphism classes of surjective G-morphisms with connected fibers from X onto another wonderful G-variety.*

Here, let X, Y_1, Y_2 be three wonderful varieties, we say two G-morphisms $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are G-isomorphic if there exists a G-morphism $\varphi : Y_1 \to Y_2$ such that $\varphi \circ f_1 = f_2$.

This proposition will be very helpful for us to study the Galois descent properties of morphisms between wonderful varieties.

2.4.2. *Parabolic induction.* In this section, we let k be an algebraically closed field of characteristic zero, we recall Luna's operation of parabolic induction. We denote G a connected reductive group over k, and B a Borel subgroup of G.

For all $\gamma = \sum n_\alpha \alpha \in \mathbb{N}\Sigma_G$, we set

$$\text{supp } \gamma = \{\alpha \in \Sigma_G \mid n_\alpha \neq 0\}$$

For all $\Sigma \subset \mathbb{N}\Sigma_G$, we set

$$\text{supp } \Sigma = \bigcup_{\gamma \in \Sigma} \text{supp } \gamma$$

Let's first discuss the parabolic induction over k, let P be a parabolic subgroup of G containing B_-, the opposite Borel subgroup. We assume $P = P_{-S'}$ for $S' \subset \Sigma_G$, here $P_{-S'}$ is the parabolic subgroup generated by B_- and U_α for $\alpha \in S'$, and $L = P_{-S'} \cap P_{S'}$.

Definition 2.22. We say a G-morphism $\phi : X \to G/P$ arises from a *parabolic induction* structure on X if the unipotent radical of P acts trivially on X' with $X' = \phi^{-1}(P/P)$ a wonderful variety for L, also X is canonically isomorphic to $G *_P X'$ which is the quotient variety of $G \times X'$ by the relation

$$p(g, x) = (gp^{-1}, p \cdot x) \; p \in P, \; g \in G, \; x \in X'$$

conversely, suppose X' is a wonderful variety for L, considered as a P-variety with trivial U_P action, then $G *_P X'$ is a G-wonderful variety and the morphism $\phi : X = G *_P X' \to G/P$ induces a parabolic induction structure on X.

Luna introduced the following type of spherical systems:

Definition 2.23. A spherical system $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ is called cuspidal if supp $\Sigma = \Sigma_G$.

We have the following proposition concerning the parabolic induction.

Proposition 2.24. *([Lun01, Proposition 3.4], [BP16, Proposition 2.1.4], [BP16, Proposition 2.1.5]) Let X be a wonderful G-variety, there exists a unique G-morphism $\phi : X \to G/G_{-S'}$ that induces a parabolic induction structure on X if and only if supp$(\Sigma) \cup S^p \subset S'$, and Δ_ϕ is the set of colors $\Delta(S') = \cup_{\alpha \in S'} \Delta(\alpha)$.*

The existence of the morphism Δ_ϕ follows from the Proposition 2.21.

The following definition reflects the geometric structure of fiber product,

Definition 2.25. Let $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ be a spherical system with set of colors Δ. Two distinguished sets of colors $\Delta_1,\ \Delta_2$ are said to decompose $\mathcal{S}$ if:

(1) $S^p/\Delta_1 \cap S^p/\Delta_2$ is equal to S^p.
(2) every connected component of $S^p/(\Delta_1 \cup \Delta_2)$ is contained in either S^p/Δ_1 or S^p/Δ_2.
(3) Σ is included in $\Sigma/\Delta_1 \cup \Sigma/\Delta_2$.

If $\mathcal{S}$ admits two non-empty such subsets of colors then $\mathcal{S}$ is called *decomposable*.

We have the following proposition mentioned in Luna's paper which is important for the reduction argument:

Proposition 2.26. *Let G be a connected reductive group over an algebraically closed k of characteristic 0, and let X be a G-wonderful variety with spherical system*

$$\mathcal{S}_X = (S^p_X, \Sigma_X, \mathscr{A}_X)$$

then X can not be obtained by parabolic induction from a proper parabolic subgroup of G if and only if

$$S^p_X \cup supp(\Sigma_X) = \Sigma_G$$

Proof. Set

$$S_0 = S^p_X \cup \mathrm{supp}(\Sigma_X)$$

If $S_0 \subsetneq \Sigma_G$, Proposition 2.24 gives an induction structure on X associated with the standard parabolic subgroup P_{-S_0}. Since $S_0 \subsetneq \Sigma_G$, this parabolic subgroup is proper.

Conversely, suppose X is obtained from the parabolic induction from a proper parabolic subgroup $P \subsetneq G$. After conjugating P, we may write $P = P_{-S'}$ for some $S' \subsetneq \Sigma_G$. By Proposition 2.24,

$$S^p_X \cup \mathrm{supp}(\Sigma_X) \subseteq S'$$

Consequently,

$$S^p_X \cup \mathrm{supp}(\Sigma_X) \neq \Sigma_G$$

this proves the equivalence.

□

2.4.3. *Classification of wonderful reductive subgroups.* In this section, k will be an algebraically closed field of characteristic zero. We will recall the classification of reductive spherical subgroups in semisimple simply-connected groups in Brion's paper [Bri87].

Definition 2.27. We say that a spherical pair (G, H) is decomposable when up to permutation of factors,

$$(G, H) \cong (G_1 \times G_2, H_1 \times H_2)$$

where $H_i \subseteq G_i$ is a spherical subgroup. Otherwise we say it is indecomposable.

We also introduce the following notion of principal spherical homogeneous space

Definition 2.28. ([Yak06, Definition 2]) Let k be an algebraically closed field of characteristic 0, let G be a connected semisimple algebraic group and let $H \subseteq G$ be a connected reductive spherical subgroup. Write

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

the the decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$ into simple ideals, and put $\mathfrak{h} = \mathrm{Lie}(H)$.

We say that $H\backslash G$ is *principal* if

$$\mathfrak{z}(\mathfrak{h}) = \bigoplus_{i=1}^{r} (\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{g}_i)$$

where $\mathfrak{z}(\mathfrak{h})$ denotes the center of $\mathfrak{h}$.

We recall Brion's main theorem below, retaining his terminology and adopting his convention of identifying isogenous algebraic group. Accordingly, the groups, product decompositions, and descriptions of subgroup embeddings in this statement are understood up to isogeny

Theorem 2.29. *([Bri87, Main theorem]) The following is the list of indecomposable principal spherical pairs (G, H), where G is semisimple and simply-connected and H is connected and reductive.*

(1) G is simple, and H occurs in Krämer's table [Krä79].

(2) H is simple,

$$G = H \times H,$$

and H is embedded diagonally in $H \times H$.

(3) $H \simeq \mathrm{SL}(2)$,

$$G = H \times H \times H,$$

and H is embedded diagonally.

(4) There exist reductive groups G', H', and K', together with an embedding

$$i\colon H' \times K' \hookrightarrow G',$$

such that

$$G = H' \times G', \qquad H = H' \times K',$$

and the embedding $H \hookrightarrow G$ is given by

$$H' \times K' \longrightarrow H' \times G', \qquad (u, v) \longmapsto \big(u, i(u, v)\big).$$

Moreover, (G', H', K') is one of the following triples:

G'	H'	K'
$\mathrm{Sp}(2n+2)$	$\mathrm{SL}(2)$	$\mathrm{Sp}(2n)$
$\mathrm{Sp}(2n+4)$	$\mathrm{Sp}(4)$	$\mathrm{Sp}(2n)$
$\mathrm{SO}(n+1)$	$\mathrm{SO}(n)$	$\{1\}$
$\mathrm{SO}(8)$	$\mathrm{Spin}(7)$	$\{1\}$.

(5)

$$G = \mathrm{Sp}(2m+2) \times \mathrm{Sp}(2n+2)$$

and

$$H = \mathrm{SL}(2) \times \mathrm{Sp}(2m) \times \mathrm{Sp}(2n),$$

where H is embedded in G by

$$(t, u, v) \longmapsto (t \oplus u,\ t \oplus v).$$

(6)

$$G = \mathrm{Sp}(4) \times \mathrm{Sp}(2m+2) \times \mathrm{Sp}(2n+2)$$

and

$$H = \mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{Sp}(2m) \times \mathrm{Sp}(2n),$$

where H is embedded in G by

$$(t, u, v, w) \longmapsto (t \oplus u,\ t \oplus v,\ u \oplus w).$$

(7)

$$G = \mathrm{Sp}(2\ell+2) \times \mathrm{Sp}(2m+2) \times \mathrm{Sp}(2n+2)$$

and

$$H = \mathrm{SL}(2) \times \mathrm{Sp}(2\ell) \times \mathrm{Sp}(2m) \times \mathrm{Sp}(2n),$$

where H is embedded in G by

$$(t, u, v, w) \longmapsto (t \oplus u,\ t \oplus v,\ t \oplus w).$$

(8)

$$G = \mathrm{SL}(n) \times \mathrm{SL}(n+1)$$

and

$$H = \mathrm{SL}(n) \times \mathbb{G}_m,$$

where H is embedded in G by

$$(u, \lambda) \longmapsto \left(u,\ (\lambda u) \oplus \lambda^{-n}\right).$$

(9)

$$G = \mathrm{SL}(m+2) \times \mathrm{Sp}(2n+2),$$

and either

$$H = \mathrm{SL}(2) \times \mathrm{SL}(m) \times \mathrm{Sp}(2n),$$

or

$$H = \mathrm{SL}(2) \times \mathrm{SL}(m) \times \mathrm{Sp}(2n) \times \mathbb{G}_m.$$

In the second case, the embedding $H \hookrightarrow G$ is

$$(u, v, w, \lambda) \longmapsto \left((\lambda^m u) \oplus (\lambda^{-2} v),\ u \oplus w\right).$$

The first case is obtained by omitting the $\mathbb{G}_m$-factor, equivalently by setting $\lambda = 1$. Here we require $m \geq 1,\ n \geq 0$ in the first case and $m \geq 3,\ n \geq 0$ in the second case (for the parameter range see [Avd10, Table 1 cases 3 and 4]).

In the paper [BP15, Section 3], Bravi and Pezzini provide the list of spherical systems for the indecomposable spherical pairs appearing in Brion's theorem. From now on we will refer to the list of wonderful reductive spherical subgroups in [BP15, Section 3] as BP's list.

2.5. **Uniqueness of homogeneous spherical varieties.** In this section, k will be an algebraically closed field of characteristic zero, G is a connected reductive group over k.

In this section we will recall Losev's result on uniqueness of the homogeneous spherical varieties and the calculations of various automorphism groups based on [BG22, Appendix C].

Losev proves the following uniqueness result for the homogeneous spherical varieties with given combinatorial invariants. One advantage of his approach is that he avoided the case-by-case considerations that were used in Luna's classification.

Given two spherical varieties $X_1,\ X_2$, denote their set of colors by Δ_{X_1} and Δ_{X_2}, we write $\Delta_{X_1} = \Delta_{X_2}$ if there is a bijection $\psi : \Delta_{X_1} \to \Delta_{X_2}$, such that $G_D = G_{\psi(D)}$, $\rho(D) = \rho(\psi(D))$, here $G_D = \{g \in G \mid\ gD = D\}$.

Theorem 2.30. *([Los09, Theorem 1]) Let H_1, H_2 be two spherical subgroups, $X_1 = H_1\backslash G,\ X_2 = H_2\backslash G$, if $\chi(X_1) = \chi(X_2),\ \Sigma_{X_1} = \Sigma_{X_2}$, $\Delta_{X_1} = \Delta_{X_2}$, then H_1, H_2 are G-conjugate.*

We will introduce the following types of spherical roots as they will play an important role in the calculation of various automorphism groups Proposition 2.32 and Proposition 2.36.

Definition 2.31. We define the following four types of spherical roots, $\Sigma_X^{(1)}, \Sigma_X^{(2)}, \Sigma_X^{(3)}, \Sigma_X^{(4)}$

(1) $\gamma \in \Sigma_G$ and there exists $D \in \Delta_X$ with $\rho(D) = \frac{1}{2}\gamma^\vee \in \chi(X)^*$.
(2) there is a subset $\Sigma = \{\alpha_1, \cdots, \alpha_n\} \subset \Sigma_G$ of type $B_n,\ n \geq 2$ such that

$$\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

and $\alpha_i \in \Sigma_X^p$ for $i > 1$.
(3) there is a subset $\{\alpha_1,\ \alpha_2\} \subset \Sigma_G$ of type G_2 with α_1 short root and $\gamma = 2\alpha_1 + \alpha_2$.

(4) $\gamma \in \Sigma_X$ but $\gamma \notin \mathbb{Z}\Phi$.

We will denote the union of type $(1), (2), (3)$ distinguished spherical roots by Σ_X^{dist}.

We will discuss some spherical varieties with type (1) and type (2) spherical roots in the future examples 3.11, 3.12 (there is also a similar description for spherical varieties with type (3) spherical roots). Type (4) spherical roots contribute the part of automorphism group arising from the center of G.

We introduce the following sets obtained from the spherical roots: Let $\mathscr{L} = (S^p, \Sigma, \mathscr{A}, \chi, \rho)$ be a homogeneous spherical datum, we call the roots of type $(1), (2), (3)$ the distinguished roots, and denote them by Σ^{dist}. We denote $\Sigma^{\#} \subset \chi$ the set of spherical roots obtained from Σ by replacing γ with 2γ for all $\gamma \in \Sigma^{(2)}, \Sigma^{(3)}, \Sigma^{(4)}$, $\Sigma^N \subset \chi$ the set of spherical roots obtained from Σ by replacing γ by 2γ for all $\gamma \in \Sigma^{(1)}, \Sigma^{(2)}, \Sigma^{(3)}, \Sigma^{(4)}$.

Following [BG22, Appendix B], we recall a combinatorial description of the automorphism group $\mathrm{Aut}_X^G := \mathrm{Aut}^G(H\backslash G)$ due to Knop and Losev. For any $a \in \mathrm{Aut}_X^G$ and $\lambda \in \chi(X)$, by definition of χ, there exists $f_\lambda \in k(H\backslash G)^{(B)}_\lambda$ of B semi-invariants in $k(H\backslash G)$, the automorphism a preserves the one-dimensional subspace, it acts on this one-dimensional space by multiplication by a scalar $d_{a,\lambda} \in k^\times$, we obtain a homomorphism

$$\begin{aligned} \iota : \mathrm{Aut}_X^G &\longrightarrow \mathrm{Hom}(\chi, k^\times) \\ a &\longmapsto (\lambda \mapsto d_{a,\lambda}) \end{aligned}$$

We denote T^χ the k-torus with character lattice χ, then the group $\mathrm{Hom}(\chi, k^\times)$ can be identified with the group $T^\chi(k)$ of k-points of T^χ.

Knop [Kno96, Theorem 5.5] showed that ι is injective and its image is closed in $T^\chi(k)$, hence $\mathrm{Aut}_X^G = \mathfrak{A}(k)$ for $\mathfrak{A}$ a group of multiplicative type and $X^*(\mathfrak{A}) = \chi/\Xi$, where Ξ is a sublattice of χ.

We have the following result:

Proposition 2.32. *([BG22, Proposition C4]) We have $X^*(\mathfrak{A}) = \chi/\langle\Sigma^N\rangle$.*

We will denote $\Lambda_X := \langle\Sigma^N\rangle$ the root lattice of X, then the result can be reformulated as $X^*(\mathfrak{A}) \cong \chi/\Lambda_X$.

For $\gamma \in \Sigma^{(1)}$, then we have $\gamma \notin \Sigma^N$ but $2\gamma \in \Sigma^N$, hence we define a homomorphism

$$\begin{aligned} \mathfrak{A}(k) = \mathrm{Hom}(\chi/\langle\Sigma^N\rangle, k^\times) &\longrightarrow \mathrm{Hom}(\langle\gamma\rangle/\langle 2\gamma\rangle, k^\times) = \{\pm 1\} \\ a &\longmapsto d_{a,\gamma} \end{aligned}$$

Proposition 2.33. *([BG22, Proposition C5]) For $\gamma \in \Sigma^{(1)}$, we denote $\Delta_X(\gamma) = \{D_\gamma^+, D_\gamma^-\}$, then an automorphism $a \in \mathfrak{A}(k) = Aut^G(H\backslash G)$ swaps D_γ^+ and D_γ^- if and only if $d_{a,\gamma} = -1$.*

Definition 2.34. For a spherical system $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$, we define its spherical closure as the spherical system $\overline{\mathcal{S}} = (S^p, \Sigma^{\#}, \mathscr{A})$. A spherical system $\mathcal{S}$ is called spherically closed if $\overline{\mathcal{S}} = \mathcal{S}$.

We denote the wonderful subgroups corresponding to the spherical systems $\mathcal{S}$, $\overline{\mathcal{S}}$ by H, $\overline{H}$, we will say that $\overline{H}$ is the spherical closure of H. More generally, since every homogeneous spherical datum is an augmentation of a spherical system, it can be shown that every spherical subgroup H is contained in a spherically closed subgroup $\overline{H}$, we will call $\overline{H}$ the spherical closure of H.

Definition 2.35. For H a spherical subgroup, we denote

$$\mathfrak{A}^{ker} = \overline{H}/H \subset N_G(H)/H$$

here $\overline{H}$ is the spherical closure of H.

The following proposition calculates the character lattice of $\mathfrak{A}^{ker}$:

Proposition 2.36. *[BG22, Theorem C7] We have* $X^*(\mathfrak{A}^{ker}) = \chi/\langle\Sigma^{\#}\rangle$.

By definition

$$\mathfrak{A}^{ker}(k) = \{a \in T^{\chi}(k) \mid a(\lambda) = 1 \text{ for all } \lambda \in \langle\Sigma^N\rangle \text{ and for all } \lambda \in \Sigma^{(1)}\}$$

then the proof follows from the calculation for the automorphism group (Proposition 2.32), the previous Proposition 2.33 on the automorphisms that switch two colors.

Corollary 2.37. *For G an adjoint group and $\overline{\mathcal{S}}$ a spherically closed spherical system for G, denote H the corresponding wonderful subgroup, then we have $\mathfrak{A}^{ker} = 1$.*

Proof. Since H is a wonderful subgroup, we have $\chi = \langle\Sigma\rangle$. We have $\langle\Sigma\rangle = \langle\Sigma^{\#}\rangle$ as $\overline{\mathcal{S}}$ is spherically closed, hence $X^*(\mathfrak{A}^{ker}) = \chi/\langle\Sigma^{\#}\rangle = 0$. □

2.6. **Classification of the spherical subgroups.** In this section, k will be an algebraically closed field of characteristic zero. We will summarize Luna's classification of the spherical subgroups.

Theorem 2.38. *For a connected reductive group G, the map φ*

$$X \mapsto (S_X^p, \Sigma_X, \mathscr{A}_X, \chi(X), \rho_X)$$

sending a homogeneous G-spherical variety X to the unique quintuple $(S_X^p, \Sigma_X, \mathscr{A}_X, \chi(X), \rho_X)$ is a bijection between the isomorphism classes of homogeneous G-spherical varieties and the homogeneous spherical data for G.

Proof. φ is well-defined by our definition of the axioms for homogeneous spherical datum. φ is injective by Losev's uniqueness result 2.30. φ is surjective, since the wonderful varieties are classified by spherical systems (Theorem 2.13), we can repeat Luna's argument for [Lun01, Theorem 2], and it is reduced to the case when the spherical systems are *spherically closed*. □

3. Forms of spherical varieties

In this section, we summarize the theory of forms for homogeneous spherical varieties, which is based on the papers [Wan18] and [Les24], [BG22].

3.1. **Inner forms of spherical varieties.** In this section k will be a characteristic zero field and G a quasisplit connected reductive group over k, X will be a homogeneous G-spherical variety over k. We recall the theory of G-forms as in part I of [Les24]. We denote $\Gamma = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k.

If $\mathrm{SAut}(G)$ denotes the set of semi-linear automorphisms of $G(\overline{k})$ (cf. [Bor20, Section 5.3]), the k-rational structure of G induces a morphism

$$\mu : \Gamma \longrightarrow \mathrm{SAut}(G)$$

which splits the sequence

$$1 \longrightarrow \mathrm{Aut}(G)(\overline{k}) \longrightarrow \mathrm{SAut}(G) \longrightarrow \Gamma$$

As explained in [Bor20], this induces a Γ-action on $G(\overline{k})$, which we refer to as μ_σ for $\sigma \in \Gamma$.

Since X is a G-variety over k, the k-rational structure induces morphisms

$$\mu : \Gamma \longrightarrow \mathrm{SAut}(G), \qquad \mu_X : \Gamma \longrightarrow \mathrm{SAut}^G(X)$$

here $\mathrm{SAut}^G(X)$ denotes the group of μ-equivariant semi-linear automorphisms of X as defined in [BG22, Section 2.10]. That is, for each $\sigma \in \Gamma$

$$\mu_{X,\sigma}(g \cdot x) = \mu_\sigma(g) \cdot \mu_{X,\sigma}(x)$$

The k-rational structure of X induces an exact sequence

$$1 \longrightarrow \mathrm{Aut}^G(X)(\overline{k}) \longrightarrow \mathrm{SAut}^G(X) \longrightarrow \Gamma$$

which μ_X splits.

