Integral relative Satake for the split torus in PGL_2

An explicit verification of BZSV (9.15)

Abstract

For every discrete coefficient ring with p invertible and a chosen square root of q, the basic orbit generates the relative spherical Hecke module freely. We identify the full derived fixed-locus comparison as an equivalence of complexes of ordinary spherical Hecke modules. In positive degrees both complexes reduce to copies of the two-term complex for multiplication by $q-1$. Projectivizing a column of the dual matrix computes the algebraic group invariants even in characteristic two. A chain homotopy identifies the different Hecke scalars on the two resolutions.

1 The statements and their normalizations

In [1, §9.4, pp. 200–201], equation (9.15) on printed page 201 is

$$C_c^\infty(X_{\mathfrak{f}};k)^{G_{\mathfrak{o}}} \stackrel{?}{=} \Big(k[g\in\check{G},\ m\in\check{M}:(g,q^{-1/2})\cdot m=m]^{\check{G}}\Big)^{/\!/}. \tag{9.15}$$

The superscript denotes the paper's shear. Its next paragraph says "On both sides we take invariants in the derived sense". An orbit through x contributes $H^*(G_{\mathfrak{o},x},k)$ in place of a single copy of k. It concludes, "The right hand side should also be interpreted in a derived sense." The preceding paragraph specifies the derived intersection of the graph and the diagonal. We use the split Hamiltonian model of [1, Ex. 8.4.4, pp. 185–186] over the coefficient ring. It distinguishes the polarization $\check{X}=\mathbb{A}^2$ from $\check{M}=T^*\mathbb{A}^2=\mathrm{Mat}_2$. Indeed the alternating form identifies the dual of the standard rank-two representation with itself over $\mathbb{Z}$.

Fix the equal-characteristic field of [1, §9.1] and the coefficient ring

$$F=\mathbb{F}_q((\varpi)),\qquad \mathcal{O}=\mathbb{F}_q[[\varpi]],\qquad R_0=\mathbb{Z}[1/p,s,s^{-1}]/(s^2-q),\qquad q=p^f.$$

In the statements R is any commutative R_0-algebra with the discrete topology. Write $m=q-1$. In the transcription of (9.15), m denotes a point of $\check{M}$.

Object	Convention
Automorphic action	$G=\mathrm{PGL}_2$, $T=\{\mathrm{diag}(a,1)\}$, $K=G(\mathcal{O})$, $(\rho(g)f)(x)=f(xg)$
Measure and Hecke	$\mathrm{vol}(K)=1$, $\mathsf{T}=1_{K\,\mathrm{diag}(\varpi,1)K}$, T sums over adjacent vertices
Dual parameters	$\check{G}=\mathrm{SL}_2$, $t=\mathrm{tr}(g)=z+z^{-1}$ on the torus, $\mathrm{Sat}_G(\mathsf{T})=st$
Derived invariants	Continuous cochains of profinite groups on the left, quasicoherent descent for the flat algebraic group scheme on the right
Analytic shear	A function of scaling weight w and cochain degree i moves to degree $i-w$ and gains internal super-parity w mod 2
Ambient category	$D(\mathcal{H}_R)$, the derived category of modules over the ordinary algebra $\mathcal{H}_R=R[\mathsf{T}]$

The normalized measure specifies finite coset sums wherever it is used on $G(F)$. A general R-valued Haar integral on all locally constant compactly supported functions with $\mathrm{vol}(K) = 1$ would require divisibility by more indices.

Theorem 1 (Ordinary relative Satake)**.** *The map $h \mapsto he_0$ identifies $\mathcal{H}_R$ with $\mathcal{M}_R = C_c^\infty(T(F)\backslash G(F)/K; R)$ as a module. The relative transform Sat_X identifies $\mathcal{M}_R$ with $R[t]$, takes e_0 to 1, and intertwines T with multiplication by st.*

Theorem 2 (Derived comparison)**.** *Let Z be the derived zero scheme of $(g - sI)M$ on $\mathrm{SL}_{2,R} \times \mathrm{Mat}_{2,R}$. There is an isomorphism in $D(R[\mathsf{T}])$*

$$\mathcal{A}_R = \mathrm{R}\Gamma_{\mathrm{sm}}(K, C_c^\infty(X(F); R)) \;\simeq\; \mathcal{B}_R = \mathrm{R}\Gamma([Z/\,\mathrm{SL}_{2,R}], \mathcal{O})^{/\!/}$$

whose map on H^0 is Sat_X. After a choice of generator of $\mathbb{F}_q^\times$, a common model is

$$R[t][0] \;\oplus \bigoplus_{j \geq 1} \left(\underset{2j-1}{R} \xrightarrow{m} \underset{2j}{R} \right). \tag{1}$$

The subscripts are cohomological degrees. The generator T acts by multiplication by st on the polynomial summand and may be represented by the scalar 2 on each pair. On the geometric spectral model it acts by $q + 1$ on these pairs. The isomorphism includes a homotopy relating the two actions. It is compatible with derived change of coefficient rings.

