Rational p-stability and stability

Simply connected split symmetric pairs in the Knop–Röhrle tables

Verdict. For every finite extension $k/\mathbb{Q}_p$, every split absolutely almost simple simply connected G/k, and every nonidentity k-involution in the screened tables with full $H = G^\theta$, rational p-stability is equivalent to stability. This includes nonsplit fixed groups and residue characteristic two. The proof establishes the following more precise classification.

- In type A both hold for AII and for AIII with unequal blocks. AI and both rational regimes of balanced AIII have neither property.
- In type C both hold for CII, including field-interchanged blocks. They both fail for CI, including unitary fixed groups.
- In the Spin rows both hold for orthogonal line stabilizers in dimension at least five and DIII with odd half-dimension, including the indicated triality transports. The other orthogonal block cases, even DIII, and the rank-one tori have neither property.
- Among the twelve exceptional types, both hold precisely for FII (F_4/Spin_9), EIII (E_6/L_{D_5}) and EIV (E_6/F_4). The other nine have several rational open flag orbits.

The decisive distinction is whether the *rational* flag orbit splits. When it does not, separate closed-orbit arguments handle nonsplit eigenspaces and singular symmetrizations. No split counterexample occurs in this scope. This is an ordinary mathematical proof; the accompanying root computations verify the stated finite lattice data. The extension register specifies what remains open outside this scope.

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1 Definitions and arithmetic scope

Standing assumptions. The field k is a finite extension of $\mathbb{Q}_p$, including $p = 2$. The group G is split, absolutely almost simple and simply connected. The automorphism θ is a nonidentity algebraic involution defined over k, $H = G^\theta$ as an algebraic group, and B is a k-Borel. The proofs let H have whichever k-form the involution gives it. The letter p in [K]'s tables denotes *field characteristic*, which here is zero.

[N] is a two-page note dated October 2026. Its Definition 2.1.1 specifies reductive groups, the full fixed group, and $\sigma(g) = \theta(g^{-1})$. Its term "connected pair" refers to the quotient $H\backslash G$. Definition 2.1.2 tests closed $H(k) \times H(k)$-orbits in $G(k)$. Definition 2.5.1 writes $H, G/B$ with no points or topology. The note does not state our local-field, splitness, rational-Borel or simply connected hypotheses there. Its Theorem 2.5.3 cites [C, Thm. 5.27]. The next sentence promises a split example, but gives no pair. Section 3 discusses the open stratum of $H\backslash G$ and states multiplicity assertions. Neither Theorem 3.0.1 nor 3.0.2 is a premise of this report.

With the analytic topology, write $D(g) = H(k)gH(k)$. Stability means that for each closed $D(g)$, $\sigma(g) \in D(g)$. The inverse anti-automorphism σ preserves $H(k)$, so this tests the whole double coset. Rational p-stability means exactly one open $H(k)$-orbit on $G(k)/B(k)$. The geometric open orbit $\Omega \subset G/B$ is unique by sphericity and is defined over k. Its complement is a proper closed subvariety. On the smooth split flag variety the rational points of any such subset have empty interior, by local analytic coordinates and the identity principle for regular functions. The split big cell also gives Zariski density and hence $\Omega(k) \neq \varnothing$. In characteristic zero stabilizers are smooth, so the analytic implicit function theorem makes every $H(k)$-orbit in $\Omega(k)$ open. These are all the open orbits.

The flag identification $G(k)/B(k) = (G/B)(k)$ follows either by conjugacy of rational Borels or from $H^1(k, B) = 1$, since B is split solvable. The orbit map is an analytic submersion and gives the indicated quotient topology. In contrast $H(k)\backslash G(k) \to (H\backslash G)(k)$ gives only the neutral lifting orbit. The other $G(k)$-orbits of rational points are indexed by $\ker[H^1(k, H) \to H^1(k, G)]$, which can have several elements even though $H^1(k, G) = 1$. Thus inversion exchanges the rational double quotient with rational flag orbits, but cannot transfer an assertion on *all* rational points of $H\backslash G$ without a lifting argument.

[C, Defs. 5.1 and 5.11] uses every rational parabolic quotient. Each rational parabolic is rationally conjugate to one containing B; the projection $G(k)/B(k) \to G(k)/P(k)$ is open and surjective. A unique open orbit upstairs is dense, since it is $\Omega(k)$. Its image is a dense open orbit downstairs and therefore meets every open orbit there. The converse follows by taking $P = B$. This explains precisely the Borel reduction in the present scope.

1.1 Two obstructions and closedness

If $x \in \Omega(k)$ and $K = H_x$ is its full stabilizer, then

$$H(k)\backslash\Omega(k) \;\simeq\; \ker[H^1(k, K) \longrightarrow H^1(k, H)]. \tag{1}$$

For $y = hx$ with $h \in H(\bar{k})$, the cocycle is $h^{-1}\gamma(h)$. Changing the lift changes it by a K-coboundary, and acting by $H(k)$ has no effect. Conversely a K-cocycle trivial in H has this form, giving a rational hx. This also proves injectivity. Formula (1) retains all components, including finite diagonal subgroups.

Put $s(g) = g\theta(g)^{-1} = r$. Then $\theta(r) = r^{-1}$, so $Z_G(r)$ is θ-stable. Use the cyclic cocycle convention $r\theta(r) = 1$, with trivial classes of the form $c\theta(c)^{-1}$. Fibers of s on $G(k)$ are exactly right $H(k)$-cosets, since $s(c) = s(g)$ is equivalent to $g^{-1}c \in H(k)$. Also $s(hg) = hrh^{-1}$ for $h \in H(k)$, and $s(\sigma(g)) = g^{-1}rg$. Hence

$$\sigma(g) \in D(g) \iff \exists h \in H(k) :\; hrh^{-1} = g^{-1}rg \iff \exists c \in Z_G(r)(k) :\; r = c\theta(c)^{-1}. \tag{2}$$

Indeed the middle condition is $gh \in Z_G(r)(k)$, and $s(gh) = r$. The right side is triviality in $H^1(\langle\theta\rangle, Z_G(r)(k))$. This is cohomology of a group of order two acting on rational points. The cohomology in (1) is Galois cohomology of algebraic groups.

A double coset $D(g)$ is analytically closed exactly when r is semisimple. For closed implies semisimple, [AG, Thm. 2.3.8 and Prop. 7.2.1(i)] applies to $H \times H$ acting on the smooth affine group G. The algebraic proof contracts the unipotent part of r by a cocharacter of $Z_G(r_s)^\theta$ obtained from an adapted $\mathfrak{sl}_2$-triple. For the reverse implication, the geometric conjugacy class $Y = G \cdot r$ is smooth and closed. The involution $y \mapsto \theta(y)^{-1}$ on Y is equivariant for θ on G. At each rational fixed point, the differential of the conjugation orbit map is surjective on fixed tangent spaces (average by two). The fixed locus is smooth in characteristic zero. Every rational $H(k)$-orbit in $Y^\sigma(k)$ is consequently open there, hence closed there because all the other orbits are open. Its preimage under s is the closed double coset. This argument applies to every multiplicity and to nonsplit tori.

Another application of (2) uses the algebraic transporter

$$\{h \in H : \ hrh^{-1} = g^{-1}rg\}.$$

For semisimple symmetrizations this is geometrically nonempty. Indeed over $\bar{k}$ a semisimple element of $s(G)$ lies in a maximal θ-split torus up to H°-conjugacy (the geometric conjugacy theorem [HS, Thm. 3.4(8),(10)]); a square root there solves (2). The transporter is a right torsor under $Z_H(r)$. A calculation $H^1(k, Z_H(r)) = 1$ trivializes this *specified* torsor, making (2) true. This use of Galois cohomology does not assert that involution cohomology vanishes.

1.2 A proof of the forward implication in the split case

[C, Thm. 5.27] treats connected reductive G over a local field of characteristic zero with its rational points and its full fixed subgroup. Its proof uses the kernel for minimal θ-split parabolics in Thm. 5.25, the semisimple representative of Lemma 5.26 and the generic torus perturbation of Lemma 5.20. Opposite parabolics in Thm. 5.25 must have their respective types (the two types need not coincide). The following proof using pairs of Borels supplies the implication required here independently of that presentation.

Proposition 1.1. *Under the standing assumptions stability implies rational p-stability.*

Proof. Let B_1, B_2 be rational Borels belonging to the geometric open flag orbit and set $D = B_1 \cap \theta B_1$. The ordered pairs $(B_i, \theta B_i)$ are geometrically conjugate, since the Borels are geometrically H-conjugate. They are conjugate by $g \in G(k)$ as well: for the split group the rational Bruhat decomposition classifies pairs of rational Borels by their relative Weyl position. In particular $u = g^{-1}\theta(g) \in D(k)$.

The intersection $D = TU$ is split connected solvable, with U its unipotent radical. Choose T rational, maximal and θ-stable. Here are the descent details. A split torus in both Borels is obtained from their rational Bruhat position. Its $U(k)$-conjugates are the rational maximal tori in D, and U acts simply transitively on this set of tori over $\bar{k}$, since no root of G is trivial on a maximal torus. The θ-action on this torsor has a rational fixed point: a cyclic cocycle in a characteristic-zero unipotent group is a coboundary, successively on the vector quotients of its lower central series by averaging. The same argument, now for unipotent groups twisted by a torus cocycle, shows that $H^1(\theta, D(k)) \to H^1(\theta, T(k))$ is bijective. Replace g by gd, $d \in D(k)$, so $g^{-1}\theta(g) = t \in T(k)$; neither Borel in the ordered image changes.

Let $A = (1 - \theta)T$ and $M = Z_G(A)$. In the root decomposition for T, B_1, the roots of M are precisely the θ-fixed roots. We claim $M^{\mathrm{der}} \subset H$. Openness of the orbit gives $\mathfrak{h} + \mathfrak{b}_1 = \mathfrak{g}$. Taking the orthogonal space for the Killing form gives $\mathfrak{u}_{B_1} \cap \mathfrak{g}^{-\theta} = 0$, where the superscript here denotes the minus eigenspace. Thus on each positive fixed root space θ is the identity, and on its negative

it has the reciprocal scalar by the bracket. The associated root groups generate M^{der}. In fact if $\alpha, \theta\alpha$ were distinct positive roots then $X - \theta X$ would contradict the same tangent condition. Consequently

$$D = T\,U_M^+, \qquad H_{B_1} = T^\theta\,U_M^+, \qquad U_M^+ = R_u(B_1 \cap M). \tag{3}$$

This computes the full stabilizer at any rational open flag. Its torus factor has character group $X^*(T)/(1-\theta)X^*(T)$, with trivial Galois action.

Choose $v \in A(k)$ so $Z_G(tv^2) \subset M$. For each root α nontrivial on A, omit the proper closed subset $\alpha(tv^2) = 1$; the split torus has dense k-points. The centralizer of a semisimple element of a simply connected semisimple group is connected (Steinberg) and generated over $\bar{k}$ by T and the root groups with eigenvalue one. This proves the inclusion. Since $s(vg^{-1}) = tv^2$ and this is semisimple, stability and (2) provide $b \in Z_G(tv^2)(k) \subset M(k)$ with $s(b) = tv^2$. As v is central in M, $c = v^{-1}b$ has $s(c) = t$. In the split Levi M, the root subgroups and $T(k)$ generate $M(k)$ by Bruhat, so $c = c_0 h$ with $c_0 \in T(k)$, $h \in M^{\mathrm{der}}(k) \subset H(k)$. Then $s(c_0) = t$ and $\theta(gc_0) = gt\theta(c_0) = gc_0$. As $c_0 \in D$, the element $gc_0 \in H(k)$ still carries B_1 to B_2. This proves transitivity. □

1.3 Rational classes and central isogenies

All verdicts refer to the displayed embedded pair and are invariant under any *k-defined* automorphism transporting the actual pair. Here is also a uniform precise parametrization at the finer level of conjugacy by $G(k)$. For each listed geometric inner-conjugacy class with rational base involution θ_0, put

$$D_0 = \{u \in G : u\theta_0(u)^{-1} \in Z(G)\}.$$

This is the algebraic stabilizer of θ_0 for conjugation on automorphisms. Parameters are continuous cocycles $z_\gamma \in D_0(\bar{k})$ up to $D_0(\bar{k})$-cohomology. Since G is simply connected over a p-adic field, $H^1(k, G) = 1$ by Kneser's theorem. Write $z_\gamma = u^{-1}\gamma(u)$. Then the involution is $\mathrm{Int}(u)\theta_0\,\mathrm{Int}(u)^{-1}$, and the fixed group is the twist of $H_0 = G^{\theta_0}$ by $h \mapsto z_\gamma\gamma(h)z_\gamma^{-1}$, embedded as uH_0u^{-1}. Its center has this same descent action. Two such parameters give the same $G(k)$-class exactly when their cohomology classes agree, by the proof of (1). This parameter set, which permits disconnected D_0, can be used whenever scalar-equivalence classes of matrices or forms are inconvenient. A uniform proof for every rational member of a geometric orbit then covers each of these parameters without assuming a rational inner conjugacy.