Let X be a homogeneous spherical variety over k with $X(k) \neq \emptyset$, we choose $x_0 \in X(k)$, which gives a k-isomorphism $X \cong H\backslash G$, then we have $\mathrm{Aut}^G(X) \cong H\backslash N_G(H)$. Having fixed x_0, since the normalizer $N_G(H)$ of H has a natural map to $\mathrm{Aut}(H)$ via conjugation, we get a map $N_G(H) \to \mathrm{Aut}(H)$, composing with $\mathrm{Aut}(H) \to \mathrm{Out}(H)$, we obtain a map

$$\mathrm{Aut}^G(X) \longrightarrow \mathrm{Out}(H)$$

this map is independent of the choice of $x_0 \in X(k)$, for other $x \in X(k)$ with $H' = \mathrm{Stab}_G(x)$, there exists $x_0 = x \cdot g^{-1}$ so that $H' = gHg^{-1}$, then $\mathrm{Ad}(g)$ induces an isomorphism $\mathrm{Aut}(H) \cong \mathrm{Aut}(H')$, we get the following commutative diagram

$$\begin{array}{ccc} H\backslash N_G(H) & \xrightarrow{\mathrm{Ad}(g)} & H'\backslash N_G(H') \\ \downarrow & & \downarrow \\ \mathrm{Out}(H) & \xrightarrow{\mathrm{Ad}(g)} & \mathrm{Out}(H') \end{array}$$

Let $\mathrm{Out}_X(H)$ be its image, and we define $\mathrm{Out}(X) = \lim_x \mathrm{Out}_X(H_x)$, then we have

$$\mathrm{Out}: \ \mathrm{Aut}^G(X) \longrightarrow \mathrm{Out}(X)$$

Since Out is independent of the choice of base point, denote by $\mathcal{A}_X^b \subset \mathrm{Aut}^G(X)$ the kernel of this map, both $\mathcal{A}_X^b$ and $\mathrm{Aut}^G(X)$ are defined over k.

Suppose X' is another G-variety such that there exists a $G_{\overline{k}}$-equivariant isomorphism

$$\psi: \ X_{\overline{k}} \cong X'_{\overline{k}}$$

Then for each $\sigma \in \Gamma$, we have

$$\psi(\mu_{X,\sigma}(g \cdot x)) = \mu_\sigma(g) \cdot \psi(\mu_{X,\sigma}(x))$$

We say that X' is a G-form of X. If $X'(k) \neq \emptyset$, then $X' \cong H'\backslash G$ for some group $H' \subset G$.

Fix a G-form X' and an isomorphism ψ as above, for any $\sigma \in \Gamma$, the automorphism $\psi^{-1} \circ {}^\sigma\psi : X_{\overline{k}} \longrightarrow X_{\overline{k}}$ is $G_{\overline{k}}$-equivariant, here ${}^\sigma\psi(x) := \tilde{\mu}'_\sigma \circ \psi(\mu_{\sigma^{-1}}(x))$, where $\tilde{\mu}'$ denotes the corresponding semi-linear automorphism of X'. We obtain a cocycle

$$c_{X,X'} := [\sigma \longrightarrow \psi^{-1} \circ {}^\sigma\psi] \in Z^1(k, \mathrm{Aut}^G(X))$$

it can be checked that different choices of ψ correspond to twisting the cocycle by a coboundary, hence we obtain a cohomology class $[c_{X,X'}] \in H^1(k, \mathrm{Aut}^G(X))$ which depends only on X'.

Definition 3.1. Suppose we have a spherical pair (G, X) with $X = H\backslash G$ and (G, X') with $X' = H'\backslash G$ is a k-form of (G, X). We say X' is a G-inner form of X there exists a $G_{\overline{k}}$-equivariant isomorphism

$$\psi: \ X_{\overline{k}} \cong X'_{\overline{k}}$$

such that the associated $c_\sigma^\psi := \psi \circ {}^\sigma\psi \in Z^1(k, \mathrm{Aut}^G(X))$ takes values in $\mathcal{A}_X^b(\overline{k})$. A G-form of X which is not a G-inner form is called a G-outer form of $X = H\backslash G$.

We include a discussion on the k-structure of automorphism group here. Let $\mathscr{G}$ be a reductive group over $\overline{k}$ and $\mathscr{X} = \mathscr{H}\backslash\mathscr{G}$ is a homogeneous spherical $\mathscr{G}$-variety, we let G be a k-form of $\mathscr{G}$ and X a k-form of $\mathscr{X}$.

There is a canonical inclusion $\mathrm{Aut}^{\mathscr{G}}(\mathscr{X}) \hookrightarrow A_{\mathscr{X}}$, for $A_{\mathscr{X}}$ the maximal torus of $\mathscr{X}$ as defined in [BG22, Appendix B], and we have the following result on the k-structure of the automorphism group determined by the quasisplit form:

Proposition 3.2. *([Les24, Proposition 3.8]) Suppose G is a quasisplit form of $\mathscr{G}$ and X is a G-model of $\mathscr{X}$. Then the model X induces a Γ-action on $Aut^{\mathscr{G}}(\mathscr{X})$ such that the embedding*

$$Aut^{\mathscr{G}}(\mathscr{X}) \hookrightarrow A_{\mathscr{X}} \tag{3.1}$$

is Γ-equivariant. $\mathcal{A}_X := Aut^G(X)$ is a k-form of $Aut^{\mathscr{G}}(\mathscr{X})$, the embedding $\mathcal{A}_X \hookrightarrow A_X$ descends from (3.1), here A_X is a k-form of $A_{\mathscr{X}}$ induced from the k-structure on X.

3.2. **Existence of equivariant models.** In this section we summarize the result in [BG22] concerning the existence of equivariant models of spherical varieties. We assume k is an algebraically closed field of characteristic zero and we fix k, G will be a connected reductive group over k, B a Borel subgroup of G with unipotent radical U, T a maximal torus of B and X will be an affine G-variety.

We let X be a homogeneous G-spherical variety over k. Let Δ_X be the set of B-stable prime divisors of X, we let G_D be the stabilizer of D in G. We have a valuation map $\rho_X : \Delta_X \to \chi(X)^*$. For $D \in \Delta_X$, we let $\Sigma(D)$ be the set of $\alpha \in \Sigma_G$ such that $P_\alpha \nsubseteq G_D$, we obtain a map

$$\zeta_X : \Delta_X \longrightarrow \mathscr{P}(\Sigma_G) \quad D \mapsto \Sigma(D)$$

for $\mathscr{P}(\Sigma_G)$ the power set of Σ_G.

We consider the map

$$\rho_X \times \zeta_X : \Delta_X \longrightarrow \chi(X)^* \times \mathscr{P}(\Sigma_G) \tag{3.2}$$

we denote its image by Ω. The fibers of this map have either one or two colors. Let $\Omega^{(1)}, \Omega^{(2)}$ denote the corresponding subsets determined by the size of the fiber.

Theorem 3.3. *We let G be a connected reductive group over k. We denote $\Omega_X = (\chi(X), \Sigma_X, \Omega^{(1)}, \Omega^{(2)})$, suppose $X_1 = H_1\backslash G$, $X_2 = H_2\backslash G$ are two homogeneous spherical varieties over k and $\Omega_{X_1} = \Omega_{X_2}$, then X_1 and X_2 are G-isomorphic.*

This is just a reformulation of Losev's uniqueness result 2.30. From now on, we assume k is a characteristic zero field, to study the existence of the k-forms, we first note the following result:

Proposition 3.4. *Suppose that G is a quasisplit connected reductive group over k and $X = H\backslash G$ is a spherical homogeneous k-variety, then the induced Γ-action on $X^*(T)$ and Σ_G preserves Ω_X.*

When G is quasisplit, Borovoi and Gagliardi prove the following result as an inverse to the previous result 3.4, when $\sigma\Omega = \Omega$, we may choose a continuous lift

$$\alpha_\Delta : \Gamma \times \Delta_X \longrightarrow \Delta_X$$

Theorem 3.5. *([BG22, Theorem 6.13, Theorem 6.15]) Suppose G is quasisplit and $\mathscr{X} = \mathscr{H}\backslash G_{\overline{k}}$ is a spherical homogeneous space of $G_{\overline{k}}$. If the Γ-action induced from G preserves Ω_X, then for any lift α_Δ of the Γ-action from Ω to Δ_X, there exists a G-equivariant k-model X of $\mathscr{X}$ inducing α_Δ and we have $X(k) \neq \emptyset$.*

The following example shows that we need to consider different lifts of α_Δ.

Example 3.6. In this example, we let $k = \mathbb{Q}_p$, for p an odd prime. For $X = \mathbb{G}_m\backslash PGL_2$ over $\overline{k}$, we have $\Delta(X) = \{D_1, D_2\}$, so there are four lifts of Γ-action from Ω to $\mathscr{D}(X)$ corresponding to k and the three quadratic extensions of k. The trivial Γ-action corresponds to the k-form $\mathbb{G}_m\backslash PGL_2$ over k, and the non-trivial Γ-action corresponding to the quadratic extension ℓ is given by the k-form $U_{1,\ell/k}\backslash PGL_2$ of X.

3.3. **Group of doubling automorphisms.** In this section, X will be a G-spherical variety over k. Following Leslie, we introduce the group of doubling automorphisms, which encodes the arithmetic information for a spherical variety X over k.

We begin with the connected case: Let $X = H\backslash G$ be a homogeneous spherical variety, $\pi_0(H_{\overline{k}}) = 1$.

We consider the set Σ_X^d obtained from the normalized spherical roots Σ_X^N by replacing 2α with α for every $\alpha \in \Sigma_X^{dist}$, and set $\Lambda_X^d := \mathbb{Z}\Sigma_X^d$. This is Γ-stable, so we obtain a short exact sequence of Γ-modules

$$0 \longrightarrow \Lambda_X^d/\Lambda_X \longrightarrow \chi/\Lambda_X \longrightarrow \chi/\Lambda_X^d \longrightarrow 0$$

and a dual sequence of k-groups of multiplicative type

$$1 \longrightarrow \mathcal{A}_X^d \longrightarrow \mathcal{A}_X \longrightarrow \mathrm{Aut}_d(X) \longrightarrow 1$$

here $\mathcal{A}_X$ is defined in Proposition 3.2 and we have the isomorphism

$$X^*((\mathcal{A}_X)_{\overline{k}}) \cong \chi/\Lambda_X$$

from Proposition 2.32.

Definition 3.7. We will call $\mathrm{Aut}_d(X)$ the group of doubling automorphisms.

The doubling automorphism group records the sign actions associated with distinguished spherical roots, and its Galois cohomology detects the different k-forms.

To motivate this definition, for any $\gamma \in X^*(T_X)$, we set $\langle \gamma, a\rangle \in \overline{k}^\times$ for $a \in T_X(\overline{k})$.

Definition 3.8. For $\alpha \in \Sigma_X^{dist}$, an automorphism $a \in \mathcal{A}_X(\overline{k})$ is said to double α if $\langle \alpha, a\rangle = -1$.

Each $\alpha \in \Sigma_X^{dist}$ determines a canonical character

$$\mu_\alpha = \langle \alpha, -\rangle : \ \mathcal{A}_X(\overline{k}) \longrightarrow \mu_2(\overline{k})$$

in particular

$$\prod_\alpha \mu_\alpha : \ \mathrm{Aut}_d(X)_{\overline{k}} \cong \prod_\alpha \ \mu_2$$

We have the following lemma:

Lemma 3.9. *([Sak21, Lemma 2.2.4]) Let k be an algebraically closed field of characteristic zero and X a homogeneous affine spherical of rank one. Let $Z \subset X$ be a closed B-orbit, then one of the following two holds:*

- *Z is of rank zero and for every simple root α such that $Y := ZP_\alpha \neq Z$, we have $Y/\mathcal{R}(P_\alpha) \cong T\backslash PGL_2$ or $N(T)\backslash PGL_2$.*
- *Or Z is of rank one and for simple roots α we have $ZP_\alpha = Z$, except for two orthogonal simple roots α, β, we can set $Y := ZP_{\alpha\beta}$, and $Y/\mathcal{R}(P_{\alpha\beta}) \cong PGL_2^{diag}\backslash PGL_2^2$.*

Definition 3.10. A pair (Y, P) as in the Lemma 3.9 where $P = P_\alpha$ in the first case and $P = P_{\alpha\beta}$ in the second case will be called a basic orbit-parabolic pair.

We give some examples of the doubling automorphism groups.

Example 3.11. For $X = \mathbb{G}_m\backslash PGL_2$, X has a type (1) spherical root, and we can calculate $\mathrm{Aut}_d(X) \cong \mathbb{Z}/2\mathbb{Z}$, it acts by permuting the set of colors $\Delta(X) = \{D_1,\ D_2\}$.

Example 3.12. For $X = SO_{2n}\backslash SO_{2n+1}$, X has a type (2) spherical root, and we can calculate $\mathrm{Aut}_d(X) \cong \mathbb{Z}/2\mathbb{Z}$, we let (Y, P_α) be a basic orbit-parabolic pair for X (Definition 3.10), then we have the following isomorphism

$$\mathrm{Aut}_d(X) \cong \mathrm{Aut}^{L_\alpha}(Y/U_{P_\alpha}) \cong \mathrm{Aut}_d(\mathbb{G}_m\backslash PGL_2) \cong \mathbb{Z}/2\mathbb{Z}$$

the doubling automorphism of X acts by permutation on the colors of $Y/\mathcal{R}(P_\alpha) \cong \mathbb{G}_m\backslash PGL_2$.

We now discuss the k-structure on $\mathrm{Aut}_d(X)$. The k-structure on $\mathcal{A}_X$ induces a k-form on the quotient. Set I_X to be the set of Γ-orbits for Σ_X^{dist}. Then we have the orbit decomposition

$$\Sigma_X^{dist} = \bigsqcup_{i \in I_X} \mathcal{O}_i = \bigsqcup_{i \in I_X} \Gamma/\Gamma_i \cdot \gamma_i$$

where $\gamma_i \in \mathcal{O}_i$ and $\Gamma_i = \mathrm{Gal}(\overline{k}/k_i)$. The orbit structure determines the Γ-action on

$$\Lambda_X^d/\Lambda_X \cong \mathbb{Z}\Sigma_X^{dist}/2\mathbb{Z}\Sigma_X^{dist}$$

More precisely

$$X^*(\mathrm{Aut}_d(X)) \cong (\mathbb{Z}/2\mathbb{Z})^{\Sigma_X^{dist}}$$

this induces a product formula

$$\mathrm{Aut}_d(X) = \prod_{i \in I_X} \mathrm{Res}_{k_i/k}(\mu_2)$$

We have the following lemma concerning the k-structure on $\mathrm{Aut}_d(X)$, we note the k-structure on $\mathrm{Aut}_d(X)$ is independent of the k-form X.

Lemma 3.13. *([Les24, Lemma 3.13]) Suppose G is a quasisplit group and $\mathscr{X} = \mathscr{H}\backslash G_{\overline{k}}$ is a spherical homogeneous space with $\mathscr{H}$ connected. Assuming the natural Γ-action determined by G preserves Ω_X, then the k-structure on $Aut_d(X)$ is independent of the choice of the G-equivariant k-model X of $\mathscr{X}$.*

This is because the Γ-structure on $\mathrm{Aut}_d(X)$ comes from the Γ-action on Σ_X^d.

Lemma 3.14. *([Les24, Lemma 3.14]) Suppose $X = H\backslash G$ is spherical and H is geometrically connected, in the long exact sequence on cohomology, the map*

$$H^1(k, \mathcal{A}_X) \longrightarrow H^1(k, Aut_d(X))$$

is surjective.

Via the long exact sequence in cohomology, this lemma follows from the injectivity of $i : H^2(k, \mathcal{A}_X^d) \to H^2(k, \mathcal{A}_X)$, and the proof is reduced to the fact that Λ_X and Λ_X^d have Γ-stable $\mathbb{Z}$-bases.

The following definition extends the doubling automorphism group to the general case:

Definition 3.15. We define the group of doubling automorphisms of $X = H\backslash G$ as $\mathrm{Aut}_d(X) := \mathrm{Aut}_d(X^\circ)/\pi_0(H)$, here $X^\circ = H^\circ\backslash G$.

where we abuse the notation to mean that we quotient out by the image of $\pi_0(H)$ under the morphism to $\mathrm{Aut}_d(X^\circ)$.

With this definition, we have the following commutative diagram for $\mathrm{Aut}_d(X)$ with exact rows

$$\begin{array}{ccccc} \pi_0(H) & \longrightarrow & \mathcal{A}_{X^\circ} & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Aut}_d(\pi_0(H)) & \longrightarrow & \mathrm{Aut}_d(X^\circ) & \longrightarrow & \mathrm{Aut}_d(X) \end{array}$$

from this we deduce that $\mathrm{Aut}_d(X^\circ) \cong \mathrm{Aut}_d(X)$ if $H^\circ \subset H \subset H^\circ \cdot Z(G)$.

Leslie introduced the following conjecture motivated by the study of G-inner forms:

Conjecture 3.16. *([Les24, Conjecture 3.17]) Suppose G is a quasisplit reductive k-group and X is a homogeneous spherical variety then the map*

$$H^1(k, \mathcal{A}_X) \longrightarrow H^1(k, Aut_d(X))$$

is surjective.

4. On the L-equivalence classes over k

In this section, we will assume k is a characteristic zero field, and introduce the notions of extended homogeneous spherical datum and L-equivalence classes, finally I will study the existence and uniqueness result over k following Luna's approach and applying the tools developed by Leslie.

4.1. **Galois characters.** In this section, k will be a p-adic local field of characteristic 0.

We will associate Galois characters to homogeneous k-spherical varieties X whose cuspidal spherical systems are of type a-A-1, b(n)-B, g-G, these are the only rank-one spherical varieties can have nontrivial doubling automorphism group according to our Definition 3.7.

a-A-1: $G = \mathrm{PGL}_2$, B the upper triangular Borel subgroup and A the maximal diagonal torus.

From the previous section, we have the spherical system $\mathcal{S} = (\emptyset, \{\alpha\}, \{D_1,\ D_2\})$ and it has a geometric realization $X_{\overline{k}} = A\backslash G$. Note $\Sigma^{\#} = \Sigma$, so $\mathcal{S}$ is spherically closed.

For a homogeneous spherical datum $\mathscr{L} = (S^p, \Sigma, \mathscr{A}, \chi, \rho)$ with spherical closure $\mathcal{S}$. In this case, $\mathscr{L}$ is unique as $\mathbb{Z}\Sigma = X^*(A)$, so there is a unique augmentation of the spherical system.

For the split form PGL_2 of G, the Γ-action preserves the homogeneous spherical data $\mathscr{L}$, we associate the following Galois characters to the k-forms of $\mathbb{G}_m\backslash\mathrm{PGL}_2$.

- the spherical variety $X_k = \mathbb{G}_m\backslash\mathrm{PGL}_2$ corresponds to the trivial Galois character $\Gamma \to \{\pm 1\}$.
- the spherical variety $X_{\ell/k} = \mathrm{U}_1\backslash\mathrm{PGL}_2$, its Galois character corresponds to the quadratic character of Γ determined by ℓ via local class field theory.

b(n)-B: Let k be a p-adic field of characteristic 0, and let (V, q) be a quadratic space of dimension n, we put

$$x \cdot y = \frac{1}{2}(q(x+y) - q(x) - q(y))$$

for $x, y \in V$.

Let $(e_i)_{1\le i\le n}$ be a basis of V, the matrix of q with respect to this basis is the matrix $A = (a_{ij})$ where $a_{ij} = e_i \cdot e_j$. If we change the basis (e_i) by an invertible matrix X, then the new matrix A' of q with respect to the new basis is $X \cdot A \cdot {}^tX$ where ${}^tX$ denotes the transpose of X. In particular

$$\det(A') = \det(A) \cdot \det(X)^2$$

which shows that $\det(A)$ is determined up to to multiplication by an element of k^*. We call $\mathrm{disc}(q) := \det(A) \in k^*/k^{*2}$ the discriminant of q.

We choose $e = (e_1, \cdots, e_n)$ an orthogonal basis of V (the existence follows from [Ser12, §1.4 Theorem 1]). If we put $a_i = e_i \cdot e_i$, then $\mathrm{disc}(q) = a_1 \cdots a_n$. We define the normalized discriminant $d(V)$ of (V, q) as

$$d(V) = (-1)^{n(n-1)/2} a_1 \cdots a_n \in k^\times/k^{\times 2}$$

For a, b elements of k^*, we define its Hilbert symbol as $(a, b) \in \pm 1$, we put

$$\epsilon(e) = \prod_{i<j} (a_i, a_j)$$

then $\epsilon(e) = \pm 1$, and Serre proves that $\epsilon(e)$ does not depend on the choice of the orthogonal basis e in [Ser12, §2.1 Theorem 5]. It is a theorem ([Ser12, §2.3 Theorem 7]) of Serre that two quadratic spaces over k are equivalent if and only if they have the same rank, same discriminant and same invariant e.

We denote $\mathbb{H} = \langle 1, -1\rangle$ the standard hyperbolic space, and put $V_{2n+1} = \mathbb{H}^n \perp \langle 1\rangle$, then $G = SO(V_{2n+1})$ is the split orthogonal group. For $d \in k^\times/k^{\times 2}$, choose a hyperbolic basis e, f of $\mathbb{H}$, normalized so that

$$q(xe + yf) = xy$$

and set $v_d = e + df$, then $q(v_d) = d$, the orthogonal complement of v_d is $V_{2n,d} = v_d^\perp \cong \mathbb{H}^{n-1} \perp \langle 1, -d\rangle$. Every isometry h of $SO(v_d^\perp)$ extends to an isometry of V_{2n+1}, hence we get an embedding

$$SO(V_{2n,d}) \hookrightarrow SO(V_{2n+1})$$

the splitting field of $SO(V_{2n,d})$ is $\ell = k(\sqrt{d})$.

