MOMENT MAPS AND GEOMETRIC INVARIANT THEORY

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1. Introduction

This is a study note for Chris Woodward's article [Woo10].

2. Background knowledge

In this section, I will recall some background knowledge from symplectic geometric and geometric representation theory to state the Kempf-Ness theorem.

Let X be a smooth manifold, a symplectic form on X is a closed non-degenerate two form $\omega \in \Omega^2(X)$, a *symplectic manifold* is a manifold equipped with a symplectic two-form. The term symplectic is the Greek translation of the Latin word *complex* and was used by Weyl to distinguish the classical groups of linear symplectomorphisms resp. complex linear transformations.

Definition 2.1. For any symplectic manifold (X, ω), let $\mathrm{Symp}(X, \omega) \subset \mathrm{Diff}(X)$ denote the group of symplectomorphisms and $\mathrm{Vect}^s(X) \subset \mathrm{Vect}(X)$ the Lie subalgebra of *symplectic vector fields* $v \in \mathrm{Vect}(X),\ L_v\omega = 0$.

Any smooth function $H \in C^\infty(X)$ defines a symplectic vector field $H^\# \in \mathrm{Vect}^s(X)$ by $\iota_{H^\#}\omega = d\,H$, and the image of $C^\infty(X)$ in $\mathrm{Vect}^s(X)$ is the space $\mathrm{Vect}^h(X)$ of *Hamiltonian* vector fields.

Definition 2.2. Let K be a Lie group acting smoothly on a manifold X, the action is called *symplectic* if it preserves the symplectic form, that is $k_X \in \mathrm{Symp}(X, \omega)$ for all $k \in K$. It is called *infinitesimally symplectic* if $\xi_X \in \mathrm{Vect}^s(X)$ for all $\xi \in \mathfrak{k}$ and *weakly Hamiltonian* if $\xi_X \in \mathrm{Vect}^h(X)$ for all $\xi \in \mathfrak{k}$.

A symplectic K-manifold is a symplectic manifold equipped with a symplectic action of K.

Definition 2.3. Let (X, ω) be a symplectic K-manifold, the action is called *Hamiltonian* if the map $\mathfrak{k} \to \mathrm{Vect}(X),\ \xi \mapsto \xi_X$ lifts to an equivariant map of Lie algebras $\mathfrak{k} \to C^\infty(X)$, such a map is called a comoment map. A *moment map* us an equivariant map $\Phi : X \to \mathfrak{k}^*$ satisfying

$$\iota_{\xi_X}\omega = -d\langle \Phi, \xi\rangle \ \forall\ \xi \in \mathfrak{k}$$

A Hamiltonian K-manifold is a datum (X, ω, Φ) consisting of a symplectic K-manifold (X, ω) equipped with an invariant closed two-form ω and a moment map Φ for the action.

Example 2.4. Let $K = V$ be a vector space acting on $X = T^*V$ by translation, after identifying $\mathfrak{k} \to V$ and $\mathfrak{k}^* \to V^*$, a moment map is given by the projection $X \cong V \times V^\vee \to V^*,\ (q, p) = p$, by the ordinary momentum, hence the terminology *moment map*.

Definition 2.5. Let X be a Hamiltonian K-manifold with the moment map $\Phi :\ X \to \mathfrak{k}^*$, a K-Lagrangian submanifold is a K-invariant Lagrangian submanifold on which Φ vanishes, let (X_j, ω_j, Φ_j) be Hamiltonian K-manifolds for $j = 0, 1$, a K-Lagrangian correspondence is a K-Lagrangian submanifold of $X_0^- \times X_1$.

Naturally, one want to study the quotient of a Hamiltonian K-manifold which should be an object in the symplectic geometry and satisfy a universal property for the morphisms in the equivariant symplectic category, however, it is easy to see that the most naive definition of the actual quotient doesn't work.