The full fixed group upstairs is connected by the fixed-point theorem for a semisimple automorphism of a simply connected semisimple group (Steinberg, *Endomorphisms*, §8). For a central cover $\pi : G \to G_0$ with kernel Z, a compatible involution lifts uniquely to G by the universal covering property. Its square is the identity by uniqueness. On geometric points a fixed point downstairs has a lift u with $u^{-1}\theta(u) \in Z$; changing lift multiplies this by a cyclic coboundary. The upstairs fixed group maps onto $(G_0^{\theta_0})^\circ$ with kernel Z^θ, by the fixed tangent spaces in characteristic zero and connectedness. On k-points one must still take the kernel of the Galois boundary for the corresponding central extension. The flag varieties are isomorphic (the central kernel is contained in every Borel, and matching root subgroups give isomorphic Bruhat cells), whereas the rational acting groups can differ. [K, p. 2, Lemma 2.7] concerns connected geometric spherical subgroups up to isogeny. The Spin calculations below compute rational images in this common flag variety.

2 Screening the two geometric tables

The IDs below are assigned here. Each left-column entry of [K] is recorded first, followed by the positive-characteristic additions on that printed page. In [K]'s notation the argument of Sp is the dimension of its vector representation. We compare embedded connected groups with the geometric involution list (Helminck's classification, or [R, Table 1]); the matrix description in the

next two dossiers also recovers the classical list. Thus a subgroup with an extra or missing torus is a different entry.

Table 1: [K], Table 1, printed p. 3. All characteristics.

ID	Displayed $H \subset G$ and range	Screen for characteristic zero
C01	$\mathrm{SO}(n) \subset \mathrm{SL}(n)$, $n \geq 2$ (1)	AI. In rank one it is the torus case of C02/C07.
C02	$\mathrm{S}(\mathrm{GL}(m) \times \mathrm{GL}(n))$ in $\mathrm{SL}(m+n)$, $m \geq n \geq 1$	AIII, with additional rational field-centralizer models when $m = n$.
C03	$\mathrm{SL}(m) \times \mathrm{SL}(n) \subset \mathrm{SL}(m+n)$, $m > n \geq 1$	Excluded. The block involution has the determinant-one block group, including a central torus. At $(2,1)$ this row also appears in C06.
C04	$\mathrm{Sp}(2n) \subset \mathrm{SL}(2n)$, $n \geq 2$	AII. At $n = 1$ the involution is the identity.
C05	$\mathbb{G}_m \cdot \mathrm{Sp}(2n) \subset \mathrm{SL}(2n+1)$, $n \geq 1$	At $n = 1$ equals C02 for $(2,1)$: (tA, t^{-2}) over $\bar{k}$ is all of $(D, \det(D)^{-1})$. For $n > 1$ the derived symplectic block is a proper subgroup of the general linear block required by an inner involution.
C06	$\mathrm{Sp}(2n) \subset \mathrm{SL}(2n+1)$, $n \geq 1$	Excluded. The fixed group in the reducible block case must include the central torus.
C07	$\mathrm{GL}(n) \subset \mathrm{Sp}(2n)$, $n \geq 1$	CI, including its unitary rational forms. $n = 1$ is the rank-one torus.
C08	$\mathbb{G}_m \times \mathrm{Sp}(2n-2) \subset \mathrm{Sp}(2n)$, $n \geq 2$	Excluded. A symplectic two-plane split off by a self-adjoint involution has full $\mathrm{Sp}(2)$. For $n = 2$ the displayed derived SL_2 acts on the vector module as $2+1+1$, whereas the CI derived SL_2 acts as $2+2$.
C09	$\mathrm{Sp}(m) \times \mathrm{Sp}(n) \subset \mathrm{Sp}(m+n)$, $m, n \geq 2$ even	CII; equal factors may be interchanged in descent.
C10	$\mathrm{GL}(n) \subset \mathrm{SO}(2n)$, $n \geq 2$ (2)	DIII on $\mathrm{Spin}(2n)$. $n = 2$ has nonsimple ambient. Even n requires two embeddings.
C11	$\mathrm{SL}(n) \subset \mathrm{SO}(2n)$, odd $n \geq 3$	Excluded. The fixed centralizer of the polarization includes its determinant torus.
C12	$\mathrm{Sp}(4) \otimes \mathrm{Sp}(2) \subset \mathrm{SO}(8)$ (3),(4)	Two distinct embeddings transported from C17 $(5,3)$ by triality. On the vector module the tensor product kills the diagonal μ_2.
C13	$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ (3),(5)	Two spin embeddings transported from C17 $(7,1)$. The simply connected fixed group upstairs is the transported vector stabilizer.
C14	$G_2 \subset \mathrm{SO}(8)$	Excluded; the triality automorphism with fixed type G_2 has order three. No involution of D_4 has that fixed type.
C15	$\mathrm{SO}(2) \times \mathrm{Spin}(7) \subset \mathrm{SO}(10)$	Excluded by the D_5 involution list; the two-dimensional block partner would have full $\mathrm{SO}(8)$.
C16	$\mathrm{GL}(n) \subset \mathrm{SO}(2n+1)$, $n \geq 2$	Excluded. The vector representation splits as $U + U^* + 1$; an odd orthogonal involution has two nondegenerate blocks. Even at $n = 2$ the BDI $(3,2)$ module restricts differently.
C17	$\mathrm{SO}(m) \times \mathrm{SO}(n) \subset \mathrm{SO}(m+n)$, $m \geq n \geq 1$	BDI, subject to absolute almost simplicity. This is the connected image downstairs; the upstairs group and the disconnected orthogonal fixed group are distinguished in the Spin dossier.
C18	$\mathrm{Spin}(7) \subset \mathrm{SO}(9)$	Excluded; not an orthogonal two-block fixed group (the 8-dimensional spin summand has full orthogonal fixed group if split off).
C19	$G_2 \subset \mathrm{SO}(7)$	Excluded by the B_3 two-block description.

Right-hand entries on p. 3. Every one is excluded by characteristic. C20 is $\mathrm{SO}(m) \times \mathrm{SO}(n) \subset \mathrm{SO}(m+n-1)$, odd $m, n \geq 3$, characteristic two. C21 is $G_2 \times \mathrm{SO}(3) \subset \mathrm{SO}(9)$, characteristic two.

C22 is $\mathrm{SO}(2m)\times\mathrm{Sp}(2n)\subset\mathrm{Sp}(2m+2n)$, $m\geq 1,\ n\geq 0$, characteristic two. C23 is $\mathrm{Spin}(7)\subset\mathrm{Sp}(8)$, C24 is $G_2\subset\mathrm{Sp}(6)$, and C25 is $G_2\times\mathrm{Sp}(2)\subset\mathrm{Sp}(8)$, all three in characteristic two. C26 is $\Delta_q\mathrm{SL}(2)\subset\mathrm{SO}(4)$ (3) with $q=p^m>1$, in positive characteristic; [K, p. 2] specifies highest weight $(q+1)\omega_1$.

Footnotes on p. 3. (1) In characteristic two and odd $n\geq 3$ there are two classes interchanged by an outer automorphism. (2) The same two-class outer interchange holds for the C10 embedding with even n. (3) Each marked row has two conjugacy classes interchanged by an outer automorphism. (4) Triality takes C12 to the $(5,3)$ orthogonal block embedding. (5) It takes C13 to the $(7,1)$ block embedding. In D_4 triality acts on Spin_8; a permutation moving the vector node need not descend to that particular SO_8 quotient.

Boundaries and coincidences. SO_1 is trivial as a factor. The ambient SO_2 is a torus, and the cover of SO_4 is $\mathrm{SL}_2\times\mathrm{SL}_2$, so neither ambient is absolutely almost simple. At $N=3$, C17 is the rank-one torus family on SL_2. On the split covers $B_2=C_2$ identifies $(4,1)$ with CII and $(3,2)$ with CI. Under $D_3=A_3$, $(5,1)$ corresponds to AII, $(4,2)$ to balanced AIII, $(3,3)$ to AI, and DIII with $n=3$ to AIII $(3,1)$. These identifications are at the geometric-type level; each dossier calculates with its actual rational inputs. For D_4, the two C10 embeddings with $n=4$ and the vector $(6,2)$ embedding form another triality triple.

Table 2: [K], Table 2, printed p. 4. The labels here retain its root-type notation.

ID	Displayed $H\subset G$	Screen
E01	$A_2\subset G_2$	Excluded; the sole nontrivial involution class has fixed type $A_1+\widetilde{A}_1$.
E02	$A_1\times\widetilde{A}_1\subset G_2$	Included, G.
E03	$B_4\subset F_4$	Included, FII.
E04	$C_3\times A_1\subset F_4$	Included, FI.
E05	$C_4\subset E_6$	Included, EI.
E06	$F_4\subset E_6$	Included, EIV.
E07	$D_5\subset E_6$	Excluded; it is only the derived subgroup of the EIII Levi.
E08	$\mathbb{G}_m\cdot D_5\subset E_6$	Included, EIII.
E09	$A_5\times A_1\subset E_6$	Included, EII.
E10	$\mathbb{G}_m\cdot E_6\subset E_7$	Included, EVII.
E11	$A_7\subset E_7$	Included, EV.
E12	$D_6\times A_1\subset E_7$	Included, EVI.
E13	$D_8\subset E_8$	Included, EVIII.
E14	$E_7\times A_1\subset E_8$	Included, EIX.
E15	$\widetilde{A}_2\subset G_2$	Characteristic three only.
E16	$C_4\subset F_4$	Characteristic two only.
E17	$B_3\times\widetilde{A}_1\subset F_4$	Characteristic two only.

The tildes in Table 2 designate subsystems consisting of short roots. Throughout both tables an included row means its geometric embedding, lifted as above, together with the rational involutions in that geometric inner-conjugacy class. It does not assert that every rational subgroup with that abstract type is an embedded fixed group.

3 Classification of the rational pairs

Let $s=|k^\times/k^{\times 2}|$. In the tables below a number counts open rational orbits. The symbol >1 is an unconditional failure of the singleton criterion. The columns St and $\Leftrightarrow$ record stability and the equivalence. Every "yes" in the last of these columns is ARGUED, including when both predicates are false. [A], [S] and [E] here refer to the following three proof dossiers. Form parameters refer to

the actual rational matrices and embeddings defined there.

Table 3: Classical cases. The letter E denotes a quadratic field extension when a field form is specified.