Denote $G = \mathrm{SO}_{2n+1} = SO(V_{2n+1})$ the split orthogonal group, B its upper triangular Borel subgroup, $A \subset B$ the maximal torus. We denote by $\alpha_1, \cdots, \alpha_{n-1}$ the long roots and α_n the short root. From previous section we have the spherical system $\mathcal{S} = (\{\alpha_2, \cdots, \alpha_n\}, \{\alpha_1 + \cdots + \alpha_n\}, \emptyset)$ and it has a geometric realization $X_{\overline{k}} = \mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}$. $\mathcal{S}$ is not spherically closed and it has a spherical closure $\mathcal{S}^\# = (\{\alpha_2, \cdots, \alpha_n\}, \{2(\alpha_1 + \cdots + \alpha_n)\}, \emptyset)$. We denote $\mathscr{L}$ the homogeneous spherical datum of $X_{\overline{k}}$.

For the split form $G_k = \mathrm{SO}_{2n+1}$ of G, the Γ-action preserves the homogeneous spherical data, we associate the following Galois characters to the k-forms of $\mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}$.

- b(n)-B: the spherical variety $X_k = \mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}$ corresponds to the trivial Galois character where SO_{2n} is the unique split form of SO_{2n}.
- b(n)-B-ℓ: the spherical variety $X_{\ell/k} = \mathrm{SO}_{2n,\ell}\backslash\mathrm{SO}_{2n+1}$, its Galois character corresponds to the quadratic character of Γ determined by ℓ, here $\mathrm{SO}_{2n,\ell}$ the special orthogonal group $SO(V_{2n,d})$ with $d(V) = d$ and $\ell = k(\sqrt{d})$.

g-G: We first provide a discussion on the isomorphism classes of Hermitian spaces. Let ℓ/k be a quadratic extension of local fields, from the discussion in [Mın11], we have the following low-dimensional Hermitian spaces:

For $n = 1$, let $a \in k^\times$, we define the Hermitian space $\ell(a)$ as the ℓ-vector space of dimension 1 with the Hermitian form h defined via

$$h(e, e') = \overline{e}ae'$$

For $n = 2$, the hyperbolic plane H over ℓ is the ℓ-vector space of dimension 2 with the product

$$h((e_1, e_2), (e_1', e_2')) = \overline{e_1}e_2' + \overline{e}_2e_1'$$

For $n = 2$, we have the anisotropic Hermitian space $W_2(a_1, a_2)$ which is a ℓ-vector space of dimension 2, $W_2(a_1, a_2) = \ell(a_1) \perp \ell(a_2)$ with a_1, a_2 not equal to zero and $-\frac{a_1}{a_2} \notin N_{\ell/k}(\ell^\times)$. All the anisotropic Hermitian spaces of dimension 2 are isometric.

For each n there are exactly two different classes of n-dimensional spaces over ℓ:

- for $n = 2m + 1$ odd, let $V^\pm \cong mH \perp W^\pm$ where $W \cong \ell(a)$ depending on $a \in N_{\ell/k}(\ell^\times)$ or not.
- for $n = 2m$ even, let $V^+ \cong mH$ and $V^- \cong (m-1)H \perp W_2^-$ where W^- is an anisotropic space of dimension 2.

If n is odd, then $\mathrm{U}(V^+)$ is isomorphic to $\mathrm{U}(V^-)$ and it is always a quasisplit group. If n is even, then $\mathrm{U}(V^+)$ is not isomorphic to $\mathrm{U}(V^-)$, and $\mathrm{U}(V^+)$ is quasisplit and $\mathrm{U}(V^-)$ is not. For the application below we take $n = 3$, to each quadratic extension ℓ/k, we can associate a unique quasisplit unitary group $\mathrm{SU}(V^+) \cong \mathrm{SU}(V^-)$.

Let $G = \mathrm{G}_2$, B the Borel subgroup, A the maximal torus, α_1, α_2 the two simple roots. From the previous section we have the spherical system $\mathcal{S} = (\{\alpha_2\}, \{2\alpha_1 + \alpha_2\}, \emptyset)$ and it has a geometric realization $X_{\overline{k}} = \mathrm{SL}_3\backslash\mathrm{G}_2$. $\mathcal{S}$ is not spherically closed and it has a spherical closure $\mathcal{S}^\# = (\{\alpha_2\}, \{4\alpha_1 + 2\alpha_2\}, \emptyset)$. We denote $\mathscr{L}$ the homogeneous spherical datum of $X_{\overline{k}}$.

For the split form $G_k = \mathrm{G}_2$ of G, the Γ-action preserves the homogeneous spherical data, we can associate the following Galois characters to the k-forms of $\mathrm{SL}_3\backslash\mathrm{G}_2$.

- the spherical variety $X_k = \mathrm{SL}_3\backslash\mathrm{G}_2$ corresponds to the trivial Galois character $\Gamma \to \{\pm 1\}$

- the spherical variety $X_{\ell/k} = \mathrm{SU}_{3,\ell}\backslash \mathrm{G}_2$, here SU_3 is the non-split unitary group associated with the extension ℓ/k, and its Galois character corresponds to the quadratic character of Γ determined by ℓ.

Based on the results of this section, we will make a definition of the L-equivalence classes in the next section for spherical varieties for quasisplit groups.

4.2. **On the L-equivalence classes.** In this section we will assume k is a characteristic zero p-adic local field.

Given Luna's theory of classification of spherical varieties by homogeneous spherical data and Losev's uniqueness result, it is natural to consider the existence and uniqueness result for homogeneous spherical varieties over k.

Let G be a connected reductive group over k, recall that there is a bijection between homogeneous spherical data $\mathscr{L} = (S^p, \Sigma, \mathscr{A}, \chi, \rho)$ and the isomorphism classes of homogeneous G-spherical varieties. Let $\mathscr{L}_X = (S^p_X, \Sigma_X, \mathscr{A}, \chi, \rho)$ be a homogeneous spherical datum associated with a spherical variety X and Δ_X its set of colors, we have a map (3.2)

$$\rho \times \zeta : \Delta_X \longrightarrow \chi^* \times \mathscr{P}(\Sigma_G) \tag{4.1}$$

here for $D \in \Delta_X$, we define $\zeta(D)$ to be the set $\alpha \in \Sigma_G$ for which P_α is not contained in $\mathrm{stab}_G(D)$.

Definition 4.1. Let $\mathscr{G}$ be a quasisplit group over $\overline{k}$, and G a quasisplit form of $\mathscr{G}$ over k, B a Borel subgroup over k, T a maximal torus of G, let $\mathscr{L} = (S^p, \Sigma, \mathscr{A}, \chi, \rho)$ be a homogeneous spherical datum associated with a spherical variety X over $\overline{k}$, we say that the Γ-action preserves $\mathscr{L}$ if the induced Γ -action preserves (S^p, Σ, χ) viewed as subsets of $X^*(T)$, and also the image of $(\rho \times \zeta)(\Delta) \subseteq \chi^* \times \mathscr{P}(\Sigma_G)$ we defined in (4.1). We say that $\mathscr{L} = (S^p, \Sigma, \mathscr{A}, \chi, \rho)$ is a k-homogeneous spherical datum for G if the induced Γ-action preserves (S^p, Σ, χ) and also the image of $(\rho \times \zeta)(\Delta) \subseteq \chi^* \times \mathscr{P}(\Sigma_G)$. We say that a spherical system $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ is a k-spherical system if the induced Γ-action preserves the homogeneous spherical datum $\mathscr{L}_{\mathcal{S}}$ obtained from $\mathcal{S}$.

The advantage of this definition is that combining with Luna's classification of the spherical subgroups over $\overline{k}$, we have a very simple criterion for the existence of the k-equivariant models. We can reformulate the existence criterion of Borovoi-Gagliardi (Theorem 3.5) as follows:

Proposition 4.2. *Let $\mathscr{G}$ be a connected reductive group over $\overline{k}$ and G a quasisplit form of $\mathscr{G}$ over k, let $\mathscr{X} = \mathscr{H}\backslash\mathscr{G}$ be a homogeneous spherical variety over $\overline{k}$ and $\mathscr{L}_{\mathscr{X}}$ its homogeneous spherical data, then there exists a k-equivariant model X of $\mathscr{X}$ with $X(k) \neq \emptyset$ if the Γ-action induced from G preserves $\mathscr{L}_{\mathscr{X}}$.*

For the purpose of application to representation theory, we introduce the following concept of L-equivalence classes among homogeneous spherical varieties, here L stands for the L-value.

Proposition 4.3. *Let k be an algebraically closed field, let H be a spherical subgroup of an adjoint group G, and let H_w be a wonderful subgroup such that H and H_w have the same spherical closure, then we have*

$$X^*(H_w/H) \cong X^*(T_{H\backslash G})/\langle \Sigma_{H\backslash G}\rangle$$

Proof. Denote $\overline{H}$ the common spherical closure of H and H_w. We have

$$X^*(\overline{H}/H) = X^*(T_{H\backslash G})/\langle \Sigma^{\#}\rangle$$

here $\Sigma^{\#}$ is the set of spherical roots of $\overline{H}\backslash G$. Similarly,

$$X^*(\overline{H}/H_w) = X^*(T_{H_w\backslash G})/\langle \Sigma^{\#}\rangle = \langle \Sigma_{H_w\backslash G}\rangle/\langle \Sigma^{\#}\rangle = \langle \Sigma_{H\backslash G}\rangle/\langle \Sigma^{\#}\rangle$$

As a consequence

$$X^*(H_w/H) = X^*(\overline{H}/H)/X^*(\overline{H}/H_w) = X^*(T_{H\backslash G})/\langle \Sigma_{H\backslash G}\rangle$$

□

Corollary 4.4. *Let k be a field of characterisitc zero, let H be a k-spherical subgroup of a quasisplit adjoint group G, then we can find a k-spherical subgroup H_w of G such that $(H_w)_{\overline{k}}$ is a wonderful subgroup of $G_{\overline{k}}$, also H and H_w have the same spherical system.*

Proof. We denote $\mathscr{L} = (S^p, \Sigma, \mathscr{A}, \chi, \rho)$ be the homogeneous spherical datum of $H\backslash G$ and $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ be the spherical system of $H\backslash G$. We denote $\overline{\mathcal{S}}$ the spherical closure of $\mathcal{S}$ and $\overline{\mathscr{H}}$ the spherical closure of $H_{\overline{k}}$. Since $\overline{\mathcal{S}}$ is Γ-stable, we can use the existence Criterion 3.5 and conclude $\overline{\mathscr{H}} = (\overline{H})_{\overline{k}}$ for $\overline{H}$ a k-spherical subgroup of G. Using the theory of augmentation of spherical subgroups, we can find $\mathscr{H}_w$ a wonderful subgroup of $G_{\overline{k}}$ with spherical system $\mathcal{S}$ such that

$$H_{\overline{k}} \subseteq \mathscr{H}_w \subseteq (\overline{H})_{\overline{k}}$$

Then from the previous calculation, we have a surjective map

$$X^*((\overline{H})_{\overline{k}}/H_{\overline{k}}) \longrightarrow X^*(\mathscr{H}_w/H_{\overline{k}})$$

Note this map is Γ-equivariant, we deduce that there exists a k-diagonlizable group $A \subseteq \overline{H}/H$ and $A_{\overline{k}} = \mathscr{H}_w/H_{\overline{k}}$.

We consider the quotient map

$$p:\ \overline{H} \longrightarrow \overline{H}/H$$

we introduce $H_w := p^{-1}(A)$, then $(H_w)_{\overline{k}} = \mathscr{H}_w$, also H and H_w have the same spherical system.

□

Definition 4.5. Let G be a quasisplit adjoint group defined over a p-adic local field k of characteristic zero. Let H be a k-spherical subgroup of G such that the spherical system of $H\backslash G$ is of type a-A-1, b(n)-B or g-G. Denote H_w the k-spherical subgroup associated with H as in Corollary 4.4 and denote $X_w = H_w\backslash G$.

We introduce a Galois character for $H\backslash G$ as

$$\mu^{Gal} := \mu^{Gal}_{X_w}:\ \Gamma \longrightarrow \{\pm 1\}$$

Here $\mu^{Gal}_{X_w}$ the Galois character for X_w is defined in Section 4.1.

Definition 4.6. Let G be a quasisplit group of adjoint type over k, for every homogeneous G-spherical variety X over k, we denote I_X the set of Γ-orbits of distinguished spherical root of type $(1), (2), (3)$. Let $\mathscr{O}_i \in I_X$ and $\gamma_i \in \mathscr{O}_i$, denote Γ_i the stabilizer of γ_i and $k_i = \overline{k}^{\Gamma_i}$.

We will canonically associate to X a set of Galois characters in $\mathrm{Hom}(\Gamma_i, \mu_2)$:

$$\mu^{Gal}_i:\ \Gamma_i \longrightarrow \{\pm 1\}$$

for every $i \in I_X$.

This can also be identified with

$$\mu^{Gal}_i:\ \Gamma^{ab}_i \longrightarrow \{\pm 1\},\ i \in I_X$$

here $\Gamma^{ab}_i = \Gamma_i/\overline{[\Gamma_i, \Gamma_i]}$ is the topological abelianzation of Γ_i.

The construction is as follows: For $\gamma_i \in \mathscr{O}_i$, we denote $P_i = P_{\mathrm{supp}(\gamma_i)}$ and let $R(P_i)$ be its radical, note that P_i is defined over k_i. We introduce

$$G_i := P_i/R(P_i), \quad X_i := (\mathring{X})_{k_i} P_{\gamma_i}/R(P_{\gamma_i})$$

we have G_i is a quasisplit adjoint group and X_i is a G_i-spherical variety defined over k_i with a spherical system of type a-A-1, b(n)-B or g-G.

$\mu^{Gal}_i:\ \Gamma^{ab}_i \to \mu_2$ is independent of the choice of γ_i: if $\gamma'_i = \sigma(\gamma_i)$, then

$$\Gamma'_i = \sigma\Gamma_i\sigma^{-1}, \quad X_{\gamma'_i} \cong {}^{\sigma}X_{\gamma_i}$$

and

$$\mu_i'^{Gal}(\sigma\tau\sigma^{-1}) = \mu_i^{Gal}(\tau)$$

we deduce $\mu_i'^{Gal} = \mu_i^{Gal} \in \mathrm{Hom}(\Gamma_i^{ab}, \mu_2)$.

For every homogeneous spherical variety X, we define

$$\mu_X^{Gal} := (\mu_i^{Gal})_{i\in I_X} \in \prod_{i\in I_X} \mathrm{Hom}(\Gamma_i, \mu_2) = \prod_{i\in I_X} \mathrm{Hom}(\Gamma_i^{ab}, \mu_2)$$

here μ_i^{Gal} is the Galois character associated with X_i as defined in Definition 4.5.

Definition 4.7. For G a connected quasisplit group with $G^{ad} = G/Z(G)$, and $X = H\backslash G$ a homogeneous G-spherical variety, we denote $X^{ad} = H^{ad}\backslash G^{ad}$ with H^{ad} the image of H in G^{ad}, we define the Galois characters associated with X as

$$\mu_X^{Gal} = (\mu_{X^{ad},i}^{Gal})_{i\in I_X}$$

We make the following definition of extended homogeneous spherical datum:

Definition 4.8. Let G be a quasisplit connected reductive group over k, and $\mathscr{L}$ a k-homogeneous spherical datum for G, then there is a Γ-action on Σ the set of spherical roots induced from the Γ-action of G on its simple roots, we denote I the set of Γ-orbits of distinguished spherical roots of type $(1), (2), (3)$, we define the extended homogeneous spherical datum $\overline{\mathscr{L}}$ with respect to $\mathscr{L}$ as $(\mathscr{L}, (\mu_i^{Gal})_{i\in I})$, here $\mu_i^{Gal} \in \mathrm{Hom}(\Gamma_i, \mu_2) = \mathrm{Hom}(\Gamma_i^{ab}, \mu_2)$ for Γ_i the stabilizer of the Γ-action on γ_i a representative from the Γ-orbit indexed by $i \in I$.

We make the following definition of extended spherical system:

Definition 4.9. Let G be a quasisplit connected group over k and $\mathcal{S}$ a k-spherical system for G, we denote I the set of Γ-orbits of distinguished spherical roots of type $(1), (2), (3)$, we define the extended spherical system $\overline{\mathcal{S}}$ with respect to $\mathcal{S}$ as $(\mathcal{S}, (\mu_i^{Gal})_{i\in I})$, here $\mu_i^{Gal} \in \mathrm{Hom}(\Gamma_i, \mu_2(\overline{k}))$ for Γ_i the stabilizer of the Γ-action on γ_i a representative from the Γ-orbit indexed by $i \in I$.

Now we can state the following conjecture as k-version of Luna's classification of the wonderful varieties:

Conjecture 4.10. *Let G be an adjoint quasisplit group over k, there is a bijection between the the isomorphism classes of G-wonderful varieties over k and the extended spherical systems of G.*

We will prove this conjecture for wonderful varieties associated with a reductive wonderful subgroup using the classification result of [BP14] for reductive wonderful subgroups over algebraic closure.

Definition 4.11. Given two homogeneous spherical varieties X_1 and X_2 for the quasisplit group G, we say that X_1 and X_2 are in the same L-equivalence class, if $\mathscr{L}_{X_1} = \mathscr{L}_{X_2}$ and $(\mathscr{L}_{X_1}, (\mu_{i,1}^{Gal})_{i\in I_{X_1}}) = (\mathscr{L}_{X_2}, (\mu_{i,2}^{Gal})_{i\in I_{X_2}})$.

Conjecture 4.12. *Let G be a quasisplit group over k, given $(\mathscr{L}, (\mu_i^{Gal})_{i\in I})$ an extended homogeneous spherical datum, then there exists a homogeneous G-spherical variety X over k with $(\mathscr{L}_X, (\mu_i^{Gal})_{i\in I_X}) = (\mathscr{L}, (\mu_i^{Gal})_{i\in I})$.*

We prove this conjecture for homogeneous affine spherical varieties for quasisplit groups in the Section 7, it would be interesting to extend Luna's various operations over k and try to prove this conjecture for homogeneous spherical varieties.

5. Spherical varieties over a p-adic local field

In this section, we recall the construction of the k-root system for spherical varieties for a quasisplit group $\mathbf{G}$ from the paper [KK16].

5.1. **k-local structure theorem.** In this section, let k be a characteristic zero field and $\mathbf{G}$ is a connected quasisplit group over k, we will fix $\mathbf{B}$ a k-Borel subgroup of $\mathbf{G}$.

The parabolic subgroup associated to $\mathbf{X}$ is the standard parabolic subgroup

$$\mathbf{P}(\mathbf{X}) := \{g \in \mathbf{G} \mid \mathring{\mathbf{X}} \cdot g = \mathring{\mathbf{X}}\}$$

Let us choose a point $x_0 \in \mathring{\mathbf{X}}(k)$ and let $\mathbf{H}$ denote its stabilizer, so $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ and $\mathbf{HB}$ is open. There is a choice of the Levi subgroup $\mathbf{L}(\mathbf{X})$ of $\mathbf{P}(\mathbf{X})$ such that $\mathbf{L}(\mathbf{X}) \cap \mathbf{H}$ contains the derived subgroup of $\mathbf{L}(\mathbf{X})$.

We recall the definition of the maximal torus of $\mathbf{X}$ as in [SV17, Section 2.1].

Definition 5.1. We fix a maximal torus $\mathbf{T}$ in $\mathbf{B} \cap \mathbf{L}(\mathbf{X})$, we define $\mathbf{T_X}$ to be the torus $\mathbf{L}(\mathbf{X})/\mathbf{L}(\mathbf{X}) \cap \mathbf{H} := \mathbf{T}/\mathbf{T} \cap \mathbf{H}$. We identify $\mathbf{T_X}$ with a subvariety of $\mathring{\mathbf{X}}$ via the orbit map $a \mapsto x_0 \cdot a$.

We can think of $\mathbf{T_X}$ as a canonical abstract torus associated to $\mathbf{X}$, more precisely for every Borel subgroup $\mathbf{B}$, we may think of $\mathbf{T_X}$ as the quotient by which $\mathbf{B}$ acts on $\mathring{\mathbf{X}}/\mathbf{U}$.

We will introduce the maximal split torus of $\mathbf{X}$.

Definition 5.2. Given the maximal torus $\mathbf{T_X} = \mathbf{T}/\mathbf{T} \cap \mathbf{H}$, we define the maximal split torus $\mathbf{A_X}$ of $\mathbf{X}$ to be the quotient of $\mathbf{A}/\mathbf{A} \cap \mathbf{H}$, which can be viewed as the quotient of $\mathbf{A}$ by which $\mathbf{A}$ acts on $\mathring{\mathbf{X}}/\mathbf{UM}$, here $\mathbf{M}$ is the maximal anisotropic torus of $\mathbf{T}$.

Remark 5.3. Using the k-local structure theorem for $\mathbf{X}$, we can identify $\mathbf{X}_{el}$ in [KK16, Corollary 4.6] with $\mathbf{T_X}$, it can be checked that our definition is compatible with the definition of $\mathbf{A_X}$ in [KK16, Page 17].

We also introduce the anisotropic kernel of $\mathbf{X}$.