The theorem concerns these complexes and the ordinary spherical Hecke action. A comparison of categorical Frobenius traces or of module categories over a derived Hecke algebra would require additional data.

2 The tree and ordinary integral relative Satake

Put $\mathcal{H}_R = C_c^\infty(K\backslash G(F)/K; R)$. Write $G_F = \mathrm{PGL}_2(F)$ and $d(a) = \mathrm{diag}(a, 1)$. The map $G_F \to X(F)$ is onto since the fiber above a rational point is a T-torsor, and a $\mathbb{G}_m$-torsor over a field is the set of frames of a one-dimensional vector space. Its fibers are left $T(F)$-cosets. The identification is also topological. For instance, the quotient can be described by the ordered pair of distinct projective row lines of an invertible matrix. Choosing a nonzero coordinate in each row gives local continuous sections.

Use $(\rho(g)f)(x) = f(xg)$ and $(h_1 * h_2)(g) = \int h_1(u)h_2(u^{-1}g)\,du$, with $\mathrm{vol}(K) = 1$. On K-invariant functions the integrated action is the finite sum

$$(hf)(x) = \sum_{gK \in G_F/K} h(g)f(xg).$$

These normalized integrals on right K-invariant integrands are counting sums over G_F/K, so they make sense over the prescribed rings. The tree has as vertices the homothety classes of rank-two $\mathcal{O}$-lattices in F^2. Neighbors correspond to index-q sublattices between a lattice and its multiple by ϖ. The vertex $o = [\mathcal{O}^2]$ has stabilizer K, and its $q+1$ neighbors form $Kd(\varpi)K/K$ by reduction to lines over $\mathbb{F}_q$. Hence T evaluates a vertex function by summing its values at the neighbors of the evaluation vertex. Functions on $X(F)/K$ are $T(F)$-invariant vertex functions.

Distance from the apartment. Scale a lattice so that its second-coordinate projection is $\mathcal{O}$. It then has the unique form

$$L(d,u) = \mathcal{O}(\varpi^d, 0) + \mathcal{O}(u, 1), \qquad d \in \mathbb{Z}, \quad u \in F/\varpi^d\mathcal{O}.$$

The remaining choice of a unit scaling does not change the lattice. The apartment is the line of $L(d,0)$. If $u \neq 0$ in $F/\varpi^d\mathcal{O}$, put $k = v(u) < d$, independent of the lift. The path $L(d,u), L(d-1,u), \ldots, L(k,u) = L(k,0)$ consists of distinct adjacent vertices and meets the apartment first at its endpoint. Since this graph is a tree, the endpoint is the unique nearest apartment vertex, at distance $d-k$. The action of $d(a)$ sends $L(d,u)$ to $L(d+v(a), au)$. In particular $a = u^{-1}$ for any such lift sends the lattice to $L(d-k,1)$. The torus also acts transitively on the apartment. Distinct distances cannot be identified, as the torus preserves the apartment. The stated matrices g_n therefore give precisely one representative per orbit, with $g_0 \in K$.

The right stabilizer is $K_{x_n} = K \cap g_n^{-1} T(F) g_n$. In matrix representatives,

$$g_n^{-1} d(a) g_n = \begin{pmatrix} a & (a-1)/\varpi^n \\ 0 & 1 \end{pmatrix}.$$

If a scalar multiple belongs to $\mathrm{GL}_2(\mathcal{O})$, both diagonal entries are integral and their product is a unit. Thus a and the scalar are units, and the remaining condition is $a - 1 \in \varpi^n\mathcal{O}$. Conversely these conditions suffice. Conjugation by g_n identifies the stabilizer with $\mathcal{O}^\times$ at $n = 0$ and with $1 + \varpi^n\mathcal{O}$ at $n \geq 1$.

Adjacency and cyclicity. An apartment vertex has two apartment neighbors and $q-1$ outside. At any vertex of positive distance there is one neighbor toward its nearest apartment vertex and q further away. Thus, for $e_n = 1_{x_n K}$,

$$\mathsf{T}e_0 = 2e_0 + e_1, \qquad \mathsf{T}e_1 = (q-1)e_0 + e_2, \qquad \mathsf{T}e_n = qe_{n-1} + e_{n+1} \quad (n \geq 2).$$

In the group case K is transitive on each sphere about o, by the elementary divisors of a pair of lattices. Writing h_i for its indicator, the same neighbor count about o gives

$$\mathsf{T}h_0 = h_1, \qquad \mathsf{T}h_1 = (q+1)h_0 + h_2, \qquad \mathsf{T}h_i = qh_{i-1} + h_{i+1} \quad (i \geq 2).$$

Here h_0 is the convolution unit. These equations make T^i equal to h_i plus a linear combination of h_j for $j < i$. The h_i form the finite-support free basis of $\mathcal{H}_R$. Hence $R[U] \to \mathcal{H}_R$, $U \mapsto \mathsf{T}$, is an algebra isomorphism, since the finite triangular changes of basis have diagonal entries 1. Likewise $\mathsf{T}^i e_0$ equals e_i plus an R-combination of the preceding e_j. The orbit space is discrete and its characteristic functions e_j form the finite-support free basis. The triangular change of basis proves that $\Phi_R(h) = he_0$ is a Hecke-module isomorphism, including for rings with zero divisors.