ID	G, H, θ and k-form	Parameters	Open	St	$\Leftrightarrow$	Evidence
C01	$\mathrm{SL}_m, \mathrm{SO}(q)$, inverse adjoint	$m \geq 2$; all q	diagonals of q, > 1	no	yes	[A] flag line norms
C04	$\mathrm{SL}_{2n}, \mathrm{Sp}(\omega)$, inverse adjoint	$n \geq 2$; all markings	1	yes	yes	[A] étale alternating form
C02	$\mathrm{SL}_{a+b}, \mathrm{S}(\mathrm{GL}_a \times \mathrm{GL}_b)$, $\mathrm{Ad}\, J$	$1 \leq a < b$	1	yes	yes	[A] determinant correction
C02	same block model	$a = b \geq 1$	s	no	yes	[A] determinant
C02	$\mathrm{SL}_{2a}, \ker(N \det_E)$, $\mathrm{Ad}\, J$	$a \geq 1$, $k[J] = E$	$s/2$	no	yes	[A] norm determinant
C07	$\mathrm{Sp}_{2n}, \mathrm{GL}_n$, $\mathrm{Ad}\, J$, $J^* = -J$	$n \geq 1$, split	s^n	no	yes	[A] symplectic flags
C07	$\mathrm{Sp}_{2n}, U(h)$, same formula	$n \geq 1$, E, h	$s^n/2$	no	yes	[A] Hermitian determinant
C09	$\mathrm{Sp}_{2(a+b)}, \mathrm{Sp}_{2a} \times \mathrm{Sp}_{2b}$, $\mathrm{Ad}\, J$, $J^* = J$	$1 \leq a \leq b$	1	yes	yes	[A] symplectic transporter
C09	$\mathrm{Sp}_{4a}, \mathrm{Res}_{E/k}\, \mathrm{Sp}_{2a}$, same formula	$a \geq 1$, E	1	yes	yes	[A] same on every twist
C17	$\mathrm{Spin}_N, \mathrm{Spin}(e^\perp)$, line reflection	$N \geq 5$; all lines of nonzero norm	1	yes	yes	[S] norm quadric
C17	$\mathrm{Spin}_N, (\mathrm{Spin}A \times \mathrm{Spin}D_0)/\Delta\mu_2$, block signs	$2 \leq a \leq b$, $N = a + b \geq 5$	> 1	no	yes	[S] projected norm
C17	Spin_{2a}, $(\mathrm{Res}_{E/k}\, \mathrm{Spin}Q)/\Delta\mu_2$, self-adjoint J	$a \geq 3$, split trace Q	> 1	no	yes	[S] $b(x, Jx)$
C10	$\mathrm{Spin}_{2n}, \widetilde{\mathrm{GL}}_n$, lifted $\mathrm{Ad}\, J$	odd $n \geq 3$	1	yes	yes	[S] connected centralizer
C10	same split-polarization model	even $n \geq 4$	s	no	yes	[S] spinor norm
C10	$\mathrm{Spin}_{2n}, \widetilde{U}(h)$, lifted $\mathrm{Ad}\, J$	even $n \geq 4$, E, h	$s/2$	no	yes	[S] binary rotations
C12	Spin_8, triality image of $(5, 3)$	two embeddings	> 1	no	yes	[S] rational transport
C13	Spin_8, triality image of $(7, 1)$	two embeddings	1	yes	yes	[S] rational transport

Here $\widetilde{\mathrm{GL}}_n, \widetilde{U}(h)$ adjoin the square root of the determinant in the indicated torus, as written in [S]. The $N = 3$ C17 case is C01 with $m = 2$. C05 at $n = 1$ uses C02 with $(a, b) = (1, 2)$.

Table 4: Exceptional cases. Involution symbols and the central kernels are specified in [E]. Every row includes every $H^1(k, D_0)$ parameter.

ID	$G, H_{\bar{k}}$	θ_0	Open	St	$\Leftrightarrow$	Evidence
E02	$G_2, (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$	I_2	> 1	no	yes	[E] finite torus factor
E03	F_4, Spin_9	I_4	1	yes	yes	[E] three strata
E04	$F_4, (\mathrm{Sp}_6 \times \mathrm{SL}_2)/\mu_2$	I_1	> 1	no	yes	[E] finite torus factor

ID	Model	θ_0	Open	St	$\Leftrightarrow$	Evidence
E05	$E_6, \mathrm{Sp}_8/\mu_2$	Chevalley	> 1	no	yes	[E] finite torus factor
E06	E_6, F_4	diagram	1	yes	yes	[E] Albert idempotents
E08	$E_6, (\mathrm{Spin}_{10} \times \mathbb{G}_m)/\mu_4$	I_1	1	yes	yes	[E] seven strata
E09	$E_6, (\mathrm{SL}_6 \times \mathrm{SL}_2)/\mu_2$	I_2	> 1	no	yes	[E] finite torus factor
E10	$E_7, (E_6^{\mathrm{sc}} \times \mathbb{G}_m)/\mu_3$	I_7	> 1	no	yes	[E] torus quotient
E11	$E_7, \mathrm{SL}_8/\mu_2$	I_2	> 1	no	yes	[E] finite torus factor
E12	$E_7, (\mathrm{Spin}_{12} \times \mathrm{SL}_2)/\mu_2$	I_1	> 1	no	yes	[E] finite torus factor
E13	E_8, HSpin_{16}	I_1	> 1	no	yes	[E] finite torus factor
E14	$E_8, (E_7^{\mathrm{sc}} \times \mathrm{SL}_2)/\mu_2$	I_8	> 1	no	yes	[E] finite torus factor

4 The special linear and symplectic dossiers

Throughout this section k is a finite extension of $\mathbb{Q}_p$, including $p = 2$. Write $S = k^\times/k^{\times 2}$. Its order is at least four: the unramified quadratic extension and a ramified quadratic extension generated by a square root of a uniformizer give distinct nontrivial square classes. In odd residue characteristic $|S| = 4$, and for $k/\mathbb{Q}_2$ it is $2^{[k:\mathbb{Q}_2]+2}$ (also obtained by squaring on a sufficiently small principal-unit subgroup, using the local logarithm and its index). For quadratic E/k the local norm index theorem gives $[k^\times : N_{E/k}(E^\times)] = 2$. In particular a nondegenerate binary quadratic form represents at least two square classes. Indeed it is a scalar times the norm of a quadratic étale algebra. This observation works in all residue characteristics under consideration.

4.1 Which rational involutions are in these dossiers?

For $\mathrm{SL}(V)$, the inner automorphisms are given by $\mathrm{PGL}(V)(k)$, hence by linear maps over k modulo scalars. An inner involution is $\mathrm{Ad}\, J$ with $J^2 = d\,1$; $d \in k^\times/k^{\times 2}$. A split J is normalized to eigenvalues ± 1, of positive multiplicities a, b. If d is nonsquare, V becomes a space over $E = k(\sqrt{d})$ and has dimension $2a$ over k. For dimension at least three the other component of the automorphism group is represented by inverse transpose (the type A diagram automorphism). An involution in that component is $g \mapsto (g^*)^{-1}$ for the adjoint of a nondegenerate symmetric or alternating bilinear form. Indeed the matrix of the form is proportional to its transpose, with proportionality $+1$ or -1. In dimension two inverse transpose is itself inner, so the symmetric-form models overlap the torus models, and the alternating-form model gives the identity.

For $\mathrm{Sp}(V, \omega)$ the diagram has no automorphism in characteristic zero. Every rational automorphism is induced by $J \in \mathrm{GSp}(V, \omega)(k)$, modulo $k^\times$ (Hilbert 90 lifts a projective similitude). For an involution $J^2 = d\,1$, and its multiplier is $\pm d$, since its square has multiplier d^2. Equivalently $J^* = \pm J$. These two signs are treated separately below. All embeddings described here are actual matrix embeddings. The displayed models are connected over $\bar{k}$, so compute the full fixed groups. The only finite quotients relevant to these matrix models are spelled out when product or covering presentations are used.

4.2 AI. Orthogonal fixed points in SL_m, $m \geq 2$

Take all nondegenerate symmetric forms q on $V = k^m$, up to $q' = t\,u^*q$ with $t \in k^\times$ and $u \in \mathrm{SL}(V)(k)$; this is exactly conjugacy of the involutions by $\mathrm{SL}(V)(k)$. The fixed group is the faithfully embedded $\mathrm{SO}(q)$, whose center for $m \geq 3$ is trivial for odd m and μ_2 for even m. At $m = 2$ it is the full one-dimensional torus. For $m \geq 3$ the spin cover has kernel μ_2; it is the image $\mathrm{SO}(q)$ that acts here.

The open flags $F_\bullet$ have $q|_{F_i}$ nondegenerate for every i. Orthogonalizing yields successive lines $L_i = F_i \cap F_{i-1}^{\perp q}$. Their ordered one-dimensional isometry classes give the exact set of rational open orbits

$$\{(s_1, \ldots, s_m) \in S^m : \langle s_1, \ldots, s_m \rangle \simeq q\}.$$

A matching isometry can be given determinant one by changing sign on one target line. The geometric stabilizer is the diagonal sign group of product one, μ_2^{m-1}, and an orthogonal basis supplies a rational base point. The binary form observation supplies two choices of the first line, each extendible by orthogonalization. Thus neither property holds. For the split $m = 2$ torus, in eigenbasis coordinates on $\mathbb{P}^1$ the action on $z \neq 0, \infty$ is $z \mapsto t^2 z$. This gives S, despite there being only one geometric open orbit. The anisotropic binary model gives $|S|/2$ open orbits.

4.3 AII. Symplectic fixed points in SL_{2n}, $n \geq 2$

Use the adjoint of an alternating form ω, and $H = \mathrm{Sp}(V, \omega)$ with center μ_2 and faithful embedding. Up to $\mathrm{GL}(V)(k)$ there is one model. Its $\mathrm{SL}(V)(k)$-conjugacy classes are indexed by $k^\times / k^{\times n}$, since the determinant image of the similitude group is $k^{\times n}$ by the Pfaffian and by the existence of all symplectic multipliers. The calculation applies to every such model.

A flag is in the open orbit when its even-dimensional stages are symplectic. Orthogonal differences of successive even stages are symplectic planes, each equipped with a line by the odd stage. Conversely this data gives the flag. A symplectic basis in each plane proves rational transitivity; the stabilizer is the product of n line Borels of SL_2. Every $r = gg^*$ is self-adjoint. For semisimple r, the algebra $R = k[r]$ is étale, a product of fields R_i, with mutually orthogonal summands V_i. The restriction of ω to each summand is the trace of a nondegenerate R_i-alternating form: apply nondegeneracy of the field trace to $t \mapsto \omega(tx, y)$, using self-adjointness of R. Choose complementary R_i-Lagrangians in V_i. Let c act by r on the first and by 1 on the second on each summand. Then c commutes with r and $cc^* = r$. Moreover $g^{-1}c$ has $(g^{-1}c)(g^{-1}c)^* = 1$ and hence is symplectic, of determinant one. Thus $c \in \mathrm{SL}(V)(k)$. The centralizer criterion gives stability for all closed double cosets.

4.4 AIII. Block and quadratic-field centralizers

For split J put $a \leq b$, $e = b - a$. The fixed group is $H = \mathrm{S}(\mathrm{GL}_a \times \mathrm{GL}_b)$ by direct block embedding. Its derived subgroup is $\mathrm{SL}_a \times \mathrm{SL}_b$ and its center consists of scalar pairs (x, y) with $x^a y^b = 1$. For $h = \gcd(a, b)$ the map from $\mathbb{G}_m \times \mathrm{SL}_a \times \mathrm{SL}_b$ sending (t, A, B) to $(t^{b/h}A, t^{-a/h}B)$ has kernel the central copy of $\mu_{ab/h}$ given by the forced scalar choices of A, B. There is one $\mathrm{SL}(k)$-conjugacy class for fixed unordered dimensions since either full block permits arbitrary determinant in the linear centralizer.

In the quadratic-field case the fixed group is

$$H = \ker\left(\mathrm{Res}_{E/k}\,\mathrm{GL}_a \xrightarrow{N_{E/k}\det_E} \mathbb{G}_m\right).$$

Its derived group is $\mathrm{Res}_{E/k}\,\mathrm{SL}_a$, its center consists of $z \in \mathrm{Res}_{E/k}\,\mathbb{G}_m$ with $N(z)^a = 1$. If $T = \mathrm{Res}^1_{E/k}\,\mathbb{G}_m$, the natural multiplication $T \times \mathrm{Res}_{E/k}\,\mathrm{SL}_a \to H$ has kernel the antidiagonal $T[a]$. The rational points impose $N(\det_E) = 1$. For fixed E all embeddings are conjugate by $\mathrm{GL}_{2a}(k)$;

the $\mathrm{SL}_{2a}(k)$-classes are indexed by $k^\times/(N(E^\times)\langle(-1)^a\rangle)$. Indeed a normalizer of the projective involution takes J to $\pm J$ after fixing $J^2 = d$. The nontrivial field automorphism applied on each of a coordinates has determinant $(-1)^a$.