Definition 2.6. Let X be a Hamiltonian K-manifold with moment map $\Phi : X \to \mathfrak{k}^*$, we can define the *symplectic quotient*

$$X//K := \Phi^{-1}(0)/K$$

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Theorem 2.7. *(Meyer, Marsden-Weinstein) Let X be a Hamiltonian K-manifold, if K acts freely and properly on $\Phi^{-1}(0)$ then $X//K$ has the structure of a smooth manifold of dimension $dim(X)-2dim(K)$ with a unique symplectic form ω_0 satisfying $i^*\omega = p^*\omega_0$ where $i: \Phi^{-1}(0) \to X$ and $p: \Phi^{-1}(0) \to X//K$ are the inclusion and projection.*

Let's recall the Borel-Weil theorem: let G be a connected complex reductive group and let λ be any dominant weight for G and V_λ a simple-G module with highest weight λ, let P_λ^- be the opposite standard parabolic corresponding to λ, G/P_λ^- the generalized flag variety corresponding to λ, let $\mathcal{O}_X(\lambda) = G\times_{P_\lambda^-} \mathbb{C}_\lambda^-$, then we have the following theorem

Theorem 2.8. *Let $X = G/P_\lambda^-$ with λ a weight, then $H^0(X, \mathcal{O}_X(\lambda)) \cong V_\lambda$ if λ is dominant and vanishes otherwise.*

Example 2.9. For $G = SL_2(\mathbb{C})$, then $H^0(\mathcal{O}_X(\lambda))$ is the set of homogeneous polynomials in two variables of degree 2λ.

From the point of view of symplectic geometry, the Borel-Weil theorem says that the geometric quantization of a coadjoint orbit equipped with an integral symplectic form is a simple K-module. Indeed, let's denote Φ the moment map induced by the K-action on $\mathcal{O}_X(\lambda)$, then Φ maps X onto the coadjoint orbit $K\lambda$ through λ.

Let's now turn to the geometric invariant theory. Let G be a complex reductive group and X a G-variety, a *polarization* of X is an ample G-line bundle $\mathcal{O}_X(1) \to X$, its d-th tensor power is denoted by $\mathcal{O}_X(d)$, let

$$R(X) = \oplus_{d\geq 0}\ H^0(X, \mathcal{O}_X(d))$$

the action of X induces an action on $R(X)$, we denote $R(X)^G \subset R(X)$ the subring of invariants and by $R(X)^G_{>0}$ the part of positive degree.

Definition 2.10. A point $x \in X$ is

- *semistable* if $s(x) \neq 0$ for some $s \in R(X)^G_{>0}$.
- *polystable* if x is semistable and $Gx \subset X^{ss}$ (the semistable locus).
- *stable* if x is polystable and has finite stabilizer.
- *unstable* if x is not semistable.

Example 2.11. Suppose $G = \mathbb{C}^\times$ acts on $\mathbb{P}^2$ by $g[z_0, z_1, z_2] = [g^{-1}z_0, z_1, gz_2]$, note $R(X)_d$ is spanned by $z_0^{d_0} z_1^{d_1} z_2^{d_2}$ with $d_0 + d_1 + d_2 = d$ which has weight $d_0 - d_2$ under $\mathbb{C}^\times$. The invariant sections are $d_0 = d_2$. Hence we have

- x is semistable iff $x \neq [1, 0, 0],\ [0, 0, 1]$.
- x is polystable iff $x \in \{[0, 1, 0]\} \cup \{[z_0, z_1, z_2] \mid z_0 z_2 \neq 0\}$.
- x is stable iff $x \in \{[z_0, z_1, z_2] \mid z_0 z_2 \neq 0\ \}$.

Definition 2.12. *orbit-equivalence* is the equivalence relation on X^{ss} defined by $x_0 \sim x_1$ iff $\overline{Gx_0} \cap \overline{Gx_1} \cap X^{ss} \neq \emptyset$.