Definition 5.4. ([KK16, Definition 4.7]) The quotient $\mathbf{M_X} := \mathbf{T_X} /\!\!/ \mathbf{A}$ is called the *anisotropic kernel of* $\mathbf{X}$.

From now on we will assume that k is a p-adic local field of characteristic zero.

Definition 5.5. For a finitely generated abelian group M, define

$$D_{\overline{k}}(M) = \text{Spec } \overline{k}[M]$$

It represents the functor

$$D_{\overline{k}}(M)(R) = \text{Hom}_{\mathbb{Z}}(M, R^{\times})$$

on $\overline{k}$-algebras R.

A group scheme D over $\overline{k}$ is called diagonalizable over $\overline{k}$, or $\overline{k}$-diagonalizable, if

$$D \cong D_{\overline{k}}(M)$$

for some finitely generated abelian group M.

We say a k-group D is of multiplicative type if its base change to $\overline{k}$ is diagonalizable:

$$D_{\overline{k}} \cong D_{\overline{k}}(M)$$

for some finite abelian group M.

Definition 5.6. For a group D of multiplicative type, we say D is anisotropic if it contains no nontrivial k-split subtorus. Equivalently

$$\mathrm{Hom}_k(\mathbb{G}_m, T) = X_*(T)^\Gamma = 0$$

Over a p-adic local field k, since $k^\times$ is noncompact, we have D is anisotropic if and only if $D(k)$ is compact.

Proposition 5.7. *Let $1 \longrightarrow D_1 \longrightarrow D_2 \longrightarrow D_3 \longrightarrow 1$ be an exact sequence of groups of multiplicative type over k, suppose D_i split over K a finite extension of k, denote $\Gamma = Gal(K/k)$, then there is a natural long exact sequence:*

$$\begin{aligned} 1 \to X_*(D_1)^\Gamma \to X_*(D_2)^\Gamma \to X_*(D_3)^\Gamma \\ \to \pi_0(D_1^\Gamma) \to \pi_0(D_2^\Gamma) \to \pi_0(D_3^\Gamma) \to \cdots \end{aligned}$$

Proof. Denote $M_i = X^*(D_i)$, then we have the short exact sequence

$$0 \longrightarrow M_3 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow 0$$

Apply the functor $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(-, \mathbb{Z})$, and note

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(X^*(D), \mathbb{Z}) \cong X_*(D)^\Gamma,\ \mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(X^*(D), \mathbb{Z}) = \pi_0(D^\Gamma)$$

We get

$$\begin{aligned} 1 \to X_*(D_1)^\Gamma \to X_*(D_2)^\Gamma \to X_*(D_3)^\Gamma \\ \to \pi_0(D_1^\Gamma) \to \pi_0(D_2^\Gamma) \to \pi_0(D_3^\Gamma) \to \cdots \end{aligned}$$

□

Proposition 5.8. *We have $\boldsymbol{A_X}$ is a maximal split torus of $\boldsymbol{T_X}$.*

Proof. Note that we have the following exact sequences associated to $\mathbf{X}$ from the construction of $\mathbf{A_X}$ and $\mathbf{T_X}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{A} & \longrightarrow & \mathbf{T} & \longrightarrow & \mathbf{M} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{A_X} & \longrightarrow & \mathbf{T_X} & \longrightarrow & \mathbf{M_X} & \longrightarrow & 1 \end{array}$$

since $\mathbf{M}$ is an anisotropic torus and $\mathbf{M} \to \mathbf{M_X}$ is surjective, from the exact sequence

$$1 \longrightarrow \ker \longrightarrow \mathbf{M} \longrightarrow \mathbf{M_X} \longrightarrow 1$$

We deduce $\mathbf{M}_X$ is anisotropic.

From the Proposition 5.7 we get

$$1 \longrightarrow X_*(A_X)^\Gamma \longrightarrow X^*(T_X)^\Gamma \longrightarrow X_*(M_X)^\Gamma \longrightarrow \cdots$$

We deduce $X_*(A_X) = X_*(T_X)^\Gamma$ as $X_*(M_X)^\Gamma = 0$. From $X_*(A_X) = X_*(T_X)^\Gamma$, we deduce $\mathbf{A_X}$ is a maximal split torus of $\mathbf{T_X}$. □

By construction we have the torus $\mathbf{A_X}$ is the image of $\mathbf{A} \subset \mathbf{T}$ in $\mathbf{T_X}$ under the quotient map

$$\mathbf{T} \longrightarrow \mathbf{T_X}$$

We introduce the k-character lattice $\chi_k(X) := X^*(A_X)$ and denote $\mathfrak{t}_X = \chi_{\overline{k}}(X) \otimes \mathbb{Q}$ and $\mathfrak{a}_X = \chi_k(X)^* \otimes \mathbb{Q} = \mathrm{Hom}(\chi_k(X), \mathbb{Q})$.

For X a $\overline{k}$-variety, we consider a particular type of valuations of $\overline{k}(X)$ of $\overline{k}$-valued rational functions on X. More precisely, a valuation is a map $v :\ \overline{k}(X) \to \mathbb{Q} \cup \{\infty\}$ satisfying

- $v(f_1 f_2) = v(f_1) + v(f_2)$.
- $v(f_1 + f_2) \geq \min\{v(f_1),\ v(f_2)\}$.
- $v(f) = \infty \Leftrightarrow f = 0$.
- $v(\overline{k}^*) = 0$.

where f, f_1, f_2 run through all elements of $\overline{k}(X)$.

We will introduce the notion of k-invariant valuation.

Definition 5.9. An invariant valuation of a k-dense G-variety X is called k-central if it is trivial on the subfield $\overline{k}(X)^{AU}$. The set of k-central valuations is denoted by $\mathscr{V}_k(X)$.

A k-central valuation induces a homomorphism

$$\lambda_v :\ \chi_k(X) \to \mathbb{Q} :\ \chi_f \mapsto v(f)$$

and we get a map $\iota_k : \mathscr{V}_k(X) \to \mathfrak{a}_X$.

We may view $\mathfrak{a}_X$ as a subspace of $\mathfrak{t}_X$.

Proposition 5.10. *([KK16, Proposition 5.2]) Let X be a k-dense G-spherical variety, then:*

- *The map ι_k is injective.*
- *Considering $\mathscr{V}_k(X), \mathscr{V}_X$ as subsets of $\mathfrak{a}_X$ and $\mathfrak{t}_X$, then $\mathfrak{a}_X = \mathscr{V}_X \cap \mathfrak{a}$.*

From Proposition 5.10, we see that $\mathscr{V}_k(X)$ is a finitely generated convex cone in $\mathfrak{a}_X$. We denote $\mathfrak{a}^-$ the antidominant Weyl chamber for $\mathbf{G}$ with respect to the restricted root system.

Proposition 5.11. *([KK16, Proposition 5.4]) Let $\pi : \mathfrak{a} \to \mathfrak{a}_X$ be the canonical projection, then $\pi(\mathfrak{a}^-) \subseteq \mathscr{V}_k(X)$.*

5.2. **The Weyl group.** In this section, we will assume $\mathbf{X}$ is a k-dense $\mathbf{G}$-spherical variety over k, and we choose the associated parabolic subgroup $\mathbf{P}(\mathbf{X})$ over k. We will associate a k-Weyl group to X.

From [KK16, Lemma 9.3] there is a Borel subgroup $\mathbf{B}$ and a point $x \in \mathbf{X}(k)$ such that

(5.1)
$$\begin{array}{ccccc} \mathbf{A} & \subseteq & \mathbf{T} & \subseteq & \mathbf{L}(\mathbf{X}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{A_X} & \subseteq & \mathbf{T_X} & \xrightarrow{\cong} & \mathbf{T}x \end{array}$$

The strategy to define the k-Weyl group for $\mathbf{X}$ is to study the valuation cone $\mathscr{V}_k(X)$, which over algebraic closure we know it is the Weyl chamber for the little Weyl group.

Definition 5.12. We define

$$W_k(X) := N_{W_{\overline{k}}(X)}(\mathfrak{a}_X)/C_{W_{\overline{k}}(X)}(\mathfrak{a}_X)$$

Theorem 5.13. *([KK16, Theorem 9.5]) Let X be a k-dense G-spherical variety, then $\mathscr{V}_k(X)$ is a fundamental domain for the action of $W_k(X)$ on $\mathfrak{a}_X$.*

5.3. **The root system over k.** In this section, we recall the theory of k-root systems, just like the group case, we want to study the k-root system for spherical varieties and the Galois action on spherical root system over $\overline{k}$. We will construct an integral root system for $W_k(X)$- the k-little Weyl group, as in the case over algebraic closure, we face an issue of normalization. We will fix X a k-dense G-spherical variety with G a connected quasisplit reductive group.

Definition 5.14. A primitive k-spherical root of X is a primitive element $\sigma \in \chi_k$ such that $\mathscr{V}_k(X) \cap \{\sigma \geq 0\}$ is a facet of $\mathscr{V}_k(X)$. Here

$$\{\sigma \geq 0\} := \{\nu \in \mathfrak{a}_X \mid \langle \nu, \sigma \rangle \geq 0\}$$

We will denote the set of primitive k-spherical roots by $\Sigma_k^{pr} := \Sigma_k^{pr}(X)$, it is in one-to-one correspondence with the facets of $\mathscr{V}_k(X)$.

The *primitive k-root system of X* is

$$\Phi_k^{pr} = \Phi_k^{pr}(X) := W_k \cdot \Sigma_k^{pr} \subseteq \chi_k$$

Definition 5.15. For G a semisimple group, we define the split rank of X to be $\#\ \Sigma_k^{pr}$.

We have the following description of $\mathscr{V}_k(X)$ based on the primitive k-spherical roots:

Proposition 5.16. *([KK16, Corollary 9.7]) $\mathscr{V}_k(X) = \{a \in \mathfrak{a}_k |\ \sigma(a) \leq 0,\ \text{for all}\ \ \sigma \in \Sigma_k^{pr}\}$.*

Lemma 5.17. *([KK16, Lemma 10.2]) Every $\sigma \in \Sigma_k^{pr}$ is a linear combination $\sigma = \sum_{\alpha \in \Sigma_k} c_\alpha \alpha$ with $c_\alpha \in \mathbb{Q}_{\geq 0}$.*

The set of $\alpha \in \Sigma_k$ with $c_\alpha > 0$ is called the *support* of σ.

Proposition 5.18. *([KK16, Proposition 10.3]) Φ_k^{pr} is a reduced root system with Weyl group $W_k(X)$, the set Σ_k^{pr} is a set of simple roots for Φ_k^{pr}, the valuation cone $\mathscr{V}_k(X)$ is an antidominant Weyl chamber with respect to Σ_k^{pr}. The lattice χ_k is a weight lattice for Φ_k^{pr}.*

Here a weight lattice for a root system with Weyl group W is a lattice χ containing Φ with $(1 - s_\sigma)\chi \subseteq \mathbb{Z}\sigma$ for all $\sigma \in \Phi$, where s_σ is the reflection about σ.

We define $\Sigma_k = \Sigma_k(X) := \mathrm{res}'_A \Sigma_{\overline{k}} = \{\overline{\sigma},\ \overline{\sigma} \neq 0\}$. In general Σ_k and Σ_k^{pr} are different.

Now we consider the Galois action on spherical roots which can be viewed as a generalization from the group case.

Proposition 5.19. *([KK16, Corollary 10.9]) The set $\Sigma_k \subseteq \chi_k$ is linearly independent, moreover, $\overline{\sigma} = \overline{\tau} \neq 0$ for $\sigma, \tau \in \Sigma_{\overline{k}}$ if and only if σ and τ are in the same Γ-orbit.*

There is a map $\Sigma_k^{pr} \to \mathbb{Z}_{>0} :\ \sigma \mapsto n_\sigma$ such that $\Sigma_k = \{n_\sigma \sigma |\ \sigma \in \Sigma_k^{pr}\}$.

Now we proceed to the construction of a root system for $W_k(X)$, we define $\Phi_k := W_k(X)\Sigma_k = W_k(X)\mathrm{res}'_A \Sigma_{\overline{k}}$, $\Phi_k^{res} := \mathrm{res}'_A \Phi_{\overline{k}} = \mathrm{res}'_A W_{\overline{k}}(X)\Sigma_{\overline{k}}$. In general, $\Phi_{\overline{k}}^{res}$ is not a root system and there is an example given in [KK16, Example 10.13].

Theorem 5.20. *([KK16, Theorem 10.14]) Let X be a k-dense G-variety, then*

- *(Φ_k, χ_k) is an integral root system, its Weyl group is $W_k(X)$ and Σ_k is a system of simple roots.*
- *$n_\sigma \in \{1, 2\}$ for all $\sigma \in \Sigma_k^{pr}$.*
- *Φ_k consists precisely of the indivisible elements of Φ_k^{res}.*

5.4. **Luna's operations over k.** In this section, we let k be a characteristic zero field, we extend several Luna's operations from $\overline{k}$ to k. We denote G a connected reductive group over k, and B a Borel subgroup of G over k.

Let's first discuss the parabolic induction over k, let P be a parabolic subgroup of G containing B_-, the opposite Borel subgroup. We assume $P = P_{-S'}$ for $S' \subset \Sigma_k(G)$, here $P_{-S'}$ is the parabolic subgroup generated by B_- and G_α for $\alpha \in -S'$, and $L = P_{-S'} \cap P_{S'}$, let X be a G-wonderful variety, here wonderful means that $X_{\overline{k}}$ is a wonderful variety over $\overline{k}$.

Definition 5.21. We say a G-morphism $\phi : X \to G/P$ arises from a *parabolic induction over* k if the unipotent radical of P acts trivially on X' with $X' = \phi^{-1}(P/P)$ a wonderful variety for L, also X is canonically isomorphic to $G *_P X'$ which is the quotient variety of $G \times X'$ by the relation

$$p(g, x) = (gp^{-1}, p \cdot x)\ p \in P,\ g \in G,\ x \in X'$$

conversely, suppose X' is a wonderful variety for L, considered as a P-variety with trivial U_P action, then $G *_P X'$ is a G-wonderful variety and the morphism $\phi : X = G *_P X' \to G/P$ induces a parabolic induction structure on X.

We recall the following proposition on Galois descent for morphisms between varieties:

Proposition 5.22. *Let X, Y be two algebraic varieties over k, then for a $\overline{k}$-morphism*

$$f :\ X_{\overline{k}} \longrightarrow Y_{\overline{k}}$$

we have f is defined over k if and only if $\sigma * f = f$ for all $\sigma \in \Gamma = Gal(\overline{k}/k)$, here $\sigma * f$ is defined as

$$\sigma * f = \sigma \circ f \circ \sigma^{-1}$$

We have the following proposition on parabolic induction over k:

Proposition 5.23. *Let G be a connected quasisplit group over k and X a G-spherical variety with $X_{\overline{k}}$ a $G_{\overline{k}}$-wonderful variety, then for $S' = supp(\Sigma_X) \cup S^p$, the G-morphism over $\overline{k}$*

$$\phi :\ X_{\overline{k}} \longrightarrow (G/P_{-S'})_{\overline{k}}$$

arises from a parabolic induction structure over k.

Proof. By Proposition 2.21, for $\Delta_\phi = \Delta(S')$, there exists a *unique* map

$$\phi : X_{\overline{k}} \longrightarrow (G/P_{-S'})_{\overline{k}}$$

we now want to use Galois descent to show ϕ is in fact defined over k.

For $\sigma \in \Gamma$, we recall the definition of $\Delta_{\sigma*\phi}$ as

$$\{\ D \in \Delta_X \mid \sigma \circ \phi \circ \sigma^{-1}(D) = G/P_{-S'}\}$$

and $f(\sigma^{-1}(D)) = G/P_{-S'}$ if and only if $\sigma^{-1}(D) \in \Delta_\phi$ that is $D \in \sigma(\Delta_\phi) = \sigma(\Delta(S'))$, and since S' is Γ-stable so $\sigma(\Delta(S')) = \Delta(S')$, hence $\Delta_{\sigma*\phi} = \Delta_\phi$. Hence from bijection $\phi \leftrightarrow \Delta_\phi$ (Proposition 2.21), we deduce $\phi = \sigma * \phi$ for all $\sigma \in \Gamma$. We see ϕ is defined over k (Proposition 5.22), that is

$$\phi : X \longrightarrow G/P_{-S'}$$

□

Definition 5.24. Let X be a G-spherical variety over k, we say X is *k-cuspidal* if there doesn't exist a map $\phi : X \to G/P$ which arises from a parabolic induction structure over k.

The following is a simple corollary of Proposition 5.23.

Corollary 5.25. *Let X be a G-spherical variety with $X_{\overline{k}}$ a wonderful variety over $\overline{k}$. Then X is k-cuspidal if and only if the spherical system of $X_{\overline{k}}$ is cuspidal.*

6. On the L-equivalence classes for spherical varieties of split rank one

In this section G will be a quasisplit group over a characteristic zero field k, we denote $\Gamma = \mathrm{Gal}(\overline{k}/k)$ the Galois group of k.

In this section I will apply the theory of spherical systems and forms of spherical varieties we developed in the previous section to classify wonderful varieties of split rank one.

6.1. **Classification of unramified groups.** Let's recall the classification of unramified groups of split rank one adjoint type, according to the theory of forms for quasisplit groups, they are classified by Galois action on the Dynkin diagram, in particular, trivial Galois action on the Dynkin diagrams corresponds to split groups.

We first recall the following result on the automorphism of Dynkin diagrams for simple Lie algebras:

Proposition 6.1. *We have the following result on the automorphism group of the Dynkin diagram*

- *type A_1 has trivial automorphism, type A_n, $n \geq 2$ has automorphism group $\cong \mathbb{Z}/2\mathbb{Z}$.*
- *type B_n has trivial automorphism group.*
- *type C_n has trivial automorphism group.*
- *type D_n, $n \geq 5$ has automorphism group $\cong \mathbb{Z}/2\mathbb{Z}$, D_4 has automorphism group S_3.*
- *type E_6 has automorphism group $\cong \mathbb{Z}/2\mathbb{Z}$.*
- *type E_7 has trivial automorphism group.*
- *type E_8 has trivial automorphism group.*
- *type F_4 has trivial automorphism group.*
- *type G_2 has trivial automorphism group.*

From this proposition, since the rank of exceptional simple groups are all greater than one, there can not exist unramified groups of split rank one which are of exceptional type over algebraic closure. Only type A_1 and A_2 Dynkin diagrams admit transitive action of autormophism group on the set of simple roots.

Definition 6.2. An adjoint group G is called *absolutely simple* if $G_{\overline{k}}$ is simple.

Definition 6.3. A connected reductive group G over k is called quasisplit over k if it contains a Borel subgroup defined over k.

Theorem 6.4. *A quasisplit adjoint semisimple group over k is isomorphic to the product of groups $Res_{\ell/k}(G')$ for ℓ/k a finite separable field extension and G' is an absolutely simple adjoint group over ℓ.*

Proposition 6.5. *In the previous theorem, if G is quasisplit, then G' is also quasisplit.*

Proof. Since $G = \mathrm{Res}_{\ell/k} G'$, we note that a Borel subgroup B of G defined over k corresponds to a Borel subgroup B' of G' defined over ℓ. □

We want to classify unramified group of adjoint type and split rank one. We may assume $G = \mathrm{Res}_{\ell/k}\ G'$, with G' being absolutely simple over ℓ. Let's consider the type of root datum of G', it must be of the type A_1 or A_2 as G' is absolutely simple. Note that G' is absolutely simple and of split rank one, since G' is also quasisplit from Proposition 6.5, we have $G' \cong PGL_2$ if the root system of G' is of type A_1, $G' \cong PU_3$ if the root system of G' is of type A_2.

6.1.1. *Spherical systems with cyclic automorphism group action.* Recall that over algebraic closure, the wonderful varieties are classified by spherical systems, which is a Dynkin diagram for the reductive group plus extra information on the colors. In Wang's thesis [Wan18, Theorem 5.5.1], a list of primitive spherical systems $\mathcal{S}$ with transitive automorphism group action is given, from this list we can obtain the list of primitive spherical systems with transitive cyclic automorphism group action.

Definition 6.6. A spherical system $\mathcal{S}$ associated to (Φ, Σ_G) is called of *adjoint type* if for every spherical root $\gamma \in \Sigma$, we have $\gamma = \sum_{\alpha \in \Sigma_G}\ n_\alpha \alpha$ with all coefficients $n_\alpha \in \mathbb{Z}$.

Definition 6.7. Let $\mathcal{S}$ be a spherical system, we say $\mathcal{S}$ is connected if either the Dynkin diagram of $\mathcal{S}$ is connected, or for each pair of distinct connected components of the Dynkin diagram with positive simple roots S_1 and S_2, there is a color $D \in \Delta(\alpha_1) \cap \Delta(\alpha_2)$, where $\alpha_1 \in S_1$ and $\alpha_2 \in S_2$.

Definition 6.8. We say a spherical system $\mathcal{S}$ is *prime* if it is cuspidal (Definition 2.23), connected (Definition 6.7) and of adjoint type (Definition 6.6).

Remark 6.9. This is a notion weaker than the notion of a primitive spherical system ([Bra13, Definition 2.5.1]) in the spherical variety literature.

One can study the intersection of the spherical roots to find the spherical system of cuspidal type with cyclic transitive automorphism group action, and this is done in Wang's thesis [Wan18, Theorem 5.5.1]. In [Wan18, Table 5.2] and [Wan18, Table 5.3] he listed all prime spherical systems with nontrivial transitive automorphism groups action on spherical roots.