Normalized transform. Use the upper triangular Borel B, $n(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, additive volume $\mathrm{vol}(\mathcal{O}) = 1$, and $v(\varpi) = 1$. The normalized constant term is

$$\mathrm{Sat}_G(h) = \sum_{k \in \mathbb{Z}} z^k s^{-k} \int_F h(d(\varpi^k) n(u))\, du.$$

Indeed the B-equivariant, right-K-invariant function f_z with $f_z(bk') = s^{-v(a)} z^{v(a)}$ for diagonal ratio a of b is well-defined by $G_F = BK$. This decomposition also follows from the above lattice

normal form. Such functions form a free line over $R[z, z^{-1}]$ determined by evaluation at o. Right convolution preserves this line and gives an algebra character. Evaluating at o gives the displayed sum since at fixed k the vertices $d(\varpi^k)n(u)K$ are indexed by u mod $\mathcal{O}$. At the neighbors of o one has $k = 1$, $u \in \varpi^{-1}\mathcal{O}/\mathcal{O}$ (there are q), or $k = -1$, $u = 0$ mod $\mathcal{O}$ (there is one). This gives

$$\mathrm{Sat}_G(\mathsf{T}) = qs^{-1}z + sz^{-1} = s(z + z^{-1}).$$

Set $t = z+z^{-1}$. Every invariant Laurent polynomial for inversion has symmetric coefficients, so is a sum of a constant and R-multiples of z^j+z^{-j} for $j \geq 1$. These sums are monic polynomials of degree j in t by their two-step recurrence. The highest Laurent exponent proves algebraic independence in this sense over any R. Thus the invariant algebra is $R[t]$, which the transform reaches since s is a unit. Section 3.2 proves integrally that the conjugation invariants on SL_2 are $R[\mathrm{tr}]$, and restriction to the diagonal sends tr to t.

Finally $\mathrm{Sat}_X = \mathrm{Sat}_G \Phi_R^{-1}$ is an isomorphism of Hecke modules. With $u = st$, its values $P_n = \mathrm{Sat}_X(e_n)$ begin

$$P_0 = 1, \quad P_1 = u - 2, \quad P_2 = u^2 - 2u - (q - 1), \quad P_3 = u^3 - 2u^2 - (2q - 1)u + 2q.$$

For $n \geq 2$ one has $P_{n+1} = uP_n - qP_{n-1}$. The entire ordinary argument is integral before normalization and uses only the valuation and residue cardinality. Its module structure is the convolution action.

3 Explicit automorphic and spectral complexes

3.1 Smooth cochains and the oriented-edge correspondence

Give R the discrete topology. Work in the abelian category of discrete R-modules with a continuous K-action, equivalently smooth $R[K]$-modules. Write $C^\bullet(L, V)$ for normalized continuous inhomogeneous group cochains of a profinite group L. They compute the right derived functor of L-invariants in this category. Indeed the homogeneous continuous bar resolution is split after forgetting the action by insertion of the identity, and its terms are continuously coinduced from the trivial subgroup. Such terms are acyclic for invariants. Continuous coinduction from the trivial subgroup is exact by lifting locally constant functions on a finite clopen partition, preserves injectives by adjunction, and its invariants recover the underlying module.

Each K-orbit in $X(F)$ is open because K is open in $G(F)$, and compact because K is compact. Consequently

$$C_c^\infty(X(F), R) = \bigoplus_{n \geq 0} C^\infty(K_{x_n} \backslash K, R).$$

A compact support has finite image in the discrete orbit space. Each summand is continuous coinduction of the trivial module from the closed subgroup K_{x_n}. For clarity, if J is closed in K, choose open normal subgroups N_i decreasing to 1 and forming a neighborhood basis. Choose representatives for the finite sets $J\backslash K/N_i$ compatible modulo N_i on refinement. This is possible by multiplying a representative on the left by an element of J. The inverse limit gives a continuous section of $K \to J\backslash K$. Using it, continuous coinduction is, on underlying modules, the functor $V \mapsto C^\infty(J\backslash K, V)$, hence is exact. It preserves injectives as right adjoint to exact restriction, and its K-invariants are the J-invariants of V by evaluation. These observations prove derived Shapiro

also for these non-open stabilizers. Finally K^i is compact, so each continuous cochain into a direct sum has image contained in a finite subsum. We obtain in $D(R)$