Theorem 2.13. *(Mumford) Let X be a projective G-variety equipped with polarization $\mathcal{O}_X(1)$, then*

- *There exists a categorical quotient $\pi:\ X^{ss} \to X//G$.*
- *$\pi(X^s) \subset X//G$ is open and $\pi|\ X^s:\ X^s \to \pi(X^s)$ is a geometric quotient.*
- *The topological space underlying $X//G$ is the space of orbits modulo the orbit-closure relation on X^{ss}.*
- *$X//G$ is isomorphic to the projective variety with the coordinate ring $R(X)^G$.*

3. Kempf-Ness theorem

Theorem 3.1. *Let K be a compact group and G its complexification, let V be a G-module equipped with a K-invariant Hermitian structure, let $X \subset \mathbb{P}(V)$ be a smooth projective G-variety, and let $\Phi: X \to \mathfrak{k}^*$ the Fubini moment map, then $\Phi^{-1}(0) \subseteq X^{ps}$ and the inclusion induces a homeomorphism $X//K \to X//G$.*

The proof uses the properties of a Kempf-Ness function for each $v \in V - \{0\}$

$$\psi_v : \ K\backslash G \to \mathbb{R}, \ [g] \mapsto \log||gv||^2/2$$

The proof of the Kempf-Ness theorem has a conceptual interpretation given by Guillemin-Sternberg in terms of geometric quantization.

Theorem 3.2. *(Quantization commutes with reduction) Let X be a compact Hamiltonian K-manifold with moment map $\Phi : X \to \mathfrak{k}^*$, polarization $\mathcal{O}_X(1) \to X$ and a compatible K-invariant Kahler structure J such that K acts freely on the zero level set $\Phi^{-1}(0)$ and let $R(X)_d$ denote the space of sections of $\mathcal{O}_X(d)$, then for each $d \geq 0$ there is a canonical isomorphism*

$$\rho : \ R(X)_d^K \to R(X//K)_d$$

Guillemin-Sternberg also proved "quantization commutes with reduction" for another class of Hamiltonian actions for which there exists a good quantization scheme, namely the cotangent bundles.

4. Moment polytopes

According to the work of Atiyah, Guillemin-Sternberg and Kirman the quotient of the image of the moment map is convex.

Let X be a Hamiltonian K-manifold with moment map Φ, then the *moment image* of X is $\Phi(X) \subset \mathfrak{k}$ and the quotient

$$\Delta(X) := \Phi(X) \subset \mathfrak{k}^*/K$$

can be identified with a subset of the convex cone $\mathfrak{t}^*_+ \cong \mathfrak{k}^*/K$.

Example 4.1. If $X = \mathbb{P}^1$ with $G = U(1)^2$ acts by the standard representation, then the moment image is the standard n-simplex

$$\Phi(X) = \{(\nu_1, \nu_2) \in \mathbb{R}^2_{\geq 0} \mid \nu_1 + \nu_2 = 1 \ \}$$

To describe the moment polytope, we will introduce the following definition

Definition 4.2. For $\lambda \in \mathfrak{k}^*$, we will call the quotient

$$X//_\lambda K := \Phi^{-1}(K\lambda)/K = (\mathcal{O}_\lambda^- \times X)//K$$

the *symplectic* quotient of X at λ.

Proposition 4.3. *Let X be a polarized projective G-variety and λ a dominant weight, then $R(X//_\lambda G)_d = Hom_G(V_{d\lambda}, R(X)_d)$ for any $d \geq 0$.*

Proof. This follows from the Borel-Weil theorem and the Kempf-Ness theorem:

$$\begin{aligned} R(X//_\lambda K)_d &= R(K\lambda^- \times X)_d^K \\ &= (V_{d\lambda}^\vee \otimes R(X)_d)^K \\ &= \mathrm{Hom}_K(V_{d\lambda}, R(X)_d) \end{aligned}$$

□

Lemma 4.4. *We have $\Delta(X) = \{\lambda \mid X//_\lambda \ K \neq 0\}$ is the set of λ for which the shifted symplectic quotient $X//_\lambda K$ is non-empty.*

This lemma tells us that the moment polytope $\Delta(X)$ is the classical analog of the simple modules appearing in a G-module.

Theorem 4.5. *Let K be a compact, connected Lie group and X a compact connected Hamiltonian K-manifold, then $\Delta(X)$ is a convex polytope. If K is abelian, then $\Delta(X)$ is the convex hull of image $\Phi(X^K)$ of the fixed point set X^K of K.*

References

[Woo10] Chris Woodward. Moment maps and geometric invariant theory. *Les cours du CIRM*, 1(1):55–98, 2010.