Let D be the linear centralizer of J in $\mathrm{GL}(V)$. In the split case choose eigenvectors u_i, v_i of opposite signs for $1 \le i \le a$ and a basis $w_1, \dots, w_e$ of the excess eigenspace. A rational flag is given by the ordered basis

$$(u_1 + v_1, \dots, u_a + v_a,\ w_1, \dots, w_e,\ u_a - v_a, \dots, u_1 - v_1).$$

In the field case use $(x_1, \dots, x_a, Jx_a, \dots, Jx_1)$ for an E-basis x_i. In each basis J reverses the paired positions up to scalar and fixes the middle positions projectively. A linear map preserving the flag and commuting with J is upper triangular also in the J-translated flag. Intersecting these two triangular conditions leaves only equal scalars on each paired set of positions and an arbitrary upper triangular middle block. Thus $D_F = (\mathbb{G}_m)^a \times B_{\mathrm{GL}_e}$ (omit the empty block). Subtracting its dimension from $\dim D$ gives $\dim(G/B)$, proving geometric openness. The stabilizer is split connected solvable and has trivial Galois H^1 by Hilbert 90 and its split unipotent filtration. Thus $D(k)$ is transitive on the rational open stratum. The full H-stabilizer is $D_F \cap \ker \det$, including the possibly disconnected part. The orbit indices are exactly $\det D(k)/\det D_F(k)$, hence

$$\begin{array}{c|c} e > 0 & 1 \\ e = 0,\ J \text{ split} & |S| \\ J^2 = d,\ k(\sqrt{d}) \text{ a field} & |S|/2. \end{array}$$

Over $\bar{k}$, H has the same open orbit since its stabilizer correction uses a surjective algebraic determinant on D_F.

When $e > 0$ here is a direct stability proof. For semisimple $r = gJg^{-1}J^{-1}$ we seek c commuting with r such that $cJc^{-1} = rJ$. On the complement to $V_{-1}(r)$ use $1 + r$, since $JrJ^{-1} = r^{-1}$. This complement is J-stable. As rJ and J are conjugate on the entire space, their ± 1 eigendimensions also agree on $V_{-1}(r)$, where $rJ = -J$. There interchange the two equal-dimensional J-eigenspaces by an arbitrary rational linear isomorphism and its inverse. This constructs a linear solution. Over $\bar{k}$, on every pair $V_\lambda(r) \oplus V_{\lambda^{-1}}(r)$ with $\lambda \neq \pm 1$, J exchanges the summands, contributing equally to its two eigenmultiplicities. Since $a \neq b$, $V_1(r)$ or $V_{-1}(r)$ is nonzero over k. It has a nonzero joint rational sign eigenspace for r, J. Scaling one basis vector there gives an element of $Z_{\mathrm{GL}}(r, J)(k)$ of any prescribed determinant. Multiplying the solution on the right adjusts its determinant to one. This proves [C, Thm. 6.13] here without using a representation-theoretic implication. For the two equal-block cases both properties fail by the flag computation and the forward implication.

4.5 CI. Skew-adjoint J on Sp_{2n}, $n \ge 1$

Put $E = k[\delta]/(\delta^2 - d)$ with $\bar{\delta} = -\delta$, including the split étale algebra. Put $\delta x = Jx$ and

$$h(x, y) = \omega(Jx, y) + \delta\, \omega(x, y).$$

This is a nondegenerate E/k-Hermitian form, linear in the first variable. Its second coefficient recovers ω. Thus the full fixed group is the faithfully embedded $\mathrm{U}(h)$ with center $\mathrm{Res}^1_{E/k} \mathbb{G}_m = T$. Its derived subgroup is $\mathrm{SU}(h)$ and $T \times \mathrm{SU}(h) \to \mathrm{U}(h)$ has antidiagonal kernel $T[n]$. For split E this gives the Levi $A \mapsto \mathrm{diag}(A, A^{-t})$ with $A \in \mathrm{GL}_n$. For fixed nonsquare d all nondegenerate n-dimensional Hermitian spaces occur, since their alternating coefficient forms are all symplectic. Two normalized J are symplectically conjugate precisely for isometric Hermitian models. For projective J, changing its sign also identifies h with $-\bar{h}$ under the semilinear change of scalar action (equivalently $-h$ in an orthogonal basis). This is an explicit isometry parameter for the rational involution classes; split E gives one class.

Take flags in a Lagrangian L. The open flags have $\omega(J\cdot,\cdot)$ nondegenerate on every stage. Since $\omega|_L = 0$, the restrictions of h there are k-valued. Nondegeneracy on L gives $V = L \oplus JL$: if $Jx \in L$ for $x \in L$ then $\omega(Jx, L) = 0$. Orthogonalizing the flag therefore gives an E-basis x_i, unique up to scalings of its k-lines, with $h(x_i, x_i) = s_i \in k^\times$. Conversely every such orthogonal E-basis produces an isotropic k-flag. The stabilizer is μ_2^n, and the orbits are exactly

$$\{(s_1, \dots, s_n) \in S^n : \operatorname{diag}(s_1, \dots, s_n) \simeq_E h\}.$$

This description gives rational base points and proves geometric transitivity. There are $|S|^n$ in the split case. In the field case changing x_1 by an E-scalar varies s_1 by all norms, giving at least $|S|/2 > 1$ orbits. More precisely there are $|S|^n/2$: Hermitian forms over a quadratic extension of a non-Archimedean local field are classified by dimension and determinant modulo norms. One can see sufficiency directly: any Hermitian binary diagonal $\langle a, b\rangle$ represents every element of $k^\times$, being a times the reduced norm of the quaternion algebra $(E/k, -b/a)$. The local quaternion reduced norm is onto (in the split case this is determinant; in the division case write the algebra as the unramified quadratic cyclic algebra with uniformizer parameter, whose generator has norm of valuation one and whose unramified subfield norms contain every unit). Then represent the desired first entry and split its line off inductively, ending in dimension one, where the determinant condition suffices. The norm family and this argument include residue characteristic two. Thus both properties fail for all CI models.

4.6 CII. Self-adjoint J on a symplectic space

If $J^2 = 1$, take the symplectically orthogonal sign spaces V_+, V_- of dimensions $2a, 2b$, with $1 \le a \le b$. Their full fixed product $\mathrm{Sp}_{2a} \times \mathrm{Sp}_{2b}$ embeds by blocks, without a kernel, has center μ_2^2 and is simply connected. This is one rational conjugacy class for each unordered pair. If $E = k[J]$ is a quadratic field, self-adjointness makes ω balanced over E. The trace pairing, as in AII, writes $\omega = \mathrm{Tr}_{E/k}\,\Omega$ for a nondegenerate E-alternating form Ω. Hence $\dim_E V = 2a$ is even; the actual fixed group is $\mathrm{Res}_{E/k}\,\mathrm{Sp}_{2a}$, with center $\mathrm{Res}_{E/k}\,\mu_2$, faithfully embedded and simply connected. The symplectic basis theorem over E gives one rational conjugacy class for each E. These are the equal-block involutions whose blocks are interchanged by Galois.

Here are rational flags and their whole stabilizers. For split J, choose matched symplectic planes in the two sign spaces, with symplectic bases (e_i, f_i) and (e_i', f_i'). In the ith four-space take the isotropic plane U_i with ordered basis $(e_i + e_i', f_i - f_i')$. The alternating form $\beta(x, y) = \omega(x, Jy)$ is nondegenerate on U_i; $U_i \oplus JU_i$ is the whole four-space. Form an isotropic full flag by concatenating these ordered pairs, followed by a complete isotropic flag in the excess symplectic summand of dimension $2(b - a)$. In the field case choose E-planes for Ω, with basis (e_i, f_i) satisfying $\Omega(e_i, f_i) = \sqrt{d}$. Their k-spans U_i are isotropic for ω and symplectic for β, and again $U_i \oplus JU_i$ is the four-space. Concatenate their ordered pairs without any excess summand.

To see the stabilizer, the even stages up through $2a$, together with their J-images, distinguish nested nondegenerate symplectic sums of four-spaces. Their orthogonal differences recover each four-space. On such a four-space, an element centralizing J and preserving U_i is determined by an automorphism of (U_i, β); each such automorphism extends by commuting with J and preserves ω (use its values on U_i and JU_i). One further line must be fixed in each plane. The final summand contributes precisely its flag Borel. Thus the k-stabilizer is

$$(B_{\mathrm{SL}_2})^a \times B_{\mathrm{Sp}_{2(b-a)}}$$

in the split model, and $(B_{\mathrm{SL}_2})^a$ in the field model. Its codimension in H is $(a+b)^2$, the dimension of the symplectic flag variety (set $b = a$ in the field case). This orbit is geometrically open. Its stabilizer is connected split solvable, hence has trivial Galois H^1, proving rational transitivity.

For stability, let $r = gJg^{-1}J^{-1}$ be any semisimple symmetrization in $\mathrm{Sp}(V)(k)$. The variety

$$\mathcal{T} = \{c \in Z_{\mathrm{Sp}(V)}(r) : cJc^{-1} = rJ\}$$

is defined over k. We first produce a point over $\bar{k}$, where we may normalize $J^2 = 1$. The ± 1 eigenspaces of r are symplectic and are stable under J. On $V_1(r)$ use the identity. On each reciprocal pair $V_\lambda \oplus V_{\lambda^{-1}}$, $\lambda \neq \pm 1$, choose $t^2 = \lambda$ and multiply the two spaces by t, t^{-1} respectively. Their symplectic pairing makes this an isometry and J interchanges them, so this meets the transporter equation. On $V_{-1}(r)$, J and $-J$ have matching sign multiplicities because J and rJ are globally conjugate and have just been conjugated on the other summands. The two sign subspaces there are symplectic of equal dimension and can therefore be exchanged symplectically. This finishes the geometric construction.

The variety $\mathcal{T}$ is now a right torsor under $D = Z_{\mathrm{Sp}(V)}(r, J)$. Over $\bar{k}$ its group D is a product of symplectic groups. On $V_{\pm 1}(r)$ this follows by joint sign spaces. On a reciprocal pair commuting with J determines the action from the action on V_λ. The latter is an arbitrary isometry of the nondegenerate alternating form $\omega(x, Jy)$ there (self-adjointness of J gives alternation). These independent choices give all of D, with no finite central quotient or other components. Thus D/k is semisimple simply connected. The Kneser–Bruhat–Tits theorem gives $H^1(k, D) = 1$ for our local fields (Bruhat–Tits, *Groupes algébriques sur un corps local, chapitre III*, Thm. 4.7(ii); the residue field is finite). Hence the specified torsor has a k-point. This Galois descent of a geometric transporter provides an actual c for the involution-coboundary criterion. It covers all semisimple elements arising as symmetrizations, including nonsplit and singular cases. Both properties hold.

Sources for the local arithmetic. The local norm index used above is the quadratic instance of local reciprocity; see Yoshida, *Local class field theory via Lubin–Tate theory*, original proof. For the cyclic description of local division quaternion algebras see Voight, *Quaternion algebras*, Ch. 13, Thm. 13.3.11. For the torsor vanishing see Bruhat–Tits, Ch. III, Thm. 4.7(ii). None of the arguments for the positive cases uses a multiplicity-one statement.

5 The orthogonal rows on the simply connected cover

Throughout this section b denotes the polarization of a nonsingular quadratic form q, V is a split quadratic space of dimension N, and $\pi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ has kernel μ_2. Write $D = k^\times / k^{\times 2}$. Apart from the rank-one overlap, $N \geq 5$. In type D, a complete flag specifies both maximal isotropic extensions of its penultimate space, one in each component. We may use either prescribed component to describe a flag.

5.1 Models, descent, and centers

Every involution preserving the geometric vector representation up to isomorphism is induced by a rational similitude J, unique up to scalar. Indeed the intertwining line in $\mathrm{End}(V_{\bar{k}})$ is Galois-stable, has a k-basis, and the invariant symmetric form is unique up to scalar. Equivalently apply this argument to the vector representation composed with the automorphism upstairs. We have $J^2 = dI$ and $J^* = \epsilon J$ for $\epsilon = \pm 1$, since $J^* J$ is scalar. Conversely such a nonscalar similitude defines an involution of $\mathrm{SO}(V)$, lifting uniquely to its simply connected cover. These are intrinsic parameters even when distinct rational conjugacy classes give the same abstract groups: use every admissible J, identifying J' with $c\pi(g)J\pi(g)^{-1}$ for $c \in k^\times$, $g \in \mathrm{Spin}(V)(k)$ for inner conjugacy by that group. Replacing $\pi(g)$ by a rational similitude gives isomorphisms of pairs. Conjugacy under a larger group is unnecessary for the conclusions.