$$\mathcal{A}_R \simeq \bigoplus_{n\geq 0} C^\bullet(K_{x_n}, R).$$

Put $m = q-1$ and $U^n = 1+\varpi^n\mathcal{O}$ for $n \geq 1$. Invariants of a pro-p group on smooth R-modules are exact. Lift an invariant vector across a surjection and average the lift over a finite p-group quotient acting on its orbit, dividing by its order. Thus the positive-radius summands reduce to $R[0]$. For radius zero inflation from $C_m = \mathbb{F}_q^\times$ is a quasi-isomorphism, since the normal kernel U^1 has exact invariants and its invariants functor, right adjoint to exact inflation, preserves injectives. Choose a generator σ of C_m. Set $N = 1+\sigma+\cdots+\sigma^{m-1}$. Resolve R by $R[C_m]$ in each homological degree, with augmentation in degree zero, differentials $\sigma - 1$ in odd degrees and N in strictly positive even degrees. Indeed the kernel of $\sigma - 1$ consists of constant-coefficient sums RN, and the kernel of N consists of the coefficient-sum-zero elements, which form $(\sigma-1)R[C_m]$. This also covers $m = 1$. Applying $\mathrm{Hom}_{R[C_m]}(-, R)$ gives

$$C^0 = R \xrightarrow{0} R \xrightarrow{m} R \xrightarrow{0} R \xrightarrow{m} R \xrightarrow{0} \cdots .$$

Write D_j for its two-term subcomplex in degrees $2j-1, 2j$, with differential m. Thus $\mathcal{A}_R$ has an underlying complex $\mathcal{M}_R[0] \oplus \bigoplus_{j\geq 1} D_j$. In particular its strictly positive even cohomology is R/mR and its odd cohomology is $\mathrm{Ann}_R(m)$, each in every degree of the indicated parity.

Here is the chain-level Hecke action in this model. Identify each K-stabilizer with its conjugate in $T(F)$ at the corresponding tree vertex. For each orbit of directed edges above an input and output orbit, pull cochains back from the input stabilizer to the edge stabilizer, then transfer to the output stabilizer. This is the restriction and finite-fiber sum defining the double-coset correspondence. The pullback records an incident edge and the pushforward sums over the finite set of its translates at a fixed output vertex. Translation identifying two representative vertices conjugates stabilizers inside the abelian $T(F)$ and hence induces the identity in these identifications. There are two apartment directions at radius zero. Both edge stabilizers are the full $\mathcal{O}^\times$, giving $2\,\mathrm{id}$ on $C^\bullet$ there. On an edge between radii 0 and 1, the stabilizer is U^1, of index m at radius zero. In the outward direction the map is restriction $C^\bullet \to R[0]$, represented by the degree-zero projection. Indeed on normalized cochains after inflation the homomorphism $U^1 \to C_m$ factors through 1, and the restriction to 1 is exactly this projection. The comparison between the free periodic and bar resolutions can be chosen to be the identity in degree zero, by lifting the identity inductively using freeness and exactness. In the inward direction the transfer $R[0] \to C^\bullet$ is multiplication by m on the degree-zero inclusion, since its class is determined by its value in $H^0(C^\bullet)$ because $R[0]$ is free. For an edge from radius n to $n+1$ with $n \geq 1$ the index is q at the inner vertex and 1 at the outer vertex. Both endpoint complexes are $R[0]$, giving outward map 1 and inward map q. Therefore on the displayed direct sum one can take the ordinary adjacency action on $\mathcal{M}_R$ and the strict scalar 2 on each D_j. In particular this describes a representative for the whole endomorphism in $D(R)$, including its components incident to the base orbit.

3.2 The spectral complex and its grading

Write $V = R^2$, $G' = \mathrm{SL}_{2,R}$ and write the matrix $M = (v_1, v_2)$ in columns. Example 8.4.4, pp. 185–186, gives $(G', \check{M}) = (\mathrm{SL}_2, \mathrm{Mat}_2)$ with left multiplication and scalar grading on matrices. The invariant alternating form identifies V^* with V over $\mathbb{Z}$, so $\mathrm{Mat}_2 = V \oplus V$ is the Hamiltonian space

T^*V after this identification. The polarization $\check{X} = V$ accounts for one of the two columns. The fixed condition is $s^{-1}gM = M$, so the four equations can be taken to be the entries of $(g - sI)M$. Conjugation on g and left multiplication on the columns give the quotient action.