6.2. **Prime spherical systems with cyclic automorphism group action.** In this section, we list prime spherical systems associated with unramified groups of split rank one and discuss their geometric realizations over algebraic closure, we also include the information of their Cartan pairings.

For a reductive group G over an algebraically closed field k of characteristic zero, we fix B a Borel subgroup, A the maximal torus of G contained in B. For X a G-spherical variety, denote by A_X the maximal torus of X.

a-A-1: This spherical system has a geometric realization $X = A\backslash PGL_2$ over algebraic closure, here A is the maximal torus of PGL_2, and we fix B the upper triangular Borel subgroup, α the simple root associated with B. Then we can calculate that $A_X \cong A$ and $\Sigma_X = \{\alpha\}$ which is a *distinguished spherical root*, $W_X \cong \mathbb{Z}/2\mathbb{Z}$. The spherical root system of X is of type A_1.

The set of colors of X is $\Delta_X = \{D_1,\ D_2\}$, the Cartan pairing $\rho_X : \Delta_X \to X_*(A_X)$ is

	α
D_1	1
D_2	1

The valuations associated with the colors are $\check{v}_{D_1} = \check{v}_{D_2} = \frac{1}{2}\alpha^\vee$.

aa-A-1: This spherical system has a geometric realization $PGL_2^{diag}\backslash PGL_2 \times PGL_2 = H\backslash G$. This is the group case, we fix the maximal torus $A = A_2 \times A_2$ of G with A_2 the maximal diagonal torus of PGL_2, and $B = B_2 \times B_2$ the product of two upper triangular Borel subgroups, α_1, α_2 the two simple roots associated with the first and second copy of PGL_2. Then we can calculate the maximal torus A_X of X is A/A_2^ι with $\iota : a \mapsto \begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & a \end{pmatrix}$, $\Sigma_X = \{\alpha_1 + \alpha_2\}$, the little Weyl group W_X of X is $\mathbb{Z}/2\mathbb{Z}$. The spherical root system of X is of type A_1.

The Cartan pairing ρ_X is the same as the Cartan pairing for group

	α
D	2

The set of color of X is $\Delta_X = \{D\}$, and the valuation associated with the color is $\alpha^\vee$.

Remark 6.10. For the group $\mathrm{G} = \mathrm{GL}_2 \times \mathrm{GL}_2$, we have two different conjugacy classes of embeddings of GL_2 as $\mathrm{Out}(\mathrm{GL}_2) \cong \mathbb{Z}/2\mathbb{Z}$, this will give rise to a different L-equivalence class.

a(2)-A-1: This spherical system has a geometric realization $X = GL_2\backslash PGL_3 = H\backslash G$, let B be the upper triangular Borel subgroup, $\alpha_1,\ \alpha_2$ the positive simple roots associated with B, A the maximal diagonal torus. We can calculate that the maximal torus $A_X \cong \mathbb{G}_m$, and the little Weyl group W_X is $\mathbb{Z}/2\mathbb{Z}$, the spherical root system of X is of type A_1.

The set of colors of X is $\Delta_X = \{D_1,\ D_2\}$ with $D_1 \in \Delta(\alpha_1)$ and $D_2 \in \Delta(\alpha_2)$

	$\alpha_1+\alpha_2$
D_1	1
D_2	1

the cocharacters associated with the colors are $\check{v}_{D_1}=\alpha_1^\vee$ and $\check{v}_{D_2}=\alpha_2^\vee$ according to Luna's axioms.

2a-A-1: This spherical system has a geometric realization $X=N(A)\backslash PGL_2$, for $N(A)$ the normalizer of the maximal diagonal torus A of PGL_2, and we note $N(A)$ is not geometrically connected. We fix B the upper triangular subgroup and α is the simple root. We can calculate that $A_X\cong A$, $\Sigma_X=\{\alpha\}$, $W_X\cong\mathbb{Z}/2\mathbb{Z}$, the spherical root system of X is of type A_1.

The set of color of X is $\Delta_X=\{D\}$, and the Cartan pairing is

	α
D	2

According to Luna's result $\check{v}_D=\alpha^\vee$.

a-A-2-n: For the group $G=(PGL_2)^n$, $B=\prod_{i=1}^n B_2$ the product of the upper triangular Borel subgroups, A the diagonal maximal torus of G, $\alpha_1,\cdots,\alpha_n$ the positive simple roots associated with B. This Luna diagram has a geometric realization $T_1U_1\backslash(PGL_2)^n$, with $T_1\cong\mathbb{G}_m$, T_1 embeds as $\prod_{i=1}^n\begin{pmatrix}a&\\&1\end{pmatrix}$ and $U\cong\mathbb{G}_a^{n-1}$ embeds as $\prod_{i=1}^n\begin{pmatrix}1&x_i\\&1\end{pmatrix}$ with $\sum_{i=1}^n x_i=0$, we can check that the torus A normalizes the unipotent subgroup U. We have $\Sigma_X=\{\alpha_1,\cdots,\alpha_n\}$, we can calculate that $A_X\cong A$, the little Weyl group is $(\mathbb{Z}/2\mathbb{Z})^n$.

The set of colors is $\Delta_X=\{D^+,\ D_1^-,\cdots D_n^-\}$ with $\Delta(\alpha_i)=\{D^+,\ D_i^-\}$. From the Luna axiom, we can calculate the Cartan pairing as

	α_1	α_2	$\cdots$	α_n
D^+	1	1	$\cdots$	1
D_1^-	1	-1	$\cdots$	0
D_n^-	0	0	$\cdots$	-1

a-A-3: For the group $G=(PGL_2)^3$, $B=B_2\times B_2\times B_2$ the product of three Borel subgroups, A the diagonal maximal torus, $\alpha_1,\alpha_2,\alpha_3$ the positive simple roots associated with each copy of PGL_2. This Luna diagram has a geometric realization $X=PGL_2\backslash PGL_2\times PGL_2\times PGL_2=H\backslash G$. We can calculate $A_X\cong A$ and $\Sigma_X=\{\alpha_1,\ \alpha_2,\ \alpha_3\}$. The little Weyl group is the whole Weyl group.

The set of colors of X is $\Delta_X=\{D_1,\ D_2,\ D_3\}$, the Cartan pairing of X:

	α_1	α_2	α_3
D_1	-1	1	1
D_2	1	-1	1
D_3	1	1	-1

The cocharacters associated with the colors D_i are $\frac{1}{2}(-\alpha_i^\vee+\sum_{j\neq i}\alpha_j^\vee)$.

a-A-4-n: For the group $G=(PGL_3)^n$, B the product of upper triangular Borel subgroups, A the diagonal maximal torus of G, $\alpha_1^i,\ \alpha_2^i$ the simple roots associated with the i-th copy of PGL_3, we denote $e_1^i,\ e_2^i,\ e_3^i$ the standard cocharacters associated with the i-th copy of PGL_3, $a\mapsto\begin{pmatrix}a&&\\&1&\\&&1\end{pmatrix}$ and $a\mapsto\begin{pmatrix}1&&\\&a&\\&&1\end{pmatrix}$, $a\mapsto\begin{pmatrix}1&&\\&1&\\&&a\end{pmatrix}$. For the geometric realization $X=T_1U_1\backslash(PGL_3)^n$, for

$T_1 \cong \mathbb{G}_m$ embeds as $a \mapsto \prod \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}$, U_1 is the unipotent subgroup

$$U_1 = \{\prod \begin{pmatrix} 1 & x_{i1} & x_i \\ & 1 & x_{i2} \\ & & 1 \end{pmatrix} \mid \sum_i (x_{i1} + x_{i2}) = 0,\ x_i \in \overline{k}\}$$

we can calculate that $A_X \cong A$, $\Sigma_X = \{\alpha_1^i,\ \alpha_2^i\}$, $W_X \cong (\mathbb{Z}/2\mathbb{Z})^n$.

The set of colors is $\Delta_X = \{D^+,\ D_1^{1,-},\ D_2^{1,-}, \cdots,\ D_1^{n,-},\ D_2^{n,-}\}$, $\Delta(\alpha_1^i) = \{D^+,\ D_1^{i,-}\}$, $\Delta(\alpha_2^i) = \{D^+,\ D_2^{i,-}\}$, from Luna's axiom, we can calculate the Cartan pairing ρ_X as

	α_1^1	α_2^1	$\cdots$	α_n^1	α_n^2
D^+	1	1	$\cdots$	1	1
$D_1^{1,-}$	1	-2	$\cdots$	0	0
$D_2^{1,-}$	-2	1	$\cdots$	0	0
$\cdots$					
$D_1^{n,-}$	0	0	$\cdots$	1	-2
$D_2^{n,-}$	0	0	$\cdots$	-2	1

The cocharacters associated with colors are $\check{v}_{D^+} = \sum_i (e_1^i)^\vee - (e_3^i)^\vee$, $\check{v}_{D_1^{i,-}} = -(e_2^i)^\vee + (e_3^i)^\vee$, $\check{v}_{D_2^{i,-}} = -(e_1^i)^\vee + (e_2^i)^\vee$.

Remark 6.11. The automorphism group for this spherical diagram is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ and can be described as follows: For σ_i belongs to the i-th copy of $\mathbb{Z}/2\mathbb{Z}$, $\sigma(\alpha_1^i) = \alpha_2^i$ and fixes other roots. For $\sigma \in S_n$, we have $\sigma(\alpha_j^i) = \alpha_j^{\sigma(i)},\ j = 1, 2$. It always contains a cycle of order $2n$.

a-A-5-n: Let $G = (PGL_3)^n$, B the product of upper triangular Borel subgroups, A diagonal maximal torus of G, $\alpha_1^i,\ \alpha_2^i$ the simple roots associated with the i-th copy of PGL_3, we denote $e_1^i,\ e_2^i,\ e_3^i$ the standard cocharacters associated with the i-th copy of PGL_3. We have the following geometric realization $X = T_2U_2\backslash(PGL_3)^n$ and U_2 is the unipotent subgroup

$$U_2 = \{\prod \begin{pmatrix} 1 & x_{i1} & x_i \\ & 1 & x_{i2} \\ & & 1 \end{pmatrix} \mid \sum_i x_{i1} = 0,\ \sum_i x_{i2} = 0,\ x_i \in \overline{k}\}$$

It can be checked that T normalizes U. We can calculate that $A_X \cong T$, $\Sigma_X = \{\alpha_1^i,\ \alpha_2^i\}$, $W_X \cong (\mathbb{Z}/2\mathbb{Z})^n$.

The set of colors is $\Delta_X = \{D_1^+,\ D_2^+,\ D_1^{i,-},\ D_2^{i,-},\ i = 1, \cdots, n\}$, $\Delta(\alpha_1^i) = \{D_1^+,\ D_1^{i,-}\}$, $\Delta(\alpha_2^i) = \{D_2^+,\ D_2^{i,-}\}$, we can calculate the cocharacters associated with D_1^+ and D_2^+ as $\check{v}_{D_1^+} = \check{v}_{D_2^+} = \sum_i (e_1^i)^\vee - (e_3^i)^\vee$, the cocharacters of the rest of the colors follow from Luna's axioms for colors.

Remark 6.12. The automorphism group for this spherical diagram is isomorphic to $S_n \times \mathbb{Z}/2\mathbb{Z}$ and can be described as follows: For $\sigma \in S_n$, the action is $\sigma(\alpha_1^i) = \alpha_1^{\sigma(i)}$ and $\sigma(\alpha_2^i) = \alpha_2^{\sigma(i)}$. For $\tau \in \mathbb{Z}/2\mathbb{Z}$ the action is $\tau(\alpha_1^i) = \alpha_2^i$.

2a-A-2: For the group $G = PGL_3$, B the upper triangular Borel subgroup, A the maximal diagonal torus, and α_1, α_2 are the two positive simple roots associated to B. The Luna diagram has a geometric realization $X = PO_3\backslash PGL_3$. The little Weyl group W_X is S_3, the root system of X is of type A_2.

The Cartan pairing is

	α_1	α_2
D_1	2	-1
D_2	-1	2

The set of cocharacters associated with colors are $\check{v}_{D_1} = \alpha_1^\vee,\ \check{v}_{D_2} = \alpha_2^\vee$.

6.3. **L-equivalence classes for cuspidal sphercal systems.** In this subsection, we will first prove an existence result for extended cuspidal spherical systems (Theorem 6.17) this will be used later to prove an existence result for all extended spherical systems with cyclic automorphism group action (Theorem 6.20).

6.3.1. *Luna datum.* We first recall the definitions of automoprhisms of a spherical pair and Luna datum from Wang's thesis [Wan18]. We work with reductive groups and spherical varieties over k an algebraically closed field.

Definition 6.13. Let (G, X) and (G', X') be two spherical pairs, an isomorphism $m: (G, X) \longrightarrow (G', X')$ is an isomorphism of group actions, i.e. it is a pair (m_G, m_X), where $m_G: G \longrightarrow G'$ is an isomorphism of groups, and $m_X: X \longrightarrow X'$ is an isomorphism of varieties, which makes the following diagram commutes:

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \downarrow{\scriptstyle (m_G, m_X)} & & \downarrow{\scriptstyle m_X} \\ G' \times X' & \longrightarrow & X' \end{array}$$

We introduce the set of automorphisms for a spherical pair:

Definition 6.14. An automorphism of a spherical pair (G, X) is an isomorphism of (G, X) to itself, we denote the set of such isomorphisms by $\mathrm{Aut}(G, X)$.

Definition 6.15. Given a root datum Ψ for a reductive group G and $\mathscr{L}$ a homogeneous spherical datum for G, we call the triple $(\Psi, \Sigma_G, \mathscr{L})$ a Luna datum, denoted by Λ.

Definition 6.16. We let $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$ and $\Psi' = (X'^*, \Phi', X'_*, \Phi'^\vee)$ be the root data of reductive groups G and G'.

Given Luna data $\Lambda = (\Psi, \Sigma_G, \mathscr{L})$ for the group G and $\Lambda' = (\Psi', \Sigma'_G, \mathscr{L}')$ for the group G', Λ and Λ' are said to be isomorphic if

- there is an isomorphism $i_R: \Psi \longrightarrow \Psi'$ consisting of $i^*: X^* \longrightarrow X'^*$ and $i_*: X_* \longrightarrow X'_*$, satisfying that $i^*(\chi) = \chi',\ i^*(\Sigma) = \Sigma'$ and $i^*(S^p) = (S^p)'$.
- there is a bijection $i_A: \mathscr{A} \longrightarrow \mathscr{A}'$, such that $\rho' \circ i_A = i_* \circ \rho$, i.e. the following diagram commutes.

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\rho} & \chi^\vee \\ \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_*|_{\chi^\vee}} \\ \mathscr{A}' & \xrightarrow{\rho'} & (\chi')^\vee \end{array}$$

And the morphism $i = (i_R, i_A)$ is called an isomorphism from Λ to Λ'. We denote $\mathrm{Aut}(\Lambda)$ the group of automophisms of the Luna datum Λ.

6.3.2. *Existence result for cuspidal spherical systems.* Let k be a characteristic zero p-adic local field with algebraic closure $\overline{k}$, and let G be a quasisplit group over k and X a G-wonderdul variety over k (We mean $X_{\overline{k}}$ is a wonderful variety over $\overline{k}$).

Theorem 6.17. *There exists a surjection from the isomorphism classes of split rank one spherical varieties to connected extended cuspidal spherical system with transitive automorphism group action.*

Proof. For extended cuspidal spherical systems with distinguished spherical roots, we will check the existence case by case later in this section: the cases for a-A-1, b(n)-B, g-G follows from our definition of the extended spherical systems. For a cuspidal spherical root system without distinguished spherical root, we only need to show the existence result over k.

Given an extended cuspidal spherical root system $\overline{\mathcal{S}}$ for a reductive group $\mathscr{G}$, we assume there is a transitive automorphism group action on $\mathcal{S}$, denote it by

$$\mu : \Gamma \longrightarrow \mathrm{Aut}(\Lambda)$$

Denote by $\mathscr{X}$ the corresponding wonderful variety for $\mathcal{S}$, and let $\mathscr{H}\backslash\mathscr{G}$ be the open $\mathscr{G}$-orbit of $\mathscr{X}$. Note we have the following commutative diagram from [Wan18, Proposition 3.2.3]:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker & \longrightarrow & \mathrm{Aut}(\mathscr{G},\mathscr{X}) & \longrightarrow & \mathrm{Aut}(\Lambda) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\ 1 & \longrightarrow & \mathrm{Inn}(\mathscr{G}) & \longrightarrow & \mathrm{Aut}(\mathscr{G}) & \longrightarrow & \mathrm{Aut}(\Psi_{\mathscr{G}}) & \longrightarrow & 1 \end{array}$$

Composing μ with ϕ, we get $\mu_{\mathscr{G}} : \Gamma \longrightarrow \mathrm{Aut}(\Psi_{\mathscr{G}})$, we denote G the quasisplit reductive group over k which induces the Galois action $\mu_{\mathscr{G}}$. Since the Γ-action induced from G preserves $\mathscr{L}_{\mathscr{H}\backslash\mathscr{G}}$, we conclude there exists a k-from $H\backslash G$ of $\mathscr{H}\backslash\mathscr{G}$ from Proposition 4.2.

□

6.4. **L-equivalence classes for split rank one wonderful varieties.** Let k be a characteristic zero p-adic local field with algebraic closure $\overline{k}$, and let G be a quasisplit group over k. Let X be a G-spherical variety.

Definition 6.18. We say X is a G-wonderdul variety over k if $X_{\overline{k}}$ is a wonderful variety over $\overline{k}$.

From now on we will assume that G is a connected quasisplit group of adjoint type.

Lemma 6.19. *For $\mathcal{S}$ a cuspidal spherical system with a distinguished root for a group G quasisplit of adjoint type, denote $\Sigma = \Gamma\cdot\gamma$ with stabilizer Γ_ℓ, then we have $G = Res_{\ell/k}\ G_\ell$ and G_ℓ is split with Dynkin diagram of type $A_1,\ B_n$ or G_2.*

Proof. We argue this case by case:
Type (1) distinguished root: For $\Sigma = \Gamma\cdot\gamma$, if γ is a distinguished spherical root of type (1), then since Γ acts transitively on Σ, we have $\Sigma = \Sigma_G$ from the cuspidality. The color axiom gives,

$$\rho(D_\alpha^{\pm}) = \frac{1}{2}\alpha^\vee|_{\mathbb{Z}\Sigma}$$

For every $\beta\in\Sigma$, we have

$$\rho(D_\alpha^{\pm})(\beta) = \frac{1}{2}\langle\alpha^\vee,\beta\rangle$$

hence every $\langle\alpha^\vee,\beta\rangle$ must be an even integer, this implies that two simple roots α,β can't be connected by an edge. Because G is adjoint, we have $G_{\overline{k}}\cong\prod_{\alpha\in\Sigma_G}\mathrm{PGL}_{2,\overline{k}}$. The Galois group acts transitively on these factors, taking $\Gamma_\ell = \mathrm{stab}_\Gamma(\gamma)$, giving $G\cong\mathrm{Res}_{\ell/k}G_\ell$. The group G_ℓ is quasisplit and adjoint of type A_1. The A_1 diagram has no nontrivial automorphisms, so the quasisplit form is split $G_\ell\cong PGL_{2,\ell}$.
Type (2) and (3) distinguished roots: For γ a distinguished spherical root of type (2) or (3). Let $C\subset\Sigma_G$ be the connected component containing $\mathrm{supp}(\gamma)$, since the Dynkin diagram of C contains a multiple edge, we have the Dynkin diagram of C must be of the type:

$$B_m,\ C_m,\ F_4,\ G_2$$

We define

$$\Gamma_C = \{\sigma\in\Gamma\ :\ \sigma(C) = C\}$$

then every element of Γ fixes every simple root in C, in particular it fixes γ. Now if $\beta \in \Sigma$ is another spherical root supported in C, then there exists $\sigma \in \Gamma$ such that $\beta = \sigma(\gamma)$, we have $\sigma(C) = C$ since σ sends the component containing $\mathrm{supp}(\gamma)$ to the component containing $\mathrm{supp}(\beta)$. We have σ fixes γ, so $\beta = \sigma(\gamma) = \gamma$.

From the cuspidality we have

$$\Sigma_G = \bigcup_{\beta \in \Sigma} \mathrm{supp}(\beta)$$

intersecting with C gives $C = \mathrm{supp}(\gamma)$. Thus C has exactly type B_n or G_2 root system. Denote $\Gamma_\ell = \mathrm{Stab}_\Gamma(\gamma)$, the classification of quasisplit adjoint groups gives $G \cong \mathrm{Res}_{\ell/k} G_\ell$ where G_ℓ is absolutely simple and quasisplit with Dynkin diagram C, which is of type B_n or G_2. □

Theorem 6.20. *Let G be a quasisplit group of adjoint type. Then there exists a surjection from the G-isomorphism classes of wonderful varieties of split rank one to extended spherical systems with transitive Galois group action on the set of spherical roots which is compatible with the Galois action on the Dynkin diagram for G.*

We will prove this theorem in this section by an explicit construction based on the L-equivalence classes of cuspidal split rank one wonderful varieties we studied in the previous section.

Proof of Theorem 6.20. We divide our arguments into cuspidal spherical systems and noncuspidal spherical systems.