For the self-adjoint case with d square, normalize J to act by the two signs on the orthogonal decomposition $V = A \perp D_0$, $a = \dim A \leq b_0 = \dim D_0$. The connected orthogonal image is $M = \mathrm{SO}(A) \times \mathrm{SO}(D_0)$, whereas the full fixed group in $\mathrm{SO}(V)$ also allows paired negative determinants. Upstairs the full fixed group is

$$H = \pi^{-1} M = (\mathrm{Spin}(A) \times \mathrm{Spin}(D_0))/\Delta\mu_2.$$

Multiplication in the even Clifford algebra has kernel $\{(1,1),(-1,-1)\}$ and covers M. For $a=1$ this formula reads $H=\mathrm{Spin}(D_0)$, with $\mathrm{Spin}_1=\mu_2$. The preimage is connected, and the commutator with the lifted involution on it has values in μ_2, so is constant. Connectedness of the full fixed group for an involution of a simply connected semisimple group in characteristic zero (Steinberg) excludes further components. For any other cases below the same connectedness and dimension argument applies. Rationally,

$$\pi(H(k)) = M(k) \cap \ker(\mathrm{sn}_V),$$

with sn_V the ordinary spinor norm. For split blocks this means the product of the two spinor norms is trivial, including when neither norm is trivial.

When $J=J^*$ and d is nonsquare the two geometric eigenspaces are exchanged, so $N=2a$. Put $E=k(\sqrt{d})$ and let $\sqrt{d}$ act as J. The form on $V=E^a$ is the trace of a nonsingular E-quadratic form Q (use the perfect trace pairing and self-adjointness). All Q with the required split trace form are allowed. Now $M=\mathrm{Res}_{E/k}\mathrm{SO}(Q)$ and $H=(\mathrm{Res}_{E/k}\mathrm{Spin}(Q))/\Delta\mu_2$, by the preceding Clifford calculation after base change. For both block models, the center is the product of the two spin centers modulo the displayed diagonal kernel, or the corresponding descended group in the exchanged case. For clarity, $Z(\mathrm{Spin}_m)$ is μ_2 in odd dimensions $m\geq 3$, the full spin torus for $m=2$, and has geometric type μ_2^2 for $m\equiv 0 \bmod 4$, μ_4 for $m\equiv 2 \bmod 4$, $m\geq 4$. Its precise rational descent is the scalar signs and their multiples by the normalized Clifford volume element over $\bar{k}$, with the action inherited from the indicated quadratic form. The derived group is the image of the factors of dimension at least three, with the induced kernel; a two-dimensional factor is a torus.

When $J^*=-J$, put $N=2n$. If d is a square we have a polarization $V=U\oplus U^*$ and may use $J=\mathrm{diag}(1,-1)$. Then $M=\mathrm{GL}(U)$ and

$$H=\{(A,z)\in \mathrm{GL}(U)\times \mathbb{G}_m : \det A = z^2\}, \qquad H^{\mathrm{der}}=\mathrm{SL}(U).$$

Indeed the restriction of the spin cover to the diagonal maximal torus adjoins the half-sum $(e_1+\cdots+e_n)/2$ to the vector character lattice; the type A root subgroups lift identically. This gives the displayed cover and is compatible with the usual formula $\mathrm{sn}_V(A)=\det(A) \bmod k^{\times 2}$. For $E=k(\sqrt{d})$ a field, skew-adjointness gives a hermitian space (E^n,h) with q the underlying quadratic form. In a diagonal basis q is a sum of n scalar multiples of the norm form. Its signed discriminant is d^n, so n must be even for split V. Here $M=U(h)$ and

$$H=\{(A,z)\in U(h)\times U(1) : \det_E A = z^2\}, \qquad H^{\mathrm{der}}=SU(h).$$

Indeed conjugation exchanges the eigencharacters and sends the half-determinant character to its negative. In both formulas the covering kernel is (I,μ_2); the center consists of (tI,z) with $z^2=t^n$, t,z in the indicated torus. Every split underlying form and every marking/conjugacy class is included.

5.2 Orthogonal blocks

For decomposed blocks one description of the geometric open flag orbit is the condition that $q_A\circ \mathrm{pr}_A$ be nondegenerate on each isotropic step of dimensions $1,\dots,a$. At a maximal step in even dimension use the marked component. Orthogonalization gives generators u_i+v_i, with the u_i an orthogonal basis of A and the v_i a basis of $A'\subset D_0$ with opposite Gram matrix. The residual isotropic flag lives in $W=(A')^{\perp}\subset D_0$. The geometric stabilizer in M acts by simultaneous signs on u_i,v_i, with product of signs one, and by a Borel of $\mathrm{SO}(W)$ on W; take its full preimage in H. Indeed preservation of a nondegenerate orthogonal flag preserves its successive orthogonal lines. Conversely all the indicated maps preserve that flag. This locus is nonempty by decomposing $D_0=-A\perp W$ using Witt cancellation: W is split of dimension b_0-a, with the same anisotropic

line as V when that difference is odd. Over $\bar{k}$ ordinary orthogonal extension carries the indicated bases and residual flags to each other, with the prescribed maximal component fixing orientation when relevant. Equivalently the computed stabilizer has dimension $\dim M - \dim(G/B)$, so these flags lie in the unique open orbit. In the exchanged case use $b(x, Jy)$ on all steps through a; after scalar extension this is a nonzero multiple of the same projected form on an isotropic flag. The diagonal signs have product one in $\mathrm{SO}(Q)$ and there is no residual space. Taking the full inverse image retains all stabilizer components. Rational open flags exist by density on the split flag variety.

For $a \geq 2$ in the decomposed case, the square class of $q_A(u)$ for the first line $k(u + v)$ is invariant. As $-A \subset D_0$, every class represented by A occurs. A nonsingular binary subform represents all nonzero classes if hyperbolic, or a scalar times the norm subgroup of a quadratic extension. That subgroup has index two by local reciprocity. Since $|D| \geq 4$, at least two classes occur. Each condition gives a nonempty analytic open set of flags and consequently meets the geometric open orbit. For the exchanged pair use $b(x, Jx)$ on the isotropic line kx. It does not vanish generically. Replacing x by Jx multiplies this invariant by d, while Jx is still isotropic. Complete either line to a flag in the prescribed component, and use the same density argument. Both cases have several open rational orbits, hence are unstable.

In fact for decomposed $a \geq 2$ one may count exactly. Let $m(A)$ be the number of ordered a-tuples of nonzero square classes whose diagonal form is isometric to q_A. The answer is $m(A)$ when $b_0 - a \geq 2$, and $|D|m(A)$ for $b_0 - a = 0, 1$. To see this, first take orbits under $M(k)$. Matching tuples give isometries of the paired projected bases. Extend on the residual split space by matching its flags, using Witt bases. In even total dimension any resulting extension preserves the ambient maximal component, so has positive determinant (in the balanced case use the unique marked component of the graph of the anti-isometry $A \to A'$). In odd total dimension the residual dimension is odd and a factor $-I_W$ fixes its flag and adjusts the total determinant. A simultaneous sign change on one matched pair adjusts the two individual determinants to $+1$. Each admissible tuple occurs by the construction, choosing the residual maximal flag component or, when $W = 0$, composing the anti-isometry with a reflection if necessary. Thus there are $m(A)$ orbits under $M(k)$. Its spinor norm is onto D because $b_0 \geq 3$ and $H^1(k, \mathrm{Spin}(D_0)) = 1$. The simultaneous sign changes of product one contribute trivial spinor norm: each individual paired sign contributes $-q_A(u_i)^2$ and their number is even. The residual Borel contributes all of D as soon as W contains a hyperbolic plane, by its diagonal scaling, and contributes nothing for $\dim W \leq 1$. Taking the index gives the stated counts.

The exchanged balanced case admits a finite square-class count as well. Write $\eta = \sqrt{d}$. Form the tuples $(s_1, \ldots, s_a) \in D^a$ for which $Q \simeq_E \langle \eta s_1, \ldots, \eta s_a \rangle$. Under any such isometry take L to be the k-span of the displayed basis, and retain the tuple if L belongs to the prescribed maximal component. This is well defined: changing isometry multiplies by an E-orthogonal transformation, which has k-determinant $N_{E/k}(\pm 1) = 1$. Equivalently the component test uses the parity of $\dim(L \cap L_*)$ relative to a reference maximal space L_* in the prescribed component. The answer is the number of retained tuples times $|D|/2$. Indeed Q on an isotropic k-space L is η times a k-quadratic form. Its flags with all steps nondegenerate give an orthogonal basis unique up to rescaling each k-line. Nondegeneracy on L implies $L \cap JL = 0$, as $b(x, Jy)$ recovers this same form up to a nonzero factor. Thus this basis is an E-basis of V. Matching tuples give an E-isometry of the marked flags whose E-determinant can be made $+1$ by a sign change. The stabilizer is the determinant-one sign group described above. For u an E-reflection in a vector of Q-value c, its action over k is minus identity on the quadratic plane $\mathrm{Tr}_{E/k}(cX^2)$, with spinor norm $dN_{E/k}(c)$ modulo squares, by the determinant in the basis $(1, \eta)$. Hence on an even product the k-spinor norm is the norm of the E-spinor norm. The latter is onto $E^\times/E^{\times 2}$ since $a \geq 3$ and $H^1(E, \mathrm{Spin}(Q)) = 1$. The determinant-one sign group has trivial k-spinor norm: each negative sign acts on one hyperbolic k-plane, contributing -1; the signs occur an even number of times. This proves the additional factor $|D|/2$.

Now suppose $a = 1$, $V = ke \perp W$, $q(e) \neq 0$. Open first lines are $k(e + w)$ with $q(w) = -q(e)$. Their stabilizer in $H = \mathrm{Spin}(W)$ is $\mathrm{Spin}(W \cap w^{\perp})$. This acts transitively on the residual flags since $(e + w)^{\perp}/k(e + w) \simeq W \cap w^{\perp}$ is split and its rational Borels are conjugate. The open flag stabilizer is its Borel. Transitivity on the w follows from the geometric transitivity and $H^1(k, \mathrm{Spin}(W \cap w^{\perp})) = 1$ by Kneser's theorem; the dimension is $N - 2 \geq 3$, including the semisimple dimension-four spin group. In particular there is exactly one open orbit.

For stability, write $\pi(g)e = te + u$, $\pi(\sigma(g))e = te + u'$. Equality of the e-coordinates follows by the definition of the isometric reflection J and adjointness of $\pi(g)^{-1}$, and $q(u) = q(u')$. The orbit map $g \mapsto \pi(g)e$ identifies $G(k)/H(k)$ topologically with its image in the norm quadric: its fibers are the cosets and smoothness gives local sections. The image is the whole quadric by Kneser applied to H. If u is nonzero isotropic, a hyperbolic-plane cocharacter in $\mathrm{SO}(W)$ scales u by squares and has trivial spinor norm for those parameters. Thus te is in the closure of the orbit and is outside it. Such a double coset is not closed. If $u = 0$, $t = \pm 1$ and also $u' = 0$. If $q(u) \neq 0$, the equal-norm vectors in W are geometrically conjugate and the stabilizer $\mathrm{Spin}(W \cap u^{\perp})$ again has trivial H^1. Hence u, u' are $H(k)$-conjugate. Equality of the images modulo $H(k)$ proves stability on every closed double coset. In $N = 3$ the simply connected pair is the rank-one torus case, for which the open-orbit counts are $|D|$ for the split torus and $|N_{E/k}E^{\times}/k^{\times 2}|$ for a nonsplit torus. The projective line parametrizes k-lines in E: the norm class of a generator classifies those torus orbits. $N = 2, 4$ are outside absolute almost simplicity.