Compute the underlying derived module of the intersection by the Koszul complex

$$L = R[G'][v_1, v_2] \otimes \Lambda(\xi_1, \xi_2), \qquad d\xi_i = (g - sI)v_i.$$

Here both ξ_i are columns of two exterior symbols. This complex is obtained by pulling back the free Koszul resolution of the origin of affine four-space. Regularity is needed for the coordinate sequence defining that origin, which is regular over every base ring. No regularity of the pulled-back four equations is assumed. Derived quotient functions are computed by taking the normalized algebraic group cobar complex of L for G'. This is the cosimplicial complex of functions of the affine action groupoid, with r copies of G' in bar degree r. Affine descent for quasi-coherent complexes applies to the flat group scheme G'. Equivalently the augmented cofree comodule resolution is split upon forgetting the coaction, and cofree comodules on modules here are acyclic for invariants (the extra codegeneracy contracts their augmented cobar complex).

	bidegree	weight	Koszul	sheared	internal p	total p
$g_{ij},\ t = \operatorname{tr} g$	$(0,0)$	0	0	0	0	0
$(v_i)_h$	$\mathbf{e}_i$	-1	0	1	1	0
$(\xi_i)_h$	$\mathbf{e}_i$	-1	-1	0	1	1
$D = \det(v_1, v_2)$	$(1,1)$	-2	0	2	0	0
Γ below	$(1,1)$	-2	-1	1	0	1

Column degree counts a polynomial variable or an exterior symbol in its respective column. Thus the ordinary polynomial degree in M is 1 for $(v_i)_h$, 0 for $(\xi_i)_h$, 2 for D and 1 for Γ. The entries of g have M-degree zero. The scalar action on points gives weight -1 on linear functions. In BZSV §6.1 the weight-w piece is shifted by $\Pi^w[w]$, hence an element of unsheared cohomological degree k goes to $k - w$ and picks up internal parity w mod 2. Total parity includes cohomological parity. Bar degree adds to cohomological degree. The Frobenius refinement of the shift multiplies Frobenius by $q^{-w/2}$ when a Frobenius action is retained. In this calculation s^{-1} is already in the fixed equation. We take the intersection first and shear its functions by their weights. The Koszul symbols have square zero over all bases. In characteristic two we retain internal parity as an explicit grading, since the sign of the scalar -1 cannot detect it. The computation concerns the underlying derived modules. For a homotopy-coherent commutative algebra interpretation one may form the derived intersection by derived tensor product of simplicial commutative rings and take its underlying module. The Koszul resolution computes this module without identifying simplicial commutative rings with ordinary commutative dg algebras.

3.3 Calculation in every nonzero column degree

Write (a, b) for the column degree and, by interchanging the two columns, take $a \geq b$. Suppose $a \geq 1$. Replace the first homogeneous column by the universal line on $\mathbb{P}(V)$. Its Koszul term with i first-column symbols is replaced by

$$\mathcal{O}_{\mathbb{P}(V)}(a - i) \otimes \Lambda^i V^*, \qquad 0 \leq i \leq 2,$$

with coefficients in $R[G']$ and the second-column strand. The differential contracts with the section $(g - sI)|_{\mathcal{O}(-1)}$ of $V \otimes \mathcal{O}(1)$. Every twist is at least -1. The two-standard-affines calculation on $\mathbb{P}^1_R$

gives zero higher cohomology for these twists, $H^0(\mathcal{O}(d)) = \mathrm{Sym}^d V^*$ for $d \geq 0$ and $H^0(\mathcal{O}(-1)) = 0$. Therefore derived pushforward gives precisely the original first-column strand, also for $a = 1$. This construction is G'-equivariant.

Fixing the line Re_1 changes the quotient by G' into the quotient by its upper triangular Borel B. Indeed $\mathbb{P}(V) = G'/B$ as a flat quotient and an equivariant object on G'/B is reconstructed from its B-equivariant fiber by descent. The equations on this fiber are $g_{11} = s$, $g_{21} = 0$. On g_{11} invertible the coordinate ring is the Laurent polynomial ring in g_{11} with free variables g_{12}, g_{21}, because $g_{22} = (1 + g_{12}g_{21})/g_{11}$. Thus the two equations form a regular sequence there even for nonreduced R. Their common zero scheme lies in this chart, and the Koszul complex is contractible locally off their common zero scheme. It follows that its augmentation is a quasi-isomorphism onto the coordinate ring of

$$g = \begin{pmatrix} s & u \\ 0 & s^{-1} \end{pmatrix}.$$

This augmentation respects B, the coefficient line $\mathcal{O}(a)|_{Re_1}$ and the residual second-column differential. The second-column terms are free as modules in their polynomial and exterior generators, so tensoring with them preserves the augmentation, termwise and on the bounded total complex. Evaluation of the first column at e_1 and setting $\xi_1 = 0$ describe this map in that fiber.

Write $B = S \ltimes U$ with $S = \{\mathrm{diag}(z, z^{-1})\}$ and U the upper unitriangular group. It remains to take derived B-invariants of the degree-b strand in

$$\mathcal{O}(a)|_{Re_1} \otimes R[u, x, y] \otimes \Lambda(\alpha, \beta), \qquad d\alpha = uy, \quad d\beta = (s^{-1} - s)y,$$

where $v_2 = (x, y)^t$ and $\xi_2 = (\alpha, \beta)^t$ are coordinate symbols. The line has S-weight $-a$, u has weight -2, x, α weight -1, and y, β weight $+1$. These are Borel torus weights, distinct from the scalar weights used in shearing.