First observe that if a spherical homogeneous space Y has a G-equivariant model over a field k, then its geometric wonderful compactification descends to k. Indeed, the wonderful compactification is projective, and its colored fan is the fan formed by the faces of its valuation cone. This fan is stable under the Galois action. Since G is quasisplit, the Tits obstruction in [BG22, Corollary 7.16] is trivial. Equivalently, the semilinear Galois action on $Y_{\bar{k}}$ extends uniquely to the wonderful compactification.

Case 1: For $\mathcal{S}$ a cuspidal spherical system with a distinguished root γ, denote $\Sigma = \Gamma \cdot \gamma$ with stabilizer Γ_ℓ. We have $G = \mathrm{Res}_{\ell/k} G_\ell$ and G_ℓ is split with Dynkin diagram of type A_1, B_n or G_2 from Lemma 6.19. We have $\mu^{Gal} \in \mathrm{Hom}(\Gamma_\ell, \mu_2(\bar{k}))$. We may apply the construction of § 1.5.3 to find a G_ℓ homogeneous spherical variety Y_ℓ with the given Galois cocycle $\mu^{Gal} \in H^1(\ell, \mu_2)$, we denote X_ℓ its wonderful compactification over ℓ, X_ℓ exists as we can apply [BG22, Corollary 7.16]. Then the Galois cocycle for $\mathrm{Res}_{\ell/k} X_\ell$ is $\mu^{Gal} \in H^1(k, \mathrm{Res}_{\ell/k}\mu_2) = H^1(\ell, \mu_2)$, and hence the extended spherical system for $\mathrm{Res}_{\ell/k} X_\ell$ is $\overline{\mathcal{S}}$. If $\mathcal{S}$ doesn't have a distinguished spherical root, we let $\mathscr{X}$ be a geometric realization of $\mathcal{S}$, and $\mathscr{H}\backslash\mathscr{G}$ the corresponding homogeneous part. Then we can first apply Theorem 3.5 to conclude that $\mathscr{H}\backslash\mathscr{G}$ is defined over k and then use [BG22, Corollary 7.16] show that the wonderful embedding is defined over k.

Case 2: Denote $\overline{\mathcal{S}} = (\mathcal{S}, \{\mu_i^{Gal}\})$ an extended spherical system with $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$. Set

$$S_0 = \mathrm{supp}(\Sigma), \quad S' = S_0 \cup S^p$$

Both S_0 and S' are Γ-stable, hence the standard parabolic $P = P_{-S'}$ and its Levi subgroup $L = P_{-S'} \cap P_{S'}$ are defined over k.

We denote L_0 the adjoint group with the set of simple roots S_0, then for $S_{triv} := S^p \backslash S_0$, we have S_{triv} is orthogonal to S_0. Hence we have $L^{ad} \cong L_0 \times L_{triv}$, here L_{triv} is an adjoint group with set of simple roots S_{triv}. So there is a natural surjection

$$\pi_0 : \ L \longrightarrow L^{ad} \longrightarrow L_0$$

Now consider the spherical system $\mathcal{S}_0 = (S^p \cap S_0, \Sigma, \mathscr{A})$ and the corresponding extended spherical system $\overline{\mathcal{S}}_0 = (\mathcal{S}_0, \{\mu_i^{Gal}\})$, this system is cuspidal as $\mathrm{supp}(\Sigma) = S_0 = \Sigma_{L_0}$. By case 1, there exists an L_0-wonderful variety Y over k realizing the extended cuspidal system $\overline{\mathcal{S}}_0$. Extend the L_0-action

on Y to an L-action using the projection map π_0 and denote the resulting L-variety by X_L, and define

$$X = G *_P X_L$$

By construction, X is a G-variety over k with extended spherical system $\overline{\mathcal{S}}$. □

6.4.1. *Strongly tempered spherical varieties.* In this section, let k be a field of characteristic zero, $\mathbf{G}$ a connected quasisplit reductive group over k, $\mathbf{X}$ a $\mathbf{G}$-spherical variety.

In this section, we will introduce a notion of strongly tempered spherical pairs as a generalization of the type T spherical subgroup for PGL_2.

Definition 6.21. Let $\Sigma_{\overline{k}}(X)$ be the set of spherical roots of X (Definition 2.5).

We say that a homogeneous spherical variety $\mathbf{X}$ over k is of *strongly tempered* type if we have $\Sigma_{\overline{k}}(X) = \Sigma_{\overline{k}}(G)$.

Definition 6.22. We say that a parabolic pair $(\mathbf{Y}, \alpha)$ is of strongly tempered type if $\mathbf{YP}_\alpha/R(\mathbf{P}_\alpha)$ is a spherical variety of strongly tempered type.

Lemma 6.23. *For X a strongly tempered spherical variety for an adjoint group $\boldsymbol{G}$, then we have $\boldsymbol{T} \cong \boldsymbol{T}_{\boldsymbol{X}}$.*

Proof. We have $\Sigma_{\overline{k}}(G) = \Sigma_{\overline{k}}(X)$, and $\langle \Sigma_{\overline{k}}(X) \rangle \subset X^*(T_X) \subseteq X^*(T) = \langle \Sigma_{\overline{k}}(G) \rangle$, we conclude $X^*(T_X) = X^*(T)$, hence $\mathbf{T}_{\mathbf{X}} \cong \mathbf{T}$. □

Lemma 6.24. *For $\boldsymbol{X}$ a strongly tempered spherical variety with $\mathring{\boldsymbol{X}} \neq \emptyset$. Suppose $\boldsymbol{T}_{\boldsymbol{X}} \cong \boldsymbol{T}$, then we have $\mathring{\boldsymbol{X}} \cong \boldsymbol{B}$ as a $\boldsymbol{B}$-variety.*

Proof. We choose $x \in \mathring{\mathbf{X}}(k)$ and obtain $\mathring{\mathbf{X}} = \mathbf{B}_x \backslash \mathbf{B}$, then we have $\mathbf{B}_x \subset \mathbf{T}$, this is because for all the rank one spherical varieties $\mathring{\mathbf{X}}\mathbf{P}_\alpha/R(\mathbf{P}_\alpha)$ for $\mathbf{PGL}_2$, we have $\mathring{\mathbf{X}}\mathbf{P}_\alpha/R(\mathbf{P}_\alpha) \cong \mathbb{G}_m \backslash \mathbf{PGL}_2$, and there is no unipotent subgroup contained in the stabilizer of the open Borel orbit for $\mathbb{G}_m \backslash \mathbf{PGL}_2$.

From $\mathbf{T}_{\mathbf{X}} \cong \mathbf{T}$, we conclude $\mathbf{B}_x = 1$ from $\mathbf{B}_x \backslash \mathbf{T} \cong \mathbf{T}_{\mathbf{X}}$. □

Proposition 6.25. *For $\boldsymbol{X}$ a strongly tempered spherical variety with $\boldsymbol{X}(k) \neq \emptyset$, let $x \in \mathring{\boldsymbol{X}}(k)$, and fix the isomorphism $\mathring{\boldsymbol{X}} = \boldsymbol{B}_x \backslash \boldsymbol{B}$, then we have $\mathring{\boldsymbol{X}} \cong \boldsymbol{B}_x \backslash \boldsymbol{B}$ with $\boldsymbol{B}_x \subset Z(G)$.*

Proof. We let $\mathbf{G}^{ad}$ be the adjoint group of $\mathbf{G}$, and $\mathbf{X}^{ad}$ the $\mathbf{G}^{ad}$-spherical variety obtained from $\mathbf{X}$ by quotienting out the $Z(G)$-action.

For $\mathbf{X}^{ad}$, we apply Lemma 6.23 to get $\mathbf{T}_{\mathbf{X}^{ad}} \cong \mathbf{T}^{ad}$, and we get $\mathring{\mathbf{X}}^{ad} \cong \mathbf{B}^{ad}$. We deduce from this for $\mathring{\mathbf{X}} = \mathbf{B}_x \backslash \mathbf{B}$, we have $\mathbf{B}_x \subset Z(G)$. □

6.4.2. *Spherical subgroups of split rank one unramified groups.* In this section k will be a finite unramified extension of $\mathbb{Q}_p$. We will present the classification of spherical subgroups for split rank one unramified groups over k in this section.

Unramified groups of split rank one of adjoint type: $\mathrm{Res}_{\ell/k}\mathrm{PGL}_2$, $\mathrm{Res}_{\ell/k}\mathrm{PU}_3$, for ℓ/k unramified extension and PU_3 a unitary group associated with a quadratic extension ℓ'/ℓ. Let G be a unramified group of split rank one of adjoint type, and let H be a spherical subgroup of G, let $\overline{H_{\overline{k}}}$ be the spherical closure of $H_{\overline{k}}$.

Given $X = H\backslash G$ a homogeneous spherical variety for G, since the split rank of X is at most 1. We have two cases to consider: the first case is $X = H\backslash G$ is of horospherical type, this corresponds to the split rank zero case by [KK16, Corollary 8.5], the second case is when X is of split rank one.

We consider X a split rank one spherical variety for G,

Lemma 6.26. *We let G be a split rank one unramified group, let $\mathscr{L}_X$ denote the homogeneous spherical data of $X = H\backslash G$ with X a split rank one spherical variety, then we must have $S_X^p = \emptyset$ and $supp(\Sigma) = \Sigma_G$.*

Proof. This is because G is quasisplit of split rank one, so the Γ-action on Σ_G is transitive. If $S_X^p \neq \emptyset$, then it contains at least one simple root, and since S^p is Γ-stable as it is the set of the simple roots of the associated parabolic over k, then we conclude $S_X^p = \Sigma_G$, but this can only happen if $H = G$, so from our assumption, we conclude $S_X^p = \emptyset$.

Next we note $\mathrm{supp}(\Sigma)$ is a Γ-stable subset of Σ_G and $\mathrm{supp}(\Sigma) \neq \emptyset$, so we conclude $\mathrm{supp}(\Sigma) = \Sigma_G$, that is the spherical system of X is cuspidal. □

First we prove the following reduction lemma:

Proposition 6.27. *Let G be a quasisplit group of adjoint type of split rank one and let X be a G-wonderful variety of split rank one. Suppose that the spherical system of X is cuspidal. Then there exists a finite separable extension ℓ/k, a quasi-split adjoint ℓ-group G_ℓ and a G_ℓ-wonderful variety X_ℓ with a cuspidal spherical system indecomposable as a direct product such that*

$$G \cong Res_{\ell/k}\, G_\ell, \quad X \cong Res_{\ell/k}\, X_\ell$$

Proof. We note the following isomorphism of automorphism groups

$$\mathrm{Aut}^G(X) \cong \mathrm{Aut}^G(H\backslash G)$$

this follows from the uniqueness of the wonderful embedding.

Hence we have the following calculation of the automorphism group of X from Lemma 7.20:

$$\mathrm{Aut}^G(X) \cong \mathrm{Aut}^G(H\backslash G) \cong \prod_{\gamma \in \Sigma_X^{dist}} \mu_2$$

here we used the assumption that G is adjoint, X is wonderful, thus $\chi = \langle \Sigma_X \rangle$. And since G is adjoint, there is no type (4) distinguished spherical roots for X.

Then we note that since G is of split rank one, we must have $\mathcal{S} = \prod \mathcal{S}'$ as a product of spherical systems $\mathcal{S}'$, the Galois group $\Gamma = \mathrm{Gal}(\overline{k}/k)$ acts transitively on $\mathcal{S}'$'s, and we have $G = \mathrm{Res}_{\ell/k} G_\ell$ for G_ℓ a quasisplit group over ℓ (not necessarily ℓ-absolutely simple), this follows from the classification of quasisplit groups.

There are two cases to consider:

The first case, X doesn't have distinguished spherical roots, then we have $\mathrm{Aut}^G(X) = 1$, we let X_ℓ be a geometric realization of $\mathcal{S}'$, using the existence criterion 3.5, we can assume X_ℓ is a G_ℓ-spherical variety over ℓ, then $X' = \mathrm{Res}_{\ell/k} X_\ell$ is a k-form of X as X and X' have the same spherical systems, since X is automorphism free, we have $X \cong X'$, that is $X = \mathrm{Res}_{\ell/k} X_\ell$, for X_ℓ a wonderful variety over ℓ.

The second case, X has distinguished spherical roots. For the spherical system $\mathcal{S}'$, we may apply the existence criterion 3.5 to conclude there exists a G_ℓ-spherical variety X_ℓ defined over ℓ, then X' is a k-form of X. X_ℓ has a single Γ-orbit of distinguished spherical roots, we denote by $\Gamma_L = \mathrm{Gal}(\overline{k}/L)$ its stabilizer with L/ℓ a finite extension. Since the spherical system of X_ℓ is indecomposable, since G is of adjoint type of split rank one, it must be of the type a-A-1, we deduce $L = \ell$ and $\Gamma_\ell = \mathrm{Gal}(\overline{k}/\ell)$ stabilizes the set of spherical roots of X_ℓ.

We can calculate

$$H^1(k, \mathrm{Aut}^G(X')) \cong H^1(k, \mathrm{Aut}^G(\mathrm{Res}_{\ell/k} X_\ell)) = H^1(\ell, \mathrm{Aut}^{G_\ell}(X_\ell)) = H^1(\ell, \mu_2) = \mathrm{Hom}(\Gamma_\ell, \mu_2)$$

Using the Theorem 6.17, for every $\mu_a^{Gal} \in \mathrm{Hom}(\Gamma_\ell, \mu_2)$, there exists an ℓ-form X_ℓ^a of X_ℓ. From the calculation of the automorphism group, all the k-forms of X' are of the form $\mathrm{Res}_{\ell/k} X_\ell^a$. As a consequence, we see that all the k-forms of X are of the form $\mathrm{Res}_{\ell/k} X_\ell^a$.

□

First we have the following list of prime spherical systems with $\Sigma \neq \emptyset$ and whose based root system is of type A_1^n or A_2^n and whose automorphism group contains a cyclic subgroup acting

transitively on the simple roots and spherical roots from Wang's tables ([Wan18, Table 5.2] and [Wan18, Table 5.3]):

(6.1)
- a-A-1 spherical system.
- a-A-2-n spherical system.
- a-A-3 spherical system.
- aa-A-1 spherical system.
- 2a-A-1 spherical system.
- 2a-A-2 spherical system.
- a-A-4-n spherical system.
- a-A-5-n spherical system, here n is odd.
- a(2)-A-1 spherical system.

Remark 6.28. The automorphism group for a-A-5-n type spherical system is $S_n \times \mathbb{Z}/2\mathbb{Z}$ (6.12), hence only for n odd there exists a cyclic automorphism group action on the simple roots.

Proposition 6.29. *Let G be a split rank one unramified group of adjoint type, $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ a spherical system with $\Sigma \neq \emptyset$, suppose that $\mathcal{S}$ is the spherical system of a G-spherical variety X over k. Then if the spherical system $\mathcal{S}$ is connected (Definition 6.7), we have $\mathcal{S}$ is one of the spherical systems in our List 6.1.*

Proof. This is because in the previous Lemma 6.26 we have shown that the spherical system $\mathcal{S}$ is of cuspidal type. $\mathcal{S}$ is a spherical system of adjoint type as G is adjoint and $\mathcal{S}$ is connected by assumption, so we have $\mathcal{S}$ is a spherical system of prime type. Since $\mathcal{S}$ is the spherical system of a G-spherical variety X, we have $\mathcal{S}$ is a prime spherical system with $\Sigma \neq \emptyset$ and whose based root system is of type A_1^n or A_2^n and whose automorphism group contains a cyclic subgroup acting transitively on the spherical roots, hence we have $\mathcal{S}$ is one of the spherical system in the List 6.1. □

We have the following complete list of spherical systems with based root system type A_1^n or A_2^n which is is a product $(\mathcal{S})^m$ for $\mathcal{S}$ a spherical system from the List 6.1.

(6.2)
- (a-A-1)^n spherical system.
- (a-A-2-n)^m spherical system.
- (a-A-3)^m spherical system.
- (aa-A-1)^m spherical system.
- (2a-A-1)^n spherical system.
- (2a-A-2)^n spherical system.
- (a-A-4-n)^m spherical system.
- (a-A-5-n)^m spherical system, here n is odd.
- (a(2)-A-1)^n spherical system.

Proposition 6.30. *Let G be a split rank one unramified group of adjoint type, $\mathcal{S} = (S^p, \Sigma, \mathscr{A})$ a spherical system with $\Sigma \neq \emptyset$, suppose $\mathcal{S}$ is the spherical system of a split rank one G-wonderful variety X over k. Then we have the spherical system is one of the spherical systems in our List 6.2.*

Proof. We proved that $\mathcal{S}$ is cuspidal in Lemma 6.26. We deduce that $\mathcal{S} = (\mathcal{S}')^m$ from Proposition 6.27. Since $\mathcal{S}'$ is a cuspidal, indecomposable spherical system of adjoint type with based root system A_1^n or A_2^n, also the automorphism group of $\mathcal{S}'$ contains a cyclic subgroup acting transitively on the simple roots and spherical roots. Then we get $\mathcal{S}'$ is a connected spherical system from Proposition **??**.

Since $\mathcal{S}'$ is a connected, cuspidal spherical system of adjoint type and whose automorphism group contains a cyclic subgroup acting transitively on the simple roots and spherical roots, we deduce

$\mathcal{S}'$ is one of the spherical system in our List 6.1. We get $\mathcal{S}$ is one of the spherical system in List 6.2 as a consequence. □

For the purpose of the reduction of local functional equation, we also prove the following proposition.

Proposition 6.31. *If H is a spherical subgroup of G where G is an unramified group of split rank one of adjoint type, suppose $X = H\backslash G$ is of split rank one of strongly tempered type and H is not reductive, then we must have $H_{\overline{k}}$ is a spherically closed subgroup of $G_{\overline{k}}$.*

Proof. We assume that $\overline{H}$ is a spherically closed subgroup of G (for definition of spherically closed subgroup see Definition 2.34), we write $\overline{H} = \overline{T}\cdot\overline{U}$ for $\overline{T}$ the reductive part of $\overline{H}$ and $\overline{U}$ the unipotent part of $\overline{H}$. We first observe that from our list 6.1, all such $\overline{H}$, we must have the spherical variety $X = \overline{H}\backslash G$ is of strongly tempered type, that is $\Sigma_X = \Sigma_G$. Since G is of adjoint type, we conclude $\mathring{X} \cong B$ from Lemma 6.23 and Lemma 6.24, hence

$$\dim \overline{H} + \dim \mathrm{B} = \dim \mathrm{G}$$

Next we want to show that for $H \subset \overline{H}$, H a spherical subgroup of G we must have $H = \overline{H}$. For this we write $H = T_1U_1$ with $T_1 \subseteq \overline{T}$ and $U_1 \subseteq \overline{U}$. Since we have

$$\dim \mathrm{H} + \dim \mathrm{B} = \dim \mathrm{G}$$

we conclude $\dim \mathrm{T}_1 = \dim \overline{\mathrm{T}}$ and $\dim \mathrm{U}_1 = \dim \overline{\mathrm{U}}$, we conclude $T_1 = \overline{T}$ and $U_1 = \overline{U}$ as $\overline{T}_{\overline{k}} \cong \mathbb{G}_m^n$ and $\overline{U}_{\overline{k}} \cong \mathbb{G}_a^m$. □

For diag : $H = SL_2 \hookrightarrow G = SL_2 \times SL_2$, we have $\overline{H} = SL_2 \cdot C_G$, denote $\overline{\mathcal{S}}$ the spherical system for $\overline{H}$, then for any spherical subgroup H_1 with spherical closure $\overline{H}$, we have $H \subseteq H_1 \subseteq \overline{H}$, this is because we have $\overline{H}/H_1$ is a diagonalizable group and $H = \overline{H}^{der} \subseteq H_1$.

For diag : $H = SL_2 \hookrightarrow SL_2 \times SL_2 \times SL_2$, we have $\overline{H} = SL_2 \cdot C_G$ and for any spherical subgroup H_1 with spherical closure $\overline{H}$, we have $H \subseteq H_1 \subseteq \overline{H}$.

For $H = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \times \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \subseteq SL_2 \times SL_2$, we have $H = \overline{H}$ is spherically closed, if we denote $\overline{\mathcal{S}}$ the spherical system of H, then for any spherical subgroup H_1 with spherical closure H, we have $H_1 = H$, this is proved in Proposition 6.31. We can generalize this argument to all the type TU spherical subgroups using Proposition 6.31.

For $H = SL_2 \hookrightarrow G = SL_3$, we have $\overline{H} = GL_2$, then for any spherical subgroup H_1 with spherical closure $\overline{H}$, we have $H \subseteq H_1 \subseteq \overline{H}$.