5.3 Even orthogonal groups with general linear geometric fixed group

For even n, take the maximal isotropic L of the complete flag in the component where $L \oplus JL = V$ generically. Such a component exists since geometrically one can take the graph of an invertible alternating matrix on U. The condition is open, so it has rational points. The form $\omega(x, y) = b(x, Jy)$ is nondegenerate alternating on L. Require each even flag step in L to be nondegenerate for ω. Any symplectic isomorphism $L \to L'$ extends uniquely to an isometry of V commuting with J by its action on JL. The resulting isometry lies in M, since the centralizer is GL_n over $\bar{k}$. Symplectic Gram–Schmidt acts transitively on the indicated subflags, with stabilizer a product of line-stabilizing Borels in the symplectic groups of the successive orthogonal two-planes. This proves transitivity for $M(k)$ on the displayed open locus. The stabilizer is inside $\mathrm{Sp}(L)$, which lifts to the spin cover by simple connectedness. Its spinor norm is trivial. Thus for $H(k)$ the exact number of open orbits is the size of $\mathrm{sn}_V(M(k))$. This is $|D|$ for split E and $|N_{E/k}E^{\times}/k^{\times 2}| = |D|/2$ for field E. For the latter assertion, the special unitary group lifts through the spin cover; multiplying by a rotation of an orthogonal E-line makes any unitary element special. On that line Cartan–Dieudonné writes a rotation as an even product of reflections, whose spinor norms generate exactly the classes of norms: a product of the reflections in 1 and x gives $N(x)$ modulo squares. Both counts exceed one. The actual flag stabilizer upstairs is the full inverse image of the product of Borels, hence includes the central kernel.

For odd n only split E occurs. Choose the component where a generic maximal L projects isomorphically to U and write L as graph of an alternating form of rank $n - 1$ on U. Require the penultimate flag space to be complementary to the radical R, and its even steps to be nondegenerate. Classification of alternating forms and the same Gram–Schmidt procedure make this a single $M(k)$-orbit. Its stabilizer is a product of the line-stabilizing symplectic Borels on the resulting two-planes and $\mathrm{GL}(R)$. The latter realizes all determinants, so the subgroup of square determinant is transitive as well. There is exactly one open $H(k)$-orbit; its stabilizer is the inverse image of the displayed subgroup.

Here is the closed-orbit argument for odd n. Write $j = J$, $j' = \pi(g)j\pi(g)^{-1}$, and $r = s(g)$. At a closed orbit jj' is semisimple by the semisimple symmetrization criterion. The common isometry group K_0 of j, j' equals the centralizer of $\pi(r) = j'j$ in M. We compute it over k, retaining singular summands. The self-adjoint operator $t = (jj' + j'j)/2$ commutes with j, j' and is semisimple.

Decompose by its irreducible polynomials. If $t = \pm 1$, semisimplicity gives $j' = \pm j$; the two j-signs are paired totally isotropic spaces, giving one ordinary GL_{m_i} factor for each nonzero such summand. Its action on $U = V^{j=1}$ is the standard one.

On a remaining factor write $F = k[t]$. Both operators are F-linear, and the F-valued symmetric form is recovered by the trace pairing. The relation $jj' + j'j = 2t$ forces the diagonal blocks of j' in the j-eigenspaces to be $t, -t$; its off-diagonal blocks multiply to $1 - t^2$, a unit. In suitable coordinates $j = \mathrm{diag}(1, -1)$ and $j' = \left(\begin{smallmatrix} t & 1 \\ 1-t^2 & -t \end{smallmatrix}\right)$ on a two-dimensional factor tensored with a multiplicity space. Both operators are skew-adjoint for the standard alternating matrix on that factor. They generate $M_2(F)$, so all invariant pairings on the tensor product come from this alternating matrix tensored with a pairing on the multiplicity space. Symmetry forces that pairing to be nondegenerate alternating. The exact isometry factor is therefore $\mathrm{Res}_{F/k} \mathrm{Sp}_{2m}$; it contributes $2m[F : k]$ to $\dim U$ and determinant one on U. This analysis proves

$$K_0 = \prod_i \mathrm{GL}_{m_i} \times \prod_j \mathrm{Res}_{F_j/k} \mathrm{Sp}_{2m_j} .$$

There is at least one general linear factor since n is odd. The inverse image of K_0 in H adjoins z with $z^2 = \prod_i \det A_i$. It is connected with simply connected derived group (the special linear and symplectic factors) and split torus quotient. In fact quotienting by those factors leaves coordinates (u_i, z) with $z^2 = \prod_i u_i$, allowing one u_i to be eliminated. The full inverse image centralizes r: the commutator is a morphism from a connected group to μ_2, trivial at identity. Consequently $Z_H(r)$ is exactly that inverse image, and its Galois H^1 is trivial by Hilbert 90 and uniqueness of symplectic forms, also after restriction of scalars. The stabilizer of the double action at g is $Z_H(r)$ by first projection. The geometric transporter and the calculation of its Galois cohomology in Section 1.1 give $\sigma(g) \in H(k)gH(k)$. This calculation includes every semisimple r even when t has field factors or sign eigenspaces. No regularity or Gelfand-property assumption enters.

5.4 Triality and low-rank identifications

For D_4 the split pinning gives all six diagram automorphisms over k. If an involution does not preserve the vector central kernel, its diagram image is a transposition, which fixes a different outer node; conjugate it by a pinned diagram automorphism so as to fix the vector node. This transports the actual involution, full fixed group, rational flags and closed cosets. Conversely transporting every marked rational vector model above gives all rational twists of the other embeddings. Table 1 footnotes (3)–(5) consequently give two transported B_3 embeddings from the $(7, 1)$ case and two transported B_2A_1 embeddings from the $(5, 3)$ case. They respectively have both properties true and both false. For $(6, 2)$, the derived root subsystem D_3 uses three coordinate directions, and is a Levi A_3 obtained by deleting the vector terminal node in D_4 (after Weyl conjugation). Permuting terminal nodes gives the two other A_3 Levis, the two GL_4 embeddings. The conclusions agree for all three. More generally for even n the two DIII embeddings in Table 1 footnote (2) are exchanged by a reflection defined over k on the split vector space; pull the actual J through this isomorphism. For $N = 5$ and 6 the split isomorphisms $B_2 = C_2$ and $D_3 = A_3$ give overlapping families; the direct spin proofs already apply. In particular Spin_6 with split DIII has the $(3, 1)$ special linear block type and both properties hold.

For the two transported classes in C13, the map to the displayed SO_8 is the faithful eight-dimensional spin representation of Spin_7, so it is an isomorphism onto the listed subgroup, also on rational points for the descended embedding. For C12 the tensor representation on $4 \otimes 2$ has exactly the diagonal sign as kernel on $\mathrm{Sp}_4 \times \mathrm{SL}_2$; it too is faithful on the transported fixed group $(\mathrm{Sp}_4 \times \mathrm{SL}_2)/\Delta\mu_2$. These kernel computations persist under descent. In both rows the μ_2 of the *chosen vector projection* of Spin_8 meets the upstairs fixed subgroup trivially, so the full inverse image of the table subgroup is its connected lift times that μ_2. The two triality-transformed involutions move the vector kernel and therefore do not descend to involutions of this particular

vector quotient. Their classification here is on Spin_8 itself. Even for these faithful images a rational point of a central product need not be represented by two rational factors.

6 Exceptional groups over the local field

In this section k is any finite extension of $\mathbb{Q}_p$, including $p = 2$, and the ambient group is split and simply connected. The three positive rows below are settled on every semisimple stratum. The other nine fail already on flags. Each verdict applies to every rational involution of the indicated geometric type.

Models and parameters. Fix a split pinning and Bourbaki numbering. Write I_j for conjugation by $\omega_j^\vee(-1)$ in the *adjoint* torus, so its sign on the root space of $\sum n_i\alpha_i$ is $(-1)^{n_j}$. Write D for pinned diagram reversal of E_6 and C for the Chevalley involution (inversion on the pinned torus and interchange of opposite simple root subgroups with the Chevalley signs). Each row gives a reference k-involution. The quotient and center columns describe the actual geometric groups, to be descended by the rational parameter below. The integer d is the number of independent μ_2 factors in the stabilizer's torus on an open flag. The last column gives both properties in the order p-stability, stability. Use $S_m = \mathrm{SL}_m$, $P_m = \mathrm{Sp}_m$, $T = \mathbb{G}_m$. A subscript 2 on a product denotes its diagonal order-two quotient as specified below.

G	type	$H_{\bar{k}}$	$Z(H)_{\bar{k}}$	(θ_0, d)	(p, s)
G_2	G	$(S_2 \times S_2)_2$	μ_2	$(I_2, 2)$	$(-,-)$
F_4	FI	$(P_6 \times S_2)_2$	μ_2	$(I_1, 4)$	$(-,-)$
F_4	FII	Spin_9	μ_2	$(I_4, 0)$	$(+,+)$
E_6	EI	P_8/μ_2	1	$(C, 6)$	$(-,-)$
E_6	EII	$(S_6 \times S_2)_2$	μ_6	$(I_2, 2)$	$(-,-)$
E_6	EIII	$(\mathrm{Spin}_{10} \times T)/\mu_4$	T	$(I_1, 0)$	$(+,+)$
E_6	EIV	F_4	1	$(D, 0)$	$(+,+)$
E_7	EV	S_8/μ_2	μ_4	$(I_2, 7)$	$(-,-)$
E_7	EVI	$(\mathrm{Spin}_{12} \times S_2)_2$	μ_2^2	$(I_1, 2)$	$(-,-)$
E_7	EVII	$(E_6^{\mathrm{sc}} \times T)/\mu_3$	T	$(I_7, 1)$	$(-,-)$
E_8	EVIII	HSpin_{16}	μ_2	$(I_1, 8)$	$(-,-)$
E_8	EIX	$(E_7^{\mathrm{sc}} \times S_2)_2$	μ_2	$(I_8, 2)$	$(-,-)$

The root signs identify each inner embedding, independent of the name assigned to its factors. In EIII and EVII the full center of the simply connected derived factor is identified with its copy in T (the graph of the inverse embedding, up to choice of generator). In EII, FI, and type G the common sign is $(-I, -I)$. In EIX it is $(z, -I)$ with z the nontrivial central element of E_7^{sc}. In EVIII divide by the order-two kernel of the selected half-spin, not the vector representation. In EVI the 56 of E_7 on the covering product is $(12 \otimes 2) \oplus S^+$, where S^+ is a half-spin of dimension 32. Divide by $(z_+, -I)$, where z_+ is killed by S^+ and acts on the vector representation as $-I$. Interchanging the half-spin naming convention exchanges the labels. These statements follow also by taking the coroot sublattice of the even roots in $Q^\vee(G)$ for each I_j. For the semisimple inner rows, the saturation of the fixed coroot lattice is $Q^\vee(G)$, so the central quotient has order the coefficient of the deleted node in the coroot of the highest root. The two Levi rows use the original cocharacter lattice and the primitive connected center instead. For the geometric classification and fixed subgroups compare Rubio, Table 1 and Section 4.6; his displayed D_8 quotient must specify the half-spin kernel. Steinberg's connected fixed-point theorem applies upstairs.

For the rational conjugacy parameters use the group $J = D_0$ of Section 1.3 for the displayed k-defined reference involution. Its action descends each displayed group, including its center, its finite central identifications, and its embedding. For clarity, $J = H_0$ in G_2, F_4, E_8 and in

EII, EIII and EVI. For inner E_6, a central symmetrization of order dividing two is trivial. In EVI the geometric normalizer has no extra component (the EVI statement of Rubio, Theorem 4.20 and Section 4.7, where pleasantness means that every inner automorphism commuting with the involution is represented by H_0). In EI, EIV one has $J = \mu_3 \times H_0$, since θ_0 inverts μ_3 and squaring there is an isomorphism. In EV and EVII $J/H_0 = \mu_2$; in the maximally inverted torus below the nontrivial ambient center belongs to A (in EVII it is $(-1,-1,-1)$ in the three displayed coroot coordinates), and a geometric square root in A represents the extra component. In these two cases the displayed full $H^1(k,J)$ parameter includes both the neutral and non-neutral component-cocycle regimes; no evaluation of their cardinalities is needed for the stated uniform verdict. For each semisimple H_0 with $J = H_0$ and covering kernel μ_2 there are exactly two rational classes by local abelianization (the geometric fundamental group is $\mathbb{Z}/2$). For G_2 these are the central products of the two copies of $\mathrm{SL}_1(Q)$ with Q the split or division quaternion algebra, via the Cayley–Dickson decomposition of the split octonions. FII and EIII have one rational class, by simply connected vanishing and, for EIII, Hilbert 90 for its split one-dimensional abelianization. EIV has the $k^\times/k^{\times 3}$ rational classes arising from the central factor of J; they are transported by $G_{\mathrm{ad}}(k)$ and have the same verdict. EI has the analogous independent cubic parameter in addition to the two classes for P_8/μ_2. The local cohomology statement used here is Borovoi, Theorem 3.2 and its cited local abelianization theorem.