For clarity, the model for derived B-invariants is the S-weight-zero summand of the normalized U-cobar complex. Indeed quotienting by the normal U and then by S gives the quotient by the semidirect product. This can also be computed by the double bar resolution for these two successive invariants. Rational S-modules are graded R-modules and taking degree zero is an exact projection. In normalized U-bar degree r, the coefficient module is tensored with $(hR[h])^{\otimes r}$, since the counit of $R[U] = R[h]$ is evaluation at 0. The conjugation action of S gives h weight -2. The maximum possible weight is therefore $-a + b - 2r$. If $a > b$, the entire invariant complex is zero. For $a = b = j > 0$ only $r = 0$ survives and the weight-zero terms, suppressing the common fiber-line factor, are

$$R\, y^{j-1}\beta \xrightarrow{s^{-1}-s} R\, y^j.$$

The symbols y, β are U-fixed. Before shearing the degrees are $-1, 0$. After shearing they are $2j - 1, 2j$. Throughout this two-term model $s\ \mathrm{tr}(g)$ acts strictly by $q + 1$.

3.4 Weight zero and determinant classes

In bidegree $(0, 0)$ the complex is the conjugation cobar complex of $R[G']$. Filter its coefficient algebra by total polynomial degree in the four matrix entries A, B', C, E. The rewrite $AE = B'C + 1$ gives an R-basis consisting of the monomials not divisible by AE (division by a monic leading term). Hence the associated graded is the rank-at-most-one matrix algebra defined by $AE = B'C$. Its degree-d part is

$$\mathrm{Sym}^d V^* \otimes \mathrm{Sym}^d V.$$

Indeed substituting $v\lambda$ with $v \in V$, $\lambda \in V^*$ identifies this part with polynomials of bidegree (d, d) on $V \times V^*$. The monomials in the displayed basis have distinct images, indexed by the two margins of a 2×2 exponent matrix of total d. For fixed margins exactly one such matrix has zero in at least one of the two diagonal positions. Every pair of margins occurs. Conjugation acts via $(v, \lambda) \mapsto (gv, \lambda g^{-1})$.

Realize the first factor by $H^0(\mathbb{P}(V), \mathcal{O}(d))$ and fix Re_1 as above. The resulting B-module is the weight-$-d$ line tensored with $\operatorname{Sym}^d V$. The normalized bar bound gives zero in positive bar degrees and exactly one copy of R in degree zero, generated on this fiber by $\lambda(e_1)^d$. It is the restriction of $\operatorname{tr}(v\lambda)^d$. The filtered exact sequences now show by induction that positive derived invariants vanish on each filtered piece and the invariants are freely spanned there by $1, t, \dots, t^d$ for $t = \operatorname{tr}(g)$. Filtered colimits commute with tensoring by group coordinate rings, finite kernels used for normalization, and cohomology of modules. This proves the equivariant quasi-isomorphism of $R[t]$-complexes

$$R[t] \longrightarrow \mathrm{R}\Gamma([G'/G'], \mathcal{O}).$$

The same bar calculation permits passage to the direct sum of column degrees. Before shear the Koszul complex has just five terms, so in each total degree only finitely many bar degrees enter. After shear the computed models are locally finite in cohomological degree.

In the classical coordinate ring, $\det(gM) = \det M$ and $gM = sM$ give $(q-1)D = 0$. There is a chain-level refinement in the full Koszul complex.

$$\Gamma = \det(\xi_1, gv_2) + s \det(v_1, \xi_2), \qquad d\Gamma = (1-q)D.$$

Both expressions are invariant because $\det g = 1$ and the two arguments in each determinant transform as vectors. In the line calculation D maps to y, and Γ maps to $s\beta$. Taking generators D^j and $-\Gamma D^{j-1}$ therefore puts the differential in the positive strand in the form

$$R \xrightarrow{q-1} R \quad \text{in degrees } 2j-1, 2j.$$

This establishes survival with exact scalar annihilator $(q-1)R$ for $[D^j]$ (including the possibility that the class is zero when $q-1$ is a unit). The full annihilator over $R[\mathsf{T}]$ is $(q-1, \mathsf{T}-2)$. The Hecke eigenvalue on all positive cohomology equals 2, the image of $q+1$ there. Every odd class is represented by $-r\Gamma D^{j-1}$ for a unique $r \in \operatorname{Ann}_R(q-1)$. Weight zero contributes exactly $R[t]$ in degree zero. There is no negative cohomology. These calculations use a normalized algebraic-group resolution over R itself. In particular they apply unchanged to nonflat R_0-algebras.