We summarize all the possible types of spherical subgroups a split rank one unramified groups of adjoint type in the following table:

Index	$\mathfrak{X}$	X
(a-A-1)^n	$(\mathbb{G}_m\backslash\mathrm{PGL}_2)^n$	$\mathrm{Res}_{\ell/k}\mathrm{T}_\ell\backslash\mathrm{Res}_{\ell/k}\mathrm{PGL}_2$
(a-A-2-n)^m	$(\mathrm{T}_1\mathrm{U}_1\backslash\mathrm{PGL}_2^n)^m$	$\mathrm{Res}_{k'/k}\mathrm{T}_{1,k'}\mathrm{U}_{1,k'}\backslash\mathrm{Res}_{k'/k}(\mathrm{Res}_{\ell/k'}\mathrm{PGL}_2)$
(a-A-3)^m	$(\mathrm{PGL}_2^{\mathrm{diag}}\backslash\mathrm{PGL}_2^3)^m$	$\mathrm{Res}_{k'/k}\mathrm{PGL}_2\backslash\mathrm{Res}_{\ell/k}\mathrm{PGL}_2$
(aa-A-1)^m	$(\mathrm{PGL}_2^{\mathrm{diag}}\backslash\mathrm{PGL}_2\times\mathrm{PGL}_2)^m$	$\mathrm{Res}_{k'/k}\mathrm{PGL}_2\backslash\mathrm{Res}_{\ell/k}\mathrm{PGL}_2$
(2a-A-1)^n	$(\mathrm{N(A)}\backslash\mathrm{PGL}_2)^n$	$\mathrm{Res}_{\ell/k}\mathrm{N(A)}\backslash\mathrm{Res}_{\ell/k}\mathrm{PGL}_2$
(2a-A-2)^n	$(\mathrm{PO}_3\backslash\mathrm{PGL}_3)^n$	$\mathrm{Res}_{\ell/k}\mathrm{PO}_3\backslash\mathrm{Res}_{\ell/k}\mathrm{PU}_3$
(a-A-4-n)^m	$(\mathrm{T}_1\mathrm{U}_1\backslash(\mathrm{PGL}_3)^n)^m$	$\mathrm{Res}_{k'/k}\mathrm{T}_{1,k'}\mathrm{U}_{1,k'}\backslash\mathrm{Res}_{k'/k}(\mathrm{Res}_{\ell/k'}\mathrm{PU}_3)$
(a-A-5-n)^m, n odd	$(\mathrm{T}_2\mathrm{U}_2\backslash(\mathrm{PGL}_3)^n)^m$	$\mathrm{Res}_{k'/k}\mathrm{T}_{2,k'}\mathrm{U}_{2,k'}\backslash\mathrm{Res}_{k'/k}(\mathrm{Res}_{\ell/k'}\mathrm{PU}_3)$
(a(2)-A-1)^n	$(\mathrm{GL}_2\backslash\mathrm{PGL}_3)^n$	$\mathrm{Res}_{\ell/k}\mathrm{U}_2\backslash\mathrm{Res}_{\ell/k}\mathrm{PU}_3$

Here ℓ/k is an unramified extension of degree n, k'/k is an unramified extension of degree m.

7. Existence and uniqueness result

In this section, we work with a characteristic zero field k and we prove an existence and uniqueness result for spherical varieties over k.

Definition 7.1. We say that a spherical system is of homogeneous reductive type if $\mathcal{S} = \mathcal{S}_X$, for the open G-orbit $H\backslash G$ of X, we have H a reductive wonderful subgroup of G.

Similarly, we say that a homogeneous spherical datum is of reductive type if $\mathscr{L} = \mathscr{L}_X$ for X a homogeneous affine spherical variety over $\overline{k}$.

The following theorem can be viewed as an existence result over k for homogeneous affine spherical varieties.

Theorem 7.2. *Let G be a quasisplit group over k. Then there exists a surjection from the G-isomorphism classes of homogeneous affine spherical varieties over k to extended homogeneous spherical data associated with homogeneous spherical data of reductive type.*

Lemma 7.3. *Suppose our Theorem 7.2 holds for all adjoint quasisplit groups G_{ad}, then our theorem holds for general quasisplit groups G.*

Proof. We denote $\pi_{ad} : G \to G_{ad}$ the quotient map by the center, then given a spherical subgroup H of G, we can associate a spherical subgroup $\pi_{ad}(H)$ of G_{ad}. We denote $\mathscr{L}_{ad}$ the homogeneous spherical datum for $X_{ad} = H_{ad}\backslash G_{ad}$ and $\overline{\mathscr{L}}_{ad}$ the extended homogeneous spherical datum associated with X_{ad}.

Given an extended homogeneous spherical datum $\overline{\mathscr{L}}$ for $\mathscr{L}$, by our construction we can obtain an extended homogeneous spherical datum $\overline{\mathscr{L}}_{ad}$ for G_{ad}, we denote $\mathscr{X}$, $\mathscr{X}_{ad}$ the homogeneous spherical varieties over $\overline{k}$ with homogeneous spherical datums $\mathscr{L}$, $\mathscr{L}_{ad}$. By our assumption, there exists a k-form X_{ad} of $\mathscr{X}_{ad}$ over k. Using the quotient map $\pi_{ad} : G \to G_{ad}$, we can obtain a k-form X of $\mathscr{X}$ and the extended homogeneous spherical datum of X is $\overline{\mathscr{L}}$ by construction. □

Lemma 7.4. *Suppose our Theorem 7.2 holds for all simply-connected groups G_{sc}, then our theorem holds for general quasisplit groups G.*

Proof. From our previous Lemma 7.3 we may assume that G is adjoint. We denote G_{sc} the simply-connected cover of G.

First we work over algebraic closure: We assume $k = \overline{k}$, given a homogeneous spherical datum $\mathscr{L}$ for G, we can obtain a homogenous spherical datum for G_{sc}: let $X = H\backslash G$ be the homogenous spherical variety with homogeneous spherical datum $\mathscr{L}$, we can define $X_{sc} = H_{sc}\backslash G_{sc}$ where $H_{sc} := \pi^{-1}(H)$ for $\pi : G_{sc} \to G$. Since $G_{sc}/Z(G_{sc}) \cong G/Z(G) = G_{ad}$, we see X_{sc} is a homogeneous spherical variety for G_{sc}. Furthermore, the homogeneous spherical datum of X_{sc} and X are compatible, the set of distinguished spherical roots of X_{sc} and X are the same. We will denote the homogenous spherical datum of X_{sc} by $\mathscr{L}_{sc}$.

Given an extended homogeneous spherical datum $\overline{\mathscr{L}}$ for $\mathscr{L}$, by the previous construction, we can obtain an extended homogeneous spherical datum $\overline{\mathscr{L}}_{sc}$ for G_{sc}, we denote $\mathscr{X}$, $\mathscr{X}_{sc}$ the homogeneous spherical varieties over $\overline{k}$ with homogeneous spherical datums $\mathscr{L}$, $\mathscr{L}_{sc}$. We have a k-form X_{sc} of $\mathscr{X}_{sc}$ with extended homogenous spherical datum $\overline{\mathscr{L}}_{sc}$, use the quotient map $\pi : G_{sc} \to G$, we obtain a k-form X of $\mathscr{X}$ and the extended homogeneous spherical datum of X is $\overline{\mathscr{L}}$ by construction. □

The following lemma reduces the proof of Theorem 7.2 to the case $X = H\backslash G$ with H a reductive wonderful subgroup.

Proposition 7.5. *Let G be a simply-connected group over $k = \overline{k}$, and H a reductive wonderful spherical subgroup of G, we denote $\mathcal{S}$ the spherical system of the wonderful variety with open G-orbit $H\backslash G$. There exists a subgroup $H^* \subseteq H$, such that for any spherical subgroup $H' \subseteq H$, if the*

spherical system of H' is of type $\mathcal{S}$ (Definition 2.16), then we have $H^* \subseteq H' \subseteq H$. Here

$$H^* = \bigcap_{\chi \in X^*(H)} ker(\chi)$$

Proof. We now want to show that the group H^* we constructed has the desired property. We put

$$\mathscr{H}_{\mathcal{S}}(H) = \left\{ H' \subseteq H : H' \text{ is spherical and } \mathcal{S}_{H'\backslash G} = \mathcal{S} \right\}.$$

We recall a consequence of the theory of augmentations. If $H' \in \mathscr{H}_{\mathcal{S}}(H)$, then H' and H have the same spherical closure, hence H/H' is diagonalizable (Proposition 2.32). In particular,

$$H^* \subseteq H' \qquad \text{for every } H' \in \mathscr{H}_{\mathcal{S}}(H). \tag{7.1}$$

□

Remark 7.6. The subgroup H^* we defined here might not be spherical in general.

Proposition 7.7. *Let G be a simply-connected quasisplit group over k, and H a reductive wonderful spherical k-subgroup of G, we assume H is of type $\mathcal{S}$. There exists a subgroup $H^* \subseteq H$, such that if the spherical system of H' is of type $\mathcal{S}$ (Definition 2.16), then we have $H^* \subseteq H' \subseteq H$.*

Proof. We have defined

$$\mathscr{H}^* = \bigcap_{\chi \in X^*(H_{\overline{k}})} \ker \chi$$

in Proposition 7.5. Now since H is defined over k, we have $\mathscr{H}^*$ is also defined over k as the Galois action permutes the algebraic characters of H, hence $\mathscr{H}^* = (H^*)_{\overline{k}}$ for some k-subgroup H^*. Now since $(H^*)_{\overline{k}} \subseteq (H')_{\overline{k}} \subseteq (H)_{\overline{k}}$ for all k-spherical subgroup $H' \subseteq H$. We deduce $H^* \subseteq H' \subseteq H$.

□

Lemma 7.8. *Let G be a connected simply-connected quasisplit group over k, H a reductive wonderful k-subgroup of G, we have defined the subgroup H^* from the previous proposition. Now let $\mathscr{H}'$ be a spherical subgroup of $G_{\overline{k}}$ contained in $H_{\overline{k}}$, we have $(H^*)_{\overline{k}} \subseteq \mathscr{H}' \subseteq H_{\overline{k}}$. Suppose the homogeneous spherical datum $\mathscr{L}$ associated with $\mathscr{H}'$ is defined over k, then we can find a k-form H' of $\mathscr{H}'$ such that $H^* \subseteq H' \subseteq H$.*

Proof. Note that by our construction we have $\Sigma_{\mathscr{H}'\backslash G_{\overline{k}}} = \Sigma_{(H\backslash G)_{\overline{k}}}$, hence we can denote $\overline{\mathscr{H}}$ the common spherical closure of $\mathscr{H}'$ and $H_{\overline{k}}$, and $\Sigma^{\#}$ the spherical roots of $\overline{\mathscr{H}}\backslash G_{\overline{k}}$.

We introduce $\mathscr{A}' := H_{\overline{k}}/\mathscr{H}'$, then as a quotient of the diagonalizable group $\mathscr{A} := H_{\overline{k}}/(H^*)_{\overline{k}}$, we see $\mathscr{A}'$ is also a diagonalizable group. We can calculate the character lattice for $\mathscr{A}'$ as

$$X^*(\mathscr{A}') = X^*(\overline{\mathscr{H}}/\mathscr{H}')/X^*(\overline{\mathscr{H}}/H_{\overline{k}}) \cong (X^*(T_{\mathscr{H}'\backslash G_{\overline{k}}})/\langle\Sigma^{\#}\rangle)/(X^*(T_{(H\backslash G)_{\overline{k}}})/\langle\Sigma^{\#}\rangle) = X^*(T_{\mathscr{H}'/G_{\overline{k}}})/\langle\Sigma_{\mathscr{H}'\backslash G_{\overline{k}}}\rangle$$

Since the homogeneous spherical datum $\mathscr{L}$ is defined over k, we conclude $X^*(\mathscr{A}')$ is Galois stable and hence $\mathscr{A}'$ is a diagonalizable group defined over k.

As the kernel of the map between algebraic groups over k,

$$0 \longrightarrow \mathscr{H}' \longrightarrow H_{\overline{k}} \longrightarrow H_{\overline{k}}/\mathscr{H}' \longrightarrow 0$$

we see $\mathscr{H}'$ is in fact defined over k, that is $\mathscr{H}' = (H')_{\overline{k}}$, and we have $H^* \subseteq H' \subseteq H$. □

Lemma 7.9. *Suppose our Theorem 7.2 holds for all extended homogeneous spherical data associated with homogeneous spherical varieties $X = H\backslash G$ with H a reductive wonderful subgroup over k, then our theorem holds.*

Proof. We let H_1 be a spherical k-subgroup for G. We let $\mathscr{L} = (S^p_{X_1}, \Sigma_{H_1\backslash G}, \mathcal{A}_{X_1}, \chi(X_1), \rho_{X_1})$ be the homogeneous spherical datum of X_1. We denote $\mathscr{H}$ the wonderful subgroup associated with spherical system $(S^p_{X_1}, \Sigma_{X_1}, \mathcal{A}_{X_1})$, then we have $\mathscr{H}$ is a reductive wonderful subgroup and $\Sigma_{(H_1\backslash G)_{\overline{k}}} = \Sigma_{\mathscr{H}\backslash G_{\overline{k}}}$.

By Lemma 7.8, every k-form of the spherical variety $(H\backslash G)_{\overline{k}}$ will give rise to a k-form of X_1, hence our theorem for wonderful reductive subgroups will imply our theorem for general homogeneous affine spherical varieties. □

From now on we will assume that our $X = H\backslash G$ with H a reductive wonderful subgroup of G. The following lemma can be viewed as a k-version of Brion's classification of reductive spherical subgroups for simply-connected groups over algebraic closure (Theorem 2.29).

Lemma 7.10. *Let G be a simply-connected quasisplit group, and $X = H\backslash G$ a homogeneous affine spherical variety with H a reductive wonderful subgroup, write $G_{\overline{k}} = \prod_{a\in A} \mathcal{G}_a$ with each $\mathcal{G}_a$ a simple group over $\overline{k}$. Then we have a partition $\mathcal{P}$ of A such that*

$$H_{\overline{k}} = \prod_{J\subset\mathcal{P}} (H_{\overline{k}} \cap G_J)$$

here $G_J = \prod_{a\in J} G_a$. We can choose $\mathcal{P}$ so that

$$\gamma\mathcal{P} = \{\gamma J\}_{J\subset\mathcal{P}} = \mathcal{P}$$

here we define γa for $a \in A$ as $G_{\gamma a} = \gamma(G_a)$.

Now denote the set of Γ-orbits in $\mathcal{P}$ by I, for each Galois orbit $\mathcal{O} \subset \mathcal{P}$, we define

$$\mathcal{G}_{\mathcal{O}} = \prod_{J\in\mathcal{O}} G_J$$

then $\mathcal{G}_{\mathcal{O}} = (G_i)_{\overline{k}}$ for a k-group G_i. Put $H_i = H \cap G_i$, then we have

$$G \cong \prod_{i\in I} G_i, \quad H\backslash G \cong \prod_{i\in I} H_i\backslash G_i$$

Proof. For $J \subseteq A$, let p_J be the projection

$$G_{\overline{k}} \to \prod_{j\in J} \mathcal{G}_j$$

Call a partition $\mathcal{A}$ of A admissible if $p_J(D) \subseteq D$ for every $J \in \mathcal{A}$. For an admissible partition, multiplication gives an isomorphism

$$\prod_{J\in\mathcal{A}} H_{\overline{k}} \cap G_J \cong H_{\overline{k}}$$

The one-block partition is admissible since p_A is the identity. If two partitions are admissible, their common refinement is admissible as well. Since A is admissible, the common refinement of all admissible partitions is the unique finest one. Denote it by $\mathcal{P}$. Let $\gamma \in \Gamma$, since γ permutes the admissible partitions, by the uniqueness of the finest one, we get $\gamma\mathcal{P} = \mathcal{P}$.

For each orbit $\mathcal{O}$ of Γ on $\mathcal{P}$, the product $\mathcal{G}_{\mathcal{O}}$ is a Galois-stable closed normal subgroup of $G_{\overline{k}}$. It descends to a closed normal k-subgroup G_i of G. Put $H_i = H \cap G_i$, we have

$$H\backslash G = \prod_i H_i\backslash G_i$$

□

Remark 7.11. The same proof works for G a quasisplit adjoint group.

Lemma 7.12. *Suppose $X = H\backslash G$ with H a reductive wonderful subgroup, then Theorem 7.2 holds for the extended homogeneous spherical datum associated with X.*

Proof. We have two cases:

Case 1: There are no distinguished spherical root, the given k-model $X = H\backslash G$ realizes the given extended homogeneous spherical datum.

Case 2: There is at least one distinguished spherical root. We introduce

$$M = \prod_{i\in\Gamma\backslash\Sigma_X^{dist}} \mathrm{Res}_{k_i/k}\ \mu_2$$

and $A = \mathrm{Aut}^G(X)$, we have

$$H^1(k, M) \cong \prod_i H^1(k_i, \mu_2) = \prod_i \mathrm{Hom}_{cont}(\Gamma_i, \{\pm 1\})$$

hence we have a bijection of $H^1(k, M)$ and the set of Galois characters for the extended homogeneous spherical data $\overline{\mathcal{L}}$ lying over $\mathcal{L}_X$.

Since the projection $p : A \to M$ has a section s defined over k, we can lift $m \in H^1(k, M)$ to $m \in H^1(k, A)$. The k-forms of X are classified by $H^1(k, A)$, we obtain an affine G-homogeneous space X' with cocycle m, the Galois character of X' is

$$m \in H^1(k, M) = \prod \mathrm{Hom}_{cont}(\Gamma_i, \{\pm 1\})$$

□

Proof of Theorem 7.2. We can reduce the proof of Theorem 7.2 to wonderful reductive subgroups via Lemma 7.9. Then our theorem follows from Lemma 7.12. □

7.1. **Galois cocycle and outer data.** In this section we will compare the Galois cocycle and the outer data.

Let $\mathscr{G}$ be a reductive group over $\overline{k}$, $\mathscr{X}$ a homogeneous affine $\mathscr{G}$-spherical variety, G is a quasisplit form of $\mathscr{G}$ over k. The k-form G induces a homomorphism:

$$\mu :\ \Gamma \longrightarrow \mathrm{SAut}(\mathscr{G})$$

and an exact sequence:

$$1 \longrightarrow \mathcal{A}_X(\overline{k}) \longrightarrow \mathrm{S}\mathcal{A}_X \longrightarrow \Gamma$$

here $\mathrm{S}\mathcal{A}_X$ is the group of μ-semilinear automorphisms of $\mathscr{X}$.

We have the projection:

$$\mathcal{A}_X \longrightarrow \mathrm{Aut}_d(X)$$

which we will denote by $a_\sigma \mapsto \overline{a_\sigma}$.

Let $X = H\backslash G$ be a homogeneous spherical G-variety, we assume that $H = H^0$ is geometrically connected. We have the following commutative diagram

$$\begin{array}{ccccc} 1 & \longrightarrow & \mathcal{A}_X(\overline{k}) & \longrightarrow & \mathrm{S}\ \mathcal{A}_X & \longrightarrow & \Gamma \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{Aut}_d(X)(\overline{k}) & \longrightarrow & \mathrm{S}\ \mathrm{Aut}_d(X) & \longrightarrow & \Gamma \end{array}$$

where $\mathrm{S}\ \mathrm{Aut}_d(X) = \mathrm{S}\ \mathcal{A}_X/\mathcal{A}_X^d(\overline{k})$. The k-structure of X gives a splitting $\mu_X :\ \Gamma \to \mathrm{S}\ \mathcal{A}_X$. We have an isomorphism $\mathrm{Aut}_d(X)(\overline{k}) \cong \mu_2(\overline{k})^{\Sigma_X^{dist}}$ independent of the choice of k-form. This gives rise to a canonical extension

$$1 \longrightarrow \mu_2(\overline{k})^{\Sigma_X^{dist}} \longrightarrow \mu_2(\overline{k})^{\Sigma_X^{dist}} \rtimes \Gamma \longrightarrow \Gamma$$

where the group law on $\mu_2(\overline{k})^{\Sigma_X^{dist}} \rtimes \Gamma$ is given by

$$[(\epsilon_\alpha^\sigma)_\alpha, \sigma][(\epsilon_\alpha^\tau)_\alpha, \tau] = [(\epsilon_\alpha^\sigma \epsilon_{\sigma(\alpha)}^\tau)_\alpha, \sigma\tau]$$

using the doubling automorphism μ_α, we obtain a homomorphism $\epsilon:\ \mathrm{S}\,\mathcal{A}_X \to \mu_2(\overline{k})^{\Sigma_X^{dist}} \rtimes \Gamma$

$$\epsilon(s) = [(\mu_\alpha(s))_\alpha, \sigma]$$

where s lies over $\sigma \in \Gamma$.

Definition 7.13. (Geometric cocycle) Suppose that X is a G-form of $\mathscr{X}$, inducing a section $\mu_X:\ \Gamma \to \mathrm{S}\mathcal{A}_X$, defining a ν-equivariant Γ-action on $\mathscr{X}(\overline{k})$, composing with ϵ, we obtain a section

$$\widetilde{\mu}_X^{dist}:\ \Gamma \xrightarrow{\mu_X} \mathrm{S}\mathcal{A}_X \xrightarrow{\epsilon} \mu_2(\overline{k})^{\Sigma_X^{dist}} \rtimes \Gamma$$

composing with the trivial section $\sigma \mapsto [(1)_\alpha, \sigma]$, we get a canonical 1-cocycle

$$\mu_X^d:\ \Gamma \longrightarrow \mu_2(\overline{k})^{\Sigma_X^{dist}} = \prod_{\alpha \in \Sigma_X^{dist}} \{\pm 1\}$$

we call this the **geometric cocycle** of X.

Remark 7.14. Suppose $s_0:\ \Gamma \to S\mathcal{A}_X$ is the trivial semilinear action, then we can associate

$$a_\sigma = \mu_X(\sigma) s_0(\sigma)^{-1} \in \mathcal{A}_X(\overline{k})$$

now a_σ is a honest G-automorphism, and one may push it forward

$$\sigma \longmapsto \overline{a_\sigma} \in \mathrm{Aut}_d(X)(\overline{k})$$

Note there is a natural isomorphism

$$\mathrm{Aut}_d(X) = \prod_{i \in I_X} \mathrm{Res}_{k_i/k}(\mu_2)$$

where I_X is the set of Γ-orbits in Σ_X^{dist}.