The open stabilizer and the arithmetic obstruction. In an adapted maximal torus T_0 let w be the action of θ on $P = X^*(T_0)$. Choose the chamber by ordering restrictions first on the anti-invariant part and then on the zero subsystem R_0. The Borel intersection with its transform has precisely the positive zero-root subgroups. In the models used below θ is the identity on the zero roots' root spaces; this follows either from the pinning calculation or from $2\dim H - \dim G = \mathrm{tr}(w) + |R_0|$. This Borel has open H-orbit by the dimension formula, and its stabilizer is the semidirect product of T_0^θ and the positive zero-root unipotent group. Its torus character group is $P/(1-w)P$. The rational version at any rational open flag has a split maximal k-torus T_0 inside the two Borels as in the flag argument. Hence the diagonalizable fixed group there is split, and its finite factors are μ_2^d. The parameter d is geometric and unchanged by every twist. Maximality of the displayed anti-invariant tori can also be seen in the Lie algebra: the zero-root spaces are fixed pointwise, so the minus eigenspace commuting with the torus is exactly its own Lie algebra. In particular $d = 0$ gives precisely one orbit by Hilbert 90 and split unipotent cohomology.

If $d > 0$, choose a direct factor $F = \mu_2$ of the fixed torus and include it in the flag stabilizer. Its H^1 injects in the stabilizer's H^1 by the torus projection and the direct factor retraction. In each semisimple negative row the universal cover of H has kernel μ_2. Its inverse image of F is a commutative group scheme (over $\bar{k}$, a central extension of a cyclic group by a cyclic group of order two). Naturality of the connecting map gives a *homomorphism* on this factor

$$k^\times/k^{\times 2} = H^1(k,F) \longrightarrow H^2(k,\mu_2) = \mathrm{Br}(k)[2].$$

The target has order two and the source has order at least four. A nonidentity element in its kernel maps to the base point of $H^1(k,H)$ by exactness and simply connected vanishing. In EVII the fixed group's derived group is simply connected; its one-dimensional abelianization is either split or a quadratic norm-one torus. Its H^1 has size at most two and the composite from $H^1(k,F)$ is a homomorphism. Its kernel again maps to the base point in $H^1(k,H)$. Thus in all nine rows at least $|k^\times/k^{\times 2}|/2$ distinct open orbits occur. This argument works for the twisted groups themselves and includes every dyadic field.

Computing d. For G, FI, EI, EV and EVIII one may take $w = -1$ by using the Chevalley representative over the algebraic closure. Its fixed Lie algebra has dimension $(\dim G - \mathrm{rank}\, G)/2$, identifying the indicated type in each ambient group, and its adapted Borel and opposite have

trivial unipotent intersection. In EII take a Chevalley representative of the longest Weyl element; its square is central and it acts as negative diagram reversal on the six fundamental weights. The two unswapped nodes give the two factors μ_2. Its fixed algebra has dimension 38 and hence type A_5A_1. In EIV the positive diagram permutation gives no torsion. For FII, on a maximally inverted torus the action is reflection in a short root, Weyl-conjugate to sign change of e_1. On $P(F_4) = \mathbb{Z}^4 + \mathbb{Z}(1/2, 1/2, 1/2, 1/2)$ the image of $1 - w$ is the primitive lattice $\mathbb{Z}e_1$.

For EVI and EIX use the E_8 lattice $D_8 \cup (D_8 + (1/2, \dots, 1/2))$ with its norm-two roots. For E_7 take the roots perpendicular to $e_7 + e_8$. The product of the four root Weyl representatives for $e_1 \pm e_2, e_3 \pm e_4$ negates the first four coordinates. Its square corresponds to $2(e_1 + e_3)$ evaluated at -1, hence is trivial. It fixes the zero-root subgroups on the other coordinates, since all four root SL_2's commute with those subgroups. The respective fixed algebra dimensions are 69, 136, identifying EVI, EIX. For E_8, $P = Q$, and for E_7 the weight lattice is the orthogonal projection of the unimodular E_8 lattice to the root hyperplane (its root sublattice is primitive). In each case the first four coordinates run over all integral or all half-integral quadruples. The pure first-four-coordinate vectors in P form D_4: integrality against roots $e_i \pm e_5$ forces integral coordinates and the half-roots force even sum. Thus the saturation quotient is $D_4/(2\mathbb{Z}^4 + \mathbb{Z}(1, 1, 1, 1))$ of order four and exponent two.

For the two Levi rows simultaneously Cayley-transform the root tori for the following orthogonal roots in Bourbaki simple-root coordinates:

$$\begin{array}{c|l} \text{EIII} & (1,2,2,3,2,1),\ (1,0,1,1,1,1) \\ \text{EVII} & (2,2,3,4,3,2,1),\ (0,1,1,2,2,2,1),\ (0,0,0,0,0,0,1). \end{array}$$

Each lies in the negative grading. In each commuting root SL_2 take the torus with matrices $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, $a^2 - b^2 = 1$, on which the grading acts by inversion; it is k-split. Under Cayley coordinates $w = \prod s_{\beta_i}$. The zero subsystems are A_3 and D_4, respectively, as root restrictions verify. Pairing weights with these coroots surjects onto $\mathbb{Z}^r$ by a unit maximal minor of the displayed rows, so $(1 - w)P = \sum \mathbb{Z}\beta_i$. In fundamental-weight coordinates the generating rows are respectively

$$\omega_2, \quad \omega_1 - \omega_2 + \omega_6; \qquad \omega_1, \quad -\omega_1 + \omega_6, \quad -\omega_6 + 2\omega_7.$$

This proves the stated values 0 and 1 on the *simply connected* lattice.

Closed semisimple strata. The following transporter calculation accounts for arbitrary rational centralizers. Geometrically every semisimple symmetrization r is H-conjugate to a member of a maximal θ-split torus A, by Helminck–Schwarz, Theorem 3.4(4),(6),(10). Taking a square root there gives u commuting with r and satisfying $u\theta(u)^{-1} = r$. Then $g = uh$ for $h \in H(\bar{k})$, and $g^{-1}rg = h^{-1}rh$. The transporter between these two rational elements in H is a torsor for $Z_H(r)$. Its triviality yields a rational conjugator and hence equality of rational double cosets. A connected semisimple simply connected stabilizer has trivial $H^1(k, -)$ for *every* k-form. The same holds for a connected group with simply connected derivative and split torus abelianization by exactness and Hilbert 90.

FII. Use the F_4 roots $\pm e_i$, $\pm e_i \pm e_j$, $(\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2$. The grading fixes the integer roots B_4 and negates the half-integer roots. The ambient cocharacter lattice is D_4, also the coroot lattice of the integer B_4, giving $H = \mathrm{Spin}_9$. For the negative short root $\beta = (e_1 + e_2 + e_3 + e_4)/2$, Cayley-transform its root torus, writing $a = \beta^\vee(z)$ after transformation. The multiplicities of restrictions $0, \pm 1, \pm 2$ are $18, 8, 7$ respectively (the latter two for each sign). The trace argument above shows that θ fixes the zero subsystem B_3 pointwise. In coordinates taking β to e_1, its coroot lattice is $D_4 \cap e_1^\perp$, so its derived group is Spin_7. For $z \neq \pm 1$, $Z_G(a)$ is the product of this root subgroup and the adapted torus. As the subgroup is pointwise fixed, its fixed group is the product with the fixed torus, connected of rank three by the $d = 0$ calculation. Thus $Z_H(a) = \mathrm{Spin}_7$. At $z = -1$ the Cayley transform leaves the central element of the root SL_2 unchanged. Its centralizer

in the original B_4 has the long D_4 roots, again with full coroot lattice D_4, and is connected by the semisimple centralizer theorem for Spin_9. Thus it is Spin_8. At $z = 1$ it is Spin_9. The transporter argument proves stability.

EIV. Equivalently, take the norm-isometry group of the split Albert algebra J and the involution $g \mapsto (g^*)^{-1}$ for the Jordan trace form. Its fixed group is $\mathrm{Aut}(J) = F_4$, the stabilizer of 1; geometrically this gives the same involution. The equivariant identification $gH \mapsto g(1)$ identifies the stabilizer $Z_H(s(g))$ with the F_4-stabilizer of $g(1)$. For a diagonal Jordan frame the quadratic representations $P(u)$ for diagonal invertible u of determinant one form a two-dimensional anti-invariant torus: $P(u)$ scales diagonal Peirce lines by u_i^2, off-diagonal Peirce spaces by $u_i u_j$, and is self-adjoint. This torus is maximal since the EIV Cartan rank is two (equivalently, the two difference cocharacters of the diagram involution have the 24 fixed zero roots, by the same trace criterion). Therefore over $\bar{k}$ every semisimple symmetrization corresponds to a diagonal Jordan element, by taking a square root in that torus. With three distinct eigenvalues its stabilizer fixes the three spectral idempotents and is Spin_8 by Thakur, Proposition 4.1. For two eigenvalues it fixes the one distinguished primitive idempotent and is Spin_9 by Karasiewicz–Savin, Lemma 5.8, citing Springer–Veldkamp 7.1.3. At a scalar it is F_4. All have vanishing local H^1 in every form. This establishes stability without any rational diagonalization hypothesis.

EIII. We give the rational-torus calculation because knowing only the geometric derived type does not suffice here. In the two β coordinates above the restrictions of the 72 roots are

$$0\ (12), \qquad \pm e_i\ (8), \qquad \pm e_1 \pm e_2\ (6), \qquad \pm 2e_i\ (1)$$

with multiplicity per weight. These coordinates are actual coordinates on A, because the two coroot vectors have a unit minor. Put $L_a = Z_G(a)$ and $J_a = Z_H(a) = L_a^\theta$ (this J_a is a stabilizer, distinct from the involution normalizer J above). Up to signed permutations, the following exhausts the equations $x = 1, y = 1, x^2 = 1, y^2 = 1, x = y^{\pm 1}$. A letter t in a row denotes a coordinate with no additional equalities.

a	$\mathrm{type}(L_a^{\mathrm{der}})$	J_a^{der}	$\dim J_a^{\mathrm{ab}}$
generic	A_3	SL_4	1
$(-1, t)$	$A_3 + A_1$	SL_4	2
$(1, t)$	A_5	SL_5	1
$(t, t),\ t \neq \pm 1$	D_4	Spin_7	1
$(-1, -1)$	D_5	Spin_8	2
$(1, -1)$	$A_5 + A_1$	SL_5	2
$(1, 1)$	E_6	Spin_{10}	1

Here are details for recovering the groups and their rational torus quotients from this finite table.

First L_a is connected by Steinberg. Its fixed group is $A[2](L_a^\theta)^\circ$ by the fixed-component theorem of Vust–Richardson (Helminck–Schwarz, Theorem 3.4(6)), since A is maximal θ-split in L_a. The generic $Z_H(A)$ is connected: the zero-root group A_3 is fixed pointwise, $Z_G(A)$ is its product with the adapted torus, and the fixed torus has $d = 0$. Since $A[2] \subset Z_H(A)$, every J_a is connected.