Finally, the alternating tensor $e_1 \otimes e_2 - e_2 \otimes e_1$ maps to 2 under determinant contraction. The full module of invariant contractions $V \otimes V \to R$ consists of the multiples of this determinant, by the diagonal torus weights and invariance under the upper unipotent subgroup. Consequently the alternating inclusion has no equivariant retraction in characteristic two, and the characteristic-zero direct-sum decomposition in Example 8.4.4 is unavailable there. The preceding proof uses the integral alternating form, monomials and Borel weights, none of which require such a retraction.

4 The derived comparison as a spherical Hecke module

Identify the spherical Hecke algebra with $S = R[\mathsf{T}]$. Our statement belongs to $D(S)$ for the ordinary spherical Hecke algebra. To spell out the comparison, on the automorphic degree-zero summand take $\mathrm{Sat}_X \colon \mathcal{M}_R \to R[t]$. On the automorphic summand D_j send its degree-$2j$ basis element to

the generator y^j on the first-column slice in bidegree (j, j). It is represented before passage to the slice by $\det(M)^j$. Send its degree-$(2j-1)$ basis element to $-s\,y^{j-1}\beta$, where $d\beta = (s^{-1} - s)y$ on this slice. Indeed $-s(s^{-1} - s) = m$. These formulas give an isomorphism f between the two explicit underlying R-complexes. In particular the periodic generator in degree two determined by our cyclic resolution corresponds to the determinant. This specification fixes the comparison with the chosen generator and column order.

On $R[t]$ the Hecke action is multiplication by st, and f is Hecke-linear there by construction. On the positive bidegrees the two chain endomorphisms U_A, U_B are respectively 2 and $s(s + s^{-1}) = q + 1$. Write h for the map of degree -1 taking the upper-degree basis element of each D_j to its lower-degree basis element, zero elsewhere. Then $dh + hd = m\,\mathrm{id}$ on D_j. Taking $H = fh$ on these pairs and $H = 0$ in the ordinary summand yields

$$U_B f - f U_A = dH + Hd.$$

For completeness, this homotopy gives a morphism in the required derived category. For a complex C with strict R-linear endomorphism U, form the S-linear cochain cone

$$P(C, U) = \mathrm{Cone}\Big(S \otimes_R C \xrightarrow{\mathsf{T}-U} S \otimes_R C\Big).$$

In each degree the arrow is injective by the leading coefficient, and its cokernel is C with T acting by U, by evaluation $\sum \mathsf{T}^i c_i \mapsto \sum U^i c_i$. Hence the augmentation of this cone to (C, U) is a quasi-isomorphism. In cone coordinates with $d(y, x) = (dy + (\mathsf{T} - U)x, -dx)$, the formula

$$(y, x) \longmapsto (fy + Hx, fx)$$

on the induced complexes (extending f, H S-linearly) defines a chain map $P(C_A, U_A) \to P(C_B, U_B)$. The displayed homotopy verifies its chain identity. The map is a quasi-isomorphism using the two-step cone filtration and the freeness of S over R. This gives the desired zigzag in $D(S)$.

The same cone argument justifies using representatives for the Hecke correspondences calculated up to homotopy. The explicit automorphic model and the reduced spectral model are direct sums of bounded complexes of free R-modules. They are homotopically projective over R, because a map from any one summand to an acyclic complex is nullhomotopic, and the homotopies assemble on a direct sum. A derived isomorphism from either model can therefore be represented by a chain quasi-isomorphism, and equality of the two composites with the generator is represented by a chain homotopy. Replacement of the target by a quasi-isomorphic model, if needed, gives the same cone construction. The single free polynomial generator introduces no further relations to check. This proves (9.15) in the stated Hecke-module sense over every discrete coefficient algebra under consideration. The structure compared is the action of the ordinary spherical Hecke algebra on the two complexes.

5 Coefficients, completion, and the local field

In every row of the table $H^{i<0} = 0$, $H^0 = R[t]$, and T acts there by st. In all nonzero positive groups, T acts by 2. The two positive columns apply for every $j \geq 1$.

Coefficient ring or residue cardinality	H^{2j-1}	H^{2j}
Arbitrary discrete R	$\mathrm{Ann}_R(m)$	R/mR
Characteristic-zero field with s	0	0
R_0	0	R_0/mR_0
Field k, $\ell \neq p$, $\ell \mid m$	k	k
Field k, $\ell \mid (q+1)$, $\ell \nmid m$	0	0
Field of characteristic 2, p odd	k	k
$p = 2$, $q = 2^f$, arbitrary R	$\mathrm{Ann}_R(2^f - 1)$	$R/(2^f - 1)R$
$q = 2$, arbitrary R	0	0

The fields in the table carry a choice of s. The monic relation and $q \in \mathbb{Z}[1/p]^\times$ make R_0 free of rank two over $\mathbb{Z}[1/p]$. Hence m is a non-zero-divisor in R_0 (a unit when $q = 2$). In characteristic two with p odd the scalar 2 is zero. The factor converting the two differentials is a unit power of s up to sign. Averaging over pro-p quotients uses powers of p. No step requires 2, $q - 1$, or $q + 1$ to be units beyond what the initial coefficient ring supplies.