We may assume that G is a simply-connected group, then there exists a set of finite separable extensions $k_i/k,\ i \in I_X$ and $k_j/k,\ j \in J_X$ with a decomposition $X = \prod_i X_i \times \prod_j X_j$ (This is proved in Lemma 7.10) with $X_i = \mathrm{Res}_{k_i/k}(X_i')$ such that $X_i' = H_i' \backslash G_i'$ is a spherical variety associated to an absolutely simple and simply-connected k_i-group G_i' and $\Sigma_{X_i'}^{dist} = \{\gamma_i\}$, and where $\Sigma_{X_j}^{dist} = \emptyset$.

For $i \in I_X$, over the field extension k_i/k, the distinguished root γ_i induces a canonical character

$$\epsilon_i := \mu_{\gamma_i}:\ \mathcal{A}_{X_i'} \longrightarrow \mu_2$$

dual to the inclusion $\langle\gamma\rangle/\langle 2\gamma\rangle \subset X^*(\mathrm{Aut}_d(X)) \subset X^*(\mathcal{A}_X)$. We have

$$p_i:\ \mathrm{Aut}_d(X) \longrightarrow \mathrm{Res}_{k_i/k}(\mathrm{Aut}_d(X_i')) \xrightarrow{\mathrm{Res}(\epsilon_i)} \mathrm{Res}_{k_i/k}(\mu_2)$$

denotes the canonical projection dual to the inclusion $\mathrm{Ind}_{\Gamma_i}^{\Gamma}(\mathbb{Z}/2\mathbb{Z}\gamma_i) \subset X^*(\mathrm{Aut}_d(X))$.

Consider now the geometric cocycle $\mu_X := \mu_X^d:\ \Gamma \to \mathrm{Aut}_d(X)(\overline{k})$. For each $i \in I_X$, consider the composition

$$\mu_{X,i} := p_i \circ \mu_X:\ \Gamma \to \mathrm{Res}_{k_i/k}(\mu_2)(\overline{k})$$

then we have $\mu_X = \prod_i \mu_{X,i}$.

Consider the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\mu_{X,i}} & \mathrm{Res}_{k_i/k}(\mu_2)(\overline{k}) \\ \uparrow & & \uparrow{\scriptstyle j} \\ \Gamma_i & \xrightarrow{\mu_i} & \mu_2(\overline{k}) \end{array}$$

where j is the canonical unit map and the cocycle μ_i is determined so that the diagram commutes. Shapiro's lemma implies that the cohomology class $[\mu_{X,i}] \in H^1(k, \mathrm{Res}_{k_i/k}(\mu_2))$ is uniquely determined by the class $[\mu_i] \in H^1(k_i, \mu_2)$. In particular, μ_X is determined up to coboundary by $\{\mu_i\}_{i\in I_X}$ via the isomorphisms

$$H^1(k, \mathrm{Aut}_d(X)) \cong \prod_{i\in I_X} H^1(k, \mathrm{Res}_{k_i/k}(\mu_2)) \tag{7.2}$$

$$\cong \prod_{i\in I_X} H^1(k_i, \mu_2) \cong \prod_{i\in I_X} \mathrm{Hom}(\Gamma_i, \mu_2(\overline{k})) \tag{7.3}$$

Definition 7.15. (Outer data) Suppose that G is quasisplit and $X = H\backslash G$ is a homogeneous spherical variety with H geometrically connected. With respect to the isomorphism (7.3), the data

$$\mathrm{out}_X := \{\mu_i : \ \Gamma_i \longrightarrow \{\pm 1\} \ : \ i \in I_X\}$$

uniquely determine the cohomology class $[\mu_X] \in H^1(k, \mathrm{Aut}_d(X))$. We refer to out_X as the **outer data** of X.

Example 7.16. (Geometric cocycle for $SO_{2n}\backslash SO_{2n+1},\ n \geq 1$) We let V be a k-vector space of dimension $2n+1$ endowed with a nondegenerate quadratic form q. For two right $G = SO(V,q)$-homogeneous spherical spaces:

$$X_1 = H_1\backslash G, \quad X_a = H_a\backslash G$$

where $H_1 = SO(v_1^{\perp}), q(v_1) = 1$ and $H_a = SO(v_a^{\perp}),\ q(v_a) = a$.

Over $\overline{k}$, choose $w_a = \frac{v_a}{\sqrt{a}}$, then $q(w_a) = 1$, choose $g \in G(\overline{k})$ such that $gv_1 = w_a$. Then $gH_1g^{-1} = H_a$. Therefore g induces a $G(\overline{k})$-equivariant isomorphism

$$\psi_g : \ X_{1,\overline{k}} \longrightarrow X_{a,\overline{k}}$$

by $\psi_g(H_1x) = H_a \cdot gx$. The descent cocycle c_σ is

$$c_\sigma = \psi_g^{-1\sigma}\psi_g$$

For $H_1x \in X_1$

$$^{\sigma}\psi_g(H_1x) = H_a\sigma(g)x$$

while

$$\psi_g^{-1}(H_ay) = H_1g^{-1}y$$

therefore,

$$c_\sigma(H_1x) = \psi_g^{-1}(H_a\sigma(g)x) = H_1g^{-1}\sigma(g)x$$

so c_σ is the $G(\overline{k})$-equivariant automorphism of X_1 represented by $g^{-1}\sigma(g) \in N_G(H_1)(\overline{k})$.

For $gv_1 = w_a = \frac{v_a}{\sqrt{a}}$, we have

$$\sigma(g)v_1 = \sigma(gv_1) = \sigma(w_a) = \sigma(\frac{v_a}{\sqrt{a}}) = \frac{v_a}{\sqrt{a}}$$

but $w_a = \frac{v_a}{\sqrt{a}}$, we get $\sigma(g)v_1 = \frac{\sigma(\sqrt{a})}{\sqrt{a}}gv_1$ and hence $g^{-1}\sigma(g)v_1 = \frac{\sigma(\sqrt{a})}{\sqrt{a}}v_1$. Thus the class of $g^{-1}\sigma(g)$ in $N_G(H_1)/H_1 \cong \mu_2$ is $\frac{\sigma(\sqrt{a})}{\sqrt{a}}$.

X_1 has a trivial Galois cocycle. The k-form X_a has a geometric cocycle

$$\mu^d_{X_a}(\sigma) = \frac{\sigma(\sqrt{a})}{\sqrt{a}}$$

so the geometric cocycle is precisely the quadratic character

$$\chi_a : \quad \Gamma_k \longrightarrow \{\pm 1\}$$

associated with the quadratic étale algebra $k(\sqrt{a})/k$.

Example 7.17. (Geometric cocycle for $SL_3\backslash G_2$) Given W a seven dimensional representation of the split group G_2 with invariant quadratic form q. Choose v_1, v_a with

$$q(v_1) = 1, \quad q(v_a) = a$$

and $g \in G_2(\overline{k})$ such that

$$gv_1 = \frac{v_a}{\sqrt{a}}$$

The stabilizers are

$$\mathrm{stab}_{G_2}(v_1) \cong \mathrm{SL}_3, \quad \mathrm{stab}_{G_2}(v_a) \cong SU_{3,\ell/k}$$

where $\ell = k(\sqrt{a})$.

$X_1 = SL_3\backslash G_2$ has a trivial Geometric cocycle. We have

$$g^{-1}\sigma(g)v_1 = \frac{\sigma(\sqrt{a})}{\sqrt{a}}v_1$$

since $N_{G_2}(SL_3)/SL_3 \cong \mu_2$, this gives

$$\mu^d_{X_\ell}(\sigma) = \frac{\sigma(\sqrt{a})}{\sqrt{a}}$$

where $X_\ell = SU_{3,\ell/k}\backslash G_2$.

Proposition 7.18. *Let G be a quasisplit group, let $X = H\backslash G$ be a homogeenous spherical variety with H a geometrically connected wonderful reductive subgroup, denote $out_X = (\mu_i)_{i\in I_X}$ the outer data of X. Choose γ_i a representative for $i \in I_X$ and denote $\Gamma_i = stab_\Gamma(\gamma_i)$, then for $X_{\gamma_i} := \mathring{X}P_{\gamma_i}/R(P_{\gamma_i})$, we have*

$$\mu_i = \mu^d_{X_{\gamma_i}} \in Hom(\Gamma_i, \mu_2)$$

Proof. Note that X_{γ_i} is a spherical variety defined over $k_i = \overline{k}^{\Gamma_i}$ with $\Sigma^{dist}_{X_{\gamma_i}} = \{\gamma_i\}$. We have a decomposition

$$X = \prod_i X_i \times \prod_j X_j$$

where $X_i = \mathrm{Res}_{k_i/k}X'_i$, from the defining property of geometric cocycles we have

$$\mu^d_{X'_i} \cong \mu^d_{X_{\gamma_i}}$$

From the commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\mu_{X,i}} & \mathrm{Res}_{k_i/k}(\mu_2)(\overline{k}) \\ \uparrow & & \uparrow{\scriptstyle j} \\ \Gamma_i & \xrightarrow{\mu_i} & \mu_2(\overline{k}) \end{array}$$

we deduce

$$\mu_i = \mu^d_{X_{\gamma_i}}$$

here $\mu^d_{X_{\gamma_i}}$ is the geometric cocycle associated with X_{γ_i} viewed as a k_i-spherical variety.

□

Proposition 7.19. *Let G be a quasisplit group. For $X = H\backslash G$ a homogeneous spherical variety with H a geometrically connected reductive wonderful subgroup, we have*

$$out_X = \mu_X^{Gal}$$

Proof. Denote I_X the set of Γ-orbits of distinguished spherical roots of X, we have $\mathrm{out}_X = (\mu_i)_{i\in I_X}$, $\mu_X^{Gal} = (\mu_i^{Gal})_{i\in I_X}$.

For each $i \in I_X$, choose a representative γ_i and denote $\Gamma_i = \mathrm{stab}_\Gamma(\gamma_i)$, consider the Levi variety $X_{\gamma_i} = \mathring{X}P_{\gamma_i}/R(P_{\gamma_i})$. From Proposition 7.18, we have

$$\mu_i = \mu^d_{X_{\gamma_i}} \in \mathrm{Hom}(\Gamma_i, \mu_2)$$

Based on our discussion on the Galois cocycles in extended spherical system in Section 6.3.2 and our previous calculations on the geometric cocycles in Examples 7.16, 7.17 we have

$$\mu_i = \mu^d_{X_{\gamma_i}} = \mu^{Gal}_{X_{\gamma_i}} = \mu_i^{Gal}$$

We conclude $\mathrm{out}_X = \mu_X^{Gal}$. □

7.2. **Existence and uniqueness result for reductive wonderful groups.** In this section, we will study the existence and uniqueness result for reductive wonderful groups.

The following lemma can be viewed as a preliminary version of the uniqueness result for wonderful varieties, and we make use of the geometric cocycle introduced by Leslie:

Lemma 7.20. *Let G be a quasisplit group of adjoint type, for $X = H\backslash G$ with $H_{\overline{k}}$ a wonderful subgroup of $G_{\overline{k}}$, then we have*

$$Aut^G(X) \cong \prod_{\gamma\in\Sigma_X^{dist}} \mu_2$$

Proof. By Losev's result 2.32, we have

$$\mathrm{Aut}^G(X) \cong \langle\Sigma_X^N\rangle\backslash\chi$$

and Σ_X^N is obtained from Σ_X by doubling the distinguished spherical roots, since H is a wonderful subgroup and G is adjoint, we have $\chi = \langle\Sigma_X\rangle$, and thus

$$\mathrm{Aut}^G(X) \cong \langle\Sigma_X^N\rangle\backslash\langle\Sigma_X\rangle \cong \prod_{\gamma\in\Sigma_X^{dist}} \mu_2$$

□

Remark 7.21. If G is not adjoint, then this lemma doesn't hold, as spherical varieties without distinguished spherical roots can also have nontrivial automorphisms come from center of G.

Lemma 7.22. *Let G be a quasisplit group of adjoint type and let $X_1 = H_1\backslash G$, $X_2 = H_2\backslash G$ be two G spherical varieties, we also assume $(H_1)_{\overline{k}}$ and $(H_2)_{\overline{k}}$ are wonderful subgroups of $G_{\overline{k}}$, we assume that H_1 and H_2 are geometrically connected, suppose that $\overline{\mathcal{S}}_{X_1} = \overline{\mathcal{S}}_{X_2}$, then we have $X_1 \cong X_2$.*

Proof. We have shown that $\mathrm{Aut}^G(X) \cong \prod_{\gamma\in\Sigma_X^{dist}} \mu_2$ where $X = H\backslash G$ is a homogeneous variety with H a wonderful reductive subgroup, and since H_1 and H_2 are geometrically connected, we proved that the Galois cocycle and the outer data are the same in Proposition 7.19, hence $\overline{\mathcal{S}}_{X_1} = \overline{\mathcal{S}}_{X_2}$ implies that $[\mu^d_{X_1}] = [\mu^d_{X_2}]$, this implies $[c_{X_1,X_2}] = 1 \in H^1(k, \mathrm{Aut}^G(X_1))$, hence X_1 and X_2 are G-isomorphic. □

Now we want to prove an existence and uniqueness result for homogeneous affine varieties associated with reductive wonderful subgroups.

Lemma 7.23. *If $G = Res_{\ell/k}\tilde{G}$ with $\tilde{G}$ a ℓ-absolutely simple group of adjoint type, and $X_1 = H_1\backslash G$, $X_2 = H_2\backslash G$ are two homogeneous affine wonderful varieties for $\tilde{G}$, if $\overline{\mathcal{S}}_{X_1} = \overline{\mathcal{S}}_{X_2}$, then we have $X_1 \cong X_2$ G-equivariantly.*

Proof. If Σ_{X_1} contains a distinguished spherical root, then we may apply Lemma 7.22 to conclude $X_1 \cong X_2$ as we can check all such H_1 are geometrically connected.

If Σ_{X_1} does not contain a distinguished spherical root, then we have $\mathrm{Aut}^G(X_1) \cong 1$ from Lemma 7.20. $\mathcal{S}_{X_1} = \mathcal{S}_{X_2}$ implies $(X_1)_{\overline{k}} \cong (X_2)_{\overline{k}}$, thus $X_1 \cong X_2$. □

Theorem 7.24. *Let G be a connected quasisplit group of adjoint type, then we have a bijection between the G-isomorphism classes of homogeneous affine varieties associated with reductive wonderful subgroups and the extended spherical systems of reductive type.*

Proof. Given the surjectivity result, we only need to show the injectivity.

We want to use the previous reduction process again. We may decompose $G = \prod \tilde{G}_i$, and we decompose X, X' as $\prod X_i$ and $\prod X_i'$ (Remark 7.11), since $\overline{\mathcal{S}}_X = \overline{\mathcal{S}}_{X'}$, this will imply that $\overline{\mathcal{S}}_{X_i} = \overline{\mathcal{S}}_{X_i'}$, from previous Lemma 7.23 this implies $X_i \cong X_i'$, hence $X \cong X'$. □

7.2.1. *Homogeneous affine spherical varieties of split rank one.* In this section, we specialize our previous results to homogeneous affine spherical varieties of split rank one.

We have the following table of homogeneous affine spherical varieties of rank one from [Sak21, Page 4, table (3)].

Index	X	γ
a-A-1	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	α
a(n)-A-1	$\mathrm{GL}_n\backslash \mathrm{PGL}_{n+1}$	$\alpha_1 + \cdots + \alpha_n$
b(n)-B	$\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1}$	$\alpha_1 + \cdots \alpha_n$
c(n)-C-1	$\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2\backslash \mathrm{Sp}_{2n}$, $(n \geq 2)$	$\alpha_1 + 2\alpha_2 + \cdots 2\alpha_{n-1} + \alpha_n$
f-F	$\mathrm{Spin}_9\backslash \mathrm{F}_4$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
g-G	$\mathrm{SL}_3\backslash \mathrm{G}_2$	$2\alpha_1 + \alpha_2$
aa-A-1	$\mathrm{SL}_2 = \mathrm{SO}_3\backslash \mathrm{SO}_4$	$\alpha_1 + \alpha_2$
d(n)-D	$\mathrm{SO}_{2n-1}\backslash \mathrm{SO}_{2n}$	$2\alpha_1 + \cdots 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
d(4)-D'	$\mathrm{Spin}_7\backslash \mathrm{Spin}_8$	$2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$
b-B	$\mathrm{G}_2\backslash \mathrm{Spin}_7$	$\alpha_1 + 2\alpha_2 + 3\alpha_3$

We summarize our result for split rank one homogeneous affine spherical varieties for an adjoint unramified group G in a table. It is characterized by the property that for $X = H_1\backslash G$ a homogeneous affine spherical variety for G, there exists a spherical subgroup H from the following table and we have $H \subseteq H_1$, this follows from the Lemma 7.8 and Lemma 7.9.

Index	$\mathfrak{X}$	X	$\Phi_{\mathfrak{X}}$
a-A-1	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	A_1
a-A-1-ℓ, ℓ/k unramified quadratic extension	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	$\mathrm{U}_{1,\ell}\backslash \mathrm{PGL}_2$	A_1
2a-A-1	$\mathrm{N}(\mathbb{G}_m)\backslash \mathrm{PGL}_2$	$\mathrm{N}(\mathbb{G}_m)\backslash \mathrm{PGL}_2$	A_1
a(n)-A-1	$\mathrm{GL}_n\backslash \mathrm{PGL}_{n+1}$	$\mathrm{U}_n\backslash \mathrm{PU}_{n+1}$	A_1
aa-A-1	$\mathrm{PGL}_2\backslash \mathrm{PGL}_2 \times \mathrm{PGL}_2$	PGL_2	A_1
aa-A-1, ℓ/k unramified quadratic extension	$\mathrm{PGL}_2^{\mathrm{diag}}\backslash \mathrm{PGL}_2 \times \mathrm{PGL}_2$	$\mathrm{PGL}_2\backslash \mathrm{Res}_{\ell/k}\mathrm{PGL}_2$	A_1
aa-A-2	$\mathrm{PGL}_3\backslash \mathrm{PGL}_3 \times \mathrm{PGL}_3$	PU_3	A_2
d(3)-A-1	$\mathrm{PSp}_4\backslash \mathrm{PGL}_4$	$\mathrm{PSp}_4\backslash \mathrm{PU}_4$	A_1
d(3)-A-2	$\mathrm{PSp}_6\backslash \mathrm{PGL}_6$	$\mathrm{PSp}_6\backslash \mathrm{PU}_6$	A_2
2a-A-2	$\mathrm{PO}_3\backslash \mathrm{PGL}_3$	$\mathrm{PO}_3\backslash \mathrm{PU}_3$	A_2
a-A-3, ℓ/k unramified cubic extension	$\mathrm{PGL}_2\backslash \mathrm{PGL}_2^3$	$\mathrm{PGL}_2\backslash \mathrm{Res}_{\ell/k}\mathrm{PGL}_2$	A_1^3
a(3)-D-3	$\mathrm{G}_2\backslash \mathrm{PSO}_8$	$\mathrm{G}_2\backslash \mathrm{PSO}_8$	A_1^3
d(n)-E	$\mathrm{F}_4\backslash \mathrm{E}_6$	$\mathrm{F}_4\backslash \mathrm{E}_6$	A_2
b(n)-B	$\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1}$	$\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1}$	A_1
b(n)-B-ℓ, ℓ/k unramified quadratic extension	$\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1}$	$\mathrm{SO}_{2n,\ell}\backslash \mathrm{SO}_{2n+1}$	A_1
c(n)-C-1	$\mathrm{Sp}_{2n-2} \times^{\mu_2} \mathrm{Sp}_2\backslash \mathrm{PSp}_{2n}$	$\mathrm{Sp}_{2n-2} \times^{\mu_2} \mathrm{Sp}_2\backslash \mathrm{PSp}_{2n}$	A_1
f-F	$\mathrm{Spin}_9\backslash \mathrm{F}_4$	$\mathrm{Spin}_9\backslash \mathrm{F}_4$	A_1
g-G	$\mathrm{SL}_3\backslash \mathrm{G}_2$	$\mathrm{SL}_3\backslash \mathrm{G}_2$	A_1
g-G-ℓ, ℓ/k unramified quadratic extension	$\mathrm{SL}_3\backslash \mathrm{G}_2$	$\mathrm{SU}_{3,\ell}\backslash \mathrm{G}_2$	A_1
d(n)-D	$\mathrm{SO}_{2n-1}\backslash \mathrm{PSO}_{2n}$	$\mathrm{SO}_{2n-1}\backslash \mathrm{PSO}_{2n}$	A_1
d(4)-D'	$\mathrm{SO}_7\backslash \mathrm{PSO}_8$	$\mathrm{SO}_7\backslash \mathrm{PSO}_8$	A_1
b-B	$\mathrm{G}_2\backslash \mathrm{SO}_7$	$\mathrm{G}_2\backslash \mathrm{SO}_7$	A_1

It is interesting to note that all the spherical root systems over algebraic closure are of type A.

It would be interesting to obtain a similar table of Sakellaridis from our table which is suitable for the study of functorial transfer in the split rank one case.

Declaration of generative AI use

The first draft of this paper is part of my thesis and it is purely human written. We used ChatGPT- 6 Astra and a verification framework OSVerify [NYL+26] with ChatGPT-6 Astra to polish the draft and improve writing. The author verified all the output. The author is responsible for the correctness of this paper.

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