The first type column is computed by retaining roots whose displayed restricted character equals one. Generating the roots from the Bourbaki Cartan matrix gives the zero A_3 on nodes $3, 4, 5$. Its coroot lattice is visibly saturated in $Q^\vee(E_6)$, giving SL_4. At $(-1, t)$ the additional rank-one block has an involution with torus fixed group. At $(1, t)$ the A_5 factor has θ-split rank one (the two-dimensional A projects with one-dimensional kernel to this factor); its fixed derived type is A_4 by the simple involution classification. This A_5 is the derived subgroup of a torus centralizer, hence simply connected in G; its standard block SL_5 embeds faithfully. At (t, t) the rank-one θ-split D_4 has fixed group of derived type B_3. It contains the preceding SL_4 of the same rank. On the maximal torus of this SL_4, its coroot lattice already fills all cocharacters in its span in the ambient torus, so the inclusion lifted to the simply connected B_3 shows the derived group is Spin_7.

At the isolated sign pairs undo both Cayley transforms (central elements of each root SL_2). Inside the original split Levi retain roots of even n_1 with $a^\alpha = 1$. At $(-1,-1)$ the simple coroots may be taken as $\alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee, \alpha_5^\vee$; at $(1,-1)$ take $\alpha_3^\vee, \alpha_4^\vee, \alpha_5^\vee, (\alpha_2+\alpha_4+\alpha_5+\alpha_6)^\vee$. They generate the 24 and 20 fixed roots respectively and each set has a unit minor. This proves the remaining derived isogenies. For the rank column, the central torus directions from $Z_G(A)$ fixed by w have rank four including the three A_3 directions. A coordinate equal to -1 adds one fixed torus direction via its root SL_2 and a coordinate equal to 1 adds one as well. A nonconstant collision $x = y$ adds zero (rank-one D_4/B_3). Subtraction of the displayed derived ranks gives the column. Equivalently, for $(1,t)$ the torus rank is five, for $(-1,t)$ five, at collisions four, and at isolated pairs six.

For descent, start with the split grading Levi, whose character ω_1 is k-defined, and the split A constructed above. On its generic centralizer the character is nonzero: the cocharacters in the adapted torus fixed by w are the kernel of

$$(v_1, \dots, v_6) \mapsto (v_2, v_1 - v_2 + v_6);$$

modulo the three A_3 coroots the generator is $\alpha_1^\vee - \alpha_6^\vee$. This direction commutes with the Cayley transforms, so ω_1 evaluates as 1 on it in the original Levi. Thus all rank-one abelianizations stay split in rational descent, since the restriction of this character of H is nonzero there. At isolated sign pairs the joint centralizer in the original split torus is split. Any rational member of such a geometric H-orbit twists that centralizer only by inner automorphisms, and consequently its abelianization remains split.

It remains to check $(-1,t)$, the varying rank-two quotient. The line from the generic centralizer survives, as witnessed by ω_1. Another line is the original fixed root torus of β_1, whose Cayley coordinate has value -1. These two lines are independent in the abelianization; modulo the A_3 coroots their directions are $\alpha_1^\vee - \alpha_6^\vee$ and $\beta_1^\vee$, and the latter has nonzero v_2. In descent, first conjugate a rational r geometrically to $a \in A$. The Galois descent maps on J_a may be represented, up to inner automorphisms, by transporters normalizing A: the two maximal θ-split tori in the relevant L_a are conjugate by J_a° (Helminck–Schwarz, Theorem 3.4(4)). The action on A is a signed permutation, since it preserves the restricted BC_2 roots. Galois fixes the constant -1; no permutation exchanges this coordinate with $t \neq \pm 1$. Coordinate inversions have representatives in the original fixed torus of their individual root SL_2 (use $\mathrm{diag}(i,-i)$ over $\bar{k}$); these normalize the anti-invariant torus and fix the original β_1 root torus pointwise since the two root groups commute. They act trivially on the generic central quotient since it carries ω_1. Two representatives of a given restricted Weyl element differ by $Z_H(A) \subset J_a$ and act identically on its abelianization. The two independent lines are therefore pointwise fixed and the rank-two torus is split. This proves $H^1(k, J_r) = 1$ for every rational semisimple r, including singular ones in nonsplit tori. The transporter proves EIII stability. Its rational involution class is unique, as established above.

The accompanying `exceptional_roots.py` checks all ten inner gradings, the FII strata, the integral minors, the restricted multiplicities and the seven EIII root subsystems. The reduction by the root equations, the fixed-group identifications and rational descent above are needed to turn the finite root checks into statements over k.

7 Claim ledger and extension register

Claim	Hypotheses, dependencies and evidence	Status
Geometric versus rational openness	Split flag variety; smooth orbit maps and rational lifting in §1. The split SL_2 torus has s open orbits.	ARGUED. Geometric uniqueness as a rational test is REFUTED.
Stable implies p-stable	Standing assumptions, formula (2), semisimple centralizers upstairs and the split Bruhat argument in §1.	ARGUED.

Claim	Hypotheses, dependencies and evidence	Status
Both properties, classical	Matrix adjoints, determinant and spinor norm, norm indices, Kneser vanishing applied to specified transporters. All parameters in the classification table.	ARGUED for every listed parameter.
Both properties, exceptional	Geometric involution classification, fixed torus lattices, local Brauer/norm obstruction, three positive stabilizer calculations.	ARGUED for all twelve geometric types and all their rational classes.
Extra split counterexample suggested in [N]	Under the standing assumptions the classification exhausts the symmetric entries and yields no p-stable unstable row.	ARGUED. Strict weakness is excluded under these hypotheses.
Multiplicity and regularity	[N, §3] also involves multiplicity of representations. The geometric transporter only proves double-coset invariance.	OPEN here; separate representation-theoretic assertions.
Calculation file	Exact integer root enumeration, multiplicities, lattice minors and seven EIII root-vanishing patterns with a proof of exhaustiveness in the text.	Reproducible certificates; no formal proof-assistant claim.

There are no unresolved equivalence cases in the declared split scope. Some numerical refinements are outside the verdict: in particular the two disconnected exceptional involution normalizers in types EV and EVII are given by exact cocycle parameter sets, without a separate numerical count of their conjugacy classes. The negative exceptional orbit counts use a uniform lower bound, sufficient to decide the singleton question.

Quasi-split nonsplit groups. These include the outer forms of types A, D, E_6 (including the trialitarian D_4 actions when applicable). A rational Borel exists, and all rational Borels are rationally conjugate. Formula (1), the closedness comparison, and (2) still apply. For the flag computation the Galois action on $X^*(T)/(1-\theta)X^*(T)$ must now be retained; even a connected torus can contribute H^1, and a geometric component need not split off over k. For the converse one needs the rational stabilizer/abelianization of each semisimple stratum of the relevant outer twist. These are the outstanding calculations for the quasi-split extension. [C]'s introduction proposes this stronger equivalence; the present split-ambient result does not settle it.

Groups with no rational Borel. There is no k-point on the variety of geometric Borels, and no quotient $G(k)/B(k)$ with such a k-Borel. A different question uses a minimal rational parabolic P, as in [C]. In the anisotropic case $P = G$, so this extended p-test is vacuous. Forms with positive relative rank but no Borel require a separate rational $G(k)/P(k)$ calculation, including anisotropic Levi factors and their action on stabilizers.

A verified example for the extension. [C, Ex. 5.30, Thm. 5.31] takes norm-one quaternions over $\mathbb{Q}_p$, $p \equiv 3 \bmod 4$. At $k = \mathbb{Q}_3$ let $D = (3,-1)_k$ with $i^2 = 3,\ j^2 = -1,\ ij = -ji$, $G = \mathrm{SL}_1(D)$, and $\theta = \mathrm{Int}(i)$. The algebra is division because $k(j)/k$ is unramified quadratic and 3 is not a norm (its valuation is odd). Thus G is anisotropic of type A_1, its rational group is compact, and H is the norm-one torus of $k(i)$. Here is a direct check of instability without importing [C]'s norm computation. Put $\ell = ij,\ \ell^2 = 3$, $t = (1+\ell)/(1-\ell)$ and $g = tj$. Both factors have reduced norm one. Since $\theta(t) = t^{-1}$ and $\theta(j) = -j$,

$$r = s(g) = -t^2, \qquad Z_G(r)(k) = k(\ell)^1.$$

Indeed $t = -2 - \ell$, so r is noncentral. On that centralizer θ is field conjugation, hence $s(c) = c^2$. The equality $c^2 = -t^2$ would put $\sqrt{-1}$ in the ramified quadratic field $k(\ell)$, whose residue field is $\mathbb{F}_3$; reduction of a square root of the unit -1 is impossible. The double coset is closed because

$H(k) \times H(k)$ is compact. Formula (2) proves it fails invariance. This gives rank one for the *minimal-parabolic extension* and no example at any rank for the split theorem. The mechanism is a nonsquare in an anisotropic centralizer that is invisible on a point flag quotient.

8 Text for the note

Rational stability over a split local group

Let k be a finite extension of $\mathbb{Q}_p$, let G/k be split, absolutely almost simple and simply connected, and let θ be a nontrivial k-involution. Set $H = G^\theta$, let B be a k-Borel, and put $\sigma(g) = \theta(g^{-1})$. Define p-stability by uniqueness of the open $H(k)$-orbit on $G(k)/B(k)$, with its local-field topology. The unique geometric open orbit can have several rational orbits.

Proposition. For the symmetric entries in Tables 1 and 2 of Knop–Röhrle, interpreted on the simply connected cover, p-stability is equivalent to σ-invariance of every closed rational $H(k)$-double coset. Both properties hold precisely in the following cases, up to the stated low-rank identifications and rational transports by triality:

- AII and the unequal-block AIII pairs;
- CII, including the equal-block form with fixed group $\mathrm{Res}_{E/k}\,\mathrm{Sp}_{2a}$ for quadratic E/k;
- orthogonal line stabilizers $(\mathrm{Spin}(V), \mathrm{Spin}(e^\perp))$ with $\dim V \geq 5$, and DIII in Spin_{2n} with odd $n \geq 3$;
- F_4/Spin_9, $E_6/((\mathrm{Spin}_{10} \times \mathbb{G}_m)/\mu_4)$, and E_6/F_4, with every rational involution of the indicated geometric type.

In every other included case the geometric open flag orbit splits over k, so both properties fail. In particular $(\mathrm{SL}_2, \mathbb{G}_m)$ has $|k^\times/k^{\times 2}|$ open rational orbits. The conclusion includes dyadic fields. It concerns the rational symmetric space with full fixed subgroup and does not supply a regularity or multiplicity-one theorem.

Sources and reproducibility

[N] Rui Chen, *Unramified multiplicity one symmetric pairs*, supplied October 2026 note, pp. 1–2. Definitions 2.1.1, 2.1.2, 2.5.1, Theorem 2.5.3 and §3.

[K] F. Knop and G. Röhrle, *Spherical subgroups in simple algebraic groups*, version 3, printed pp. 2–5, Tables 1–2, Lemma 2.7. Original paper.

[C] S. Carmeli, *On the stability and Gelfand property of symmetric pairs*, v3, July 2019. Introduction, §§4–5, Example 5.30 and Theorem 6.13. The author flags inaccuracies without singling out Theorem 5.27. Paper. Author's notice.

[AG] A. Aizenbud and D. Gourevitch, *Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet–Rallis' theorem.* In their terminology stability is goodness. See §2.3 and Proposition 7.2.1. Original paper.

[R] R. Rubio, *On the Gelfand property for complex symmetric pairs*, Table 1 and §4. Only geometric classification and normalizers are used. See also A. G. Helminck, *A classification of semisimple symmetric pairs and their restricted root system.* Rubio. Helminck.

[**HS**] A. G. Helminck and G. W. Schwarz, *Orbits and invariants associated with a pair of commuting involutions*, Theorem 3.4 for a single involution over an algebraically closed field. Paper.

Connectedness and local cohomology.

R. Steinberg, *Endomorphisms of linear algebraic groups*, §8. For the local vanishing used on possibly nonsplit semisimple simply connected groups, see Bruhat–Tits, Ch. III, Thm. 4.7(ii). The matrix and exceptional sections give the additional arithmetic and stabilizer references where used. Steinberg. Bruhat–Tits.

The companion file `exceptional_roots.py` runs under standard Python 3 with no additional packages. It prints its weights and subsystem types, asserts the integral certificates, and exits with an error if a check fails. The source file `research-report.tex` is standalone. The separate `note-insert.tex` is an editable copy of the preceding note section.