Finite checks. For $q = 7$ in $R = \mathbb{Z}/9$ take $s = 4$. The automorphic differential is 6, the geometric differential is $s^{-1} - s = 3$, and multiplication by $-s$ converts the latter to 6. The odd group is $3R$ and the even group is $R/3R$. The scalar 8 on the geometric complex induces 2 on both groups. For $q = 4$, $R = \mathbb{F}_5$, $s = 2$, the differential 3 is a unit, although $q + 1 = 0$ in R. For $q = 3$ over $\mathbb{F}_2$ the differential and the cohomological Hecke operator both vanish. The accompanying `check_examples.py` counts lattice neighbors through depth four for $q = 2, 3, 5$ and enumerates kernels and images of the positive pair over five finite rings. These checks are independent of the proofs.

For a homomorphism $R \to S$ of discrete coefficient rings the displayed free pairs and the polynomial summand give a K-flat complex over R (a direct sum of bounded complexes of free modules). All reductions of the stabilizer, quotient-stack and incidence models are defined by the same constructions over S. In particular

$$\mathcal{A}_R \otimes^{\mathrm{L}}_R S \simeq \mathcal{A}_S, \qquad \mathcal{B}_R \otimes^{\mathrm{L}}_R S \simeq \mathcal{B}_S.$$

This includes nonflat reduction. The odd groups that appear on reduction of the universal pairs are exactly $\mathrm{Tor}_1^{R_0}(R_0/mR_0, S)$, including the zero group for $m = 1$. Taking underived tensor products of the universal cohomology groups would miss them.

For example, let Λ be an ℓ-adically complete separated coefficient ring with the specified s, $\ell \neq p$, and apply the discrete construction to $R_a = \Lambda/\ell^a\Lambda$. Define the completed objects by $\mathrm{R}\varprojlim_a \mathcal{A}_{R_a}$ and $\mathrm{R}\varprojlim_a \mathcal{B}_{R_a}$. The same choices give compatible equivalences at finite levels. The transition maps in (1) are degreewise surjective, so the ordinary termwise limit computes this derived limit (the difference map on the product of the tower is surjective degreewise). Each positive pair is the pair over Λ. Degree zero is

$$\varprojlim_a R_a[t] = \left\{ \textstyle\sum_{n \geq 0} c_n t^n : c_n \in \Lambda,\ c_n \to 0 \text{ } \ell\text{-adically} \right\}.$$

In the orbit basis this condition is finite support modulo every ℓ^a. This distinguishes completion from the discrete compactly supported module with coefficients in the underlying ring of Λ.

For a general nonarchimedean local field with residue field $\mathbb{F}_q$, the lattice tree, the orbit computation, the unit filtrations, transfers, and constant terms used here have the same proofs. The dual split integral model and its fixed equations depend only on q, s. Both theorems therefore extend as statements about these explicit objects, including in mixed characteristic. BZSV's geometric setup in §9.1 is the equal-characteristic one.

6 Proof dependencies

PROVED. Section 2 gives the free cyclic ordinary module by tree counts. Its class-function interpretation uses the weight-zero calculation of Section 3.

PROVED. Section 3 computes compact stabilizer cochains and rational algebraic group cochains separately. The projective-line twist and the strict inequality for positive unipotent bar degree give the full spectral calculation over every R.

PROVED. Section 4 compares the resolutions in $D(R[\mathsf{T}])$. It identifies every power of the determinant with the corresponding even periodic generator in the sense of modules.

REFUTED in characteristic two. The symmetric and alternating tensors displayed in [1, Ex. 8.4.4] do not give the stated direct decomposition integrally at 2. In the middle weight their change-of-basis determinant is ± 2. Our use of the example is its identification of the split representation and grading.

REFUTED for pointwise products. The disjoint orbit indicators satisfy $e_0 e_1 = 0$, whereas $\mathrm{Sat}_X(e_0)\,\mathrm{Sat}_X(e_1) = st - 2$ over any nonzero R.

CONDITIONAL in the source. The general passage by categorical Frobenius trace discussed in [1, §9.4] is motivational here. The proof computes the explicit right-hand side of (9.15) directly.

7 Verdict

Equation (9.15) holds for $T\backslash \mathrm{PGL}_2$ as a comparison of derived modules for the ordinary spherical Hecke algebra over every discrete R_0-algebra, with the neutral shear and the derived fixed locus specified above. No further localization is needed.

References

[1] D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh, *Relative Langlands Duality*, arXiv:2409.04677v1, 7 September 2024. §9.4 and (9.15), Example 8.4.4, §6.1, §6.8.1–6.8.2.