L-FUNCTORIALITY FOR DUAL PAIRS

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1. Introduction

This is a study note for Adams' paper "L-functoriality for dual pairs" [Ada89]. The Adams' conjecture play a very important role for the connection of theta correspondence with Langlands program.

2. Arthur packets

In this section, we introduce the Arthur packets following Adams' original definition. Of course, we have the ABV packets nowadays.

Let Ψ be an admissible homomorphism, the definition of $\Pi(\Psi)$ has two steps: construction of a unipotent Arthur-packet for G for the Levi component L of a parabolic subgroup of $G(\mathbb{C})$ and induction from L to G, the induction step is a combination of real parabolic induction, and cohomological induction from a θ-stable parabolic subalgebra of $\mathfrak{g}$.

We want to use the E-groups, and the conjugacy classes of admissible homomorphisms $\varphi : W_{\mathbb{R}} \to {}^E G$ parametrizes the L-packet of the genuine representation of $\tilde{G}$, a certain two fold covering group of G.

Let Ψ be an admissible homomorphism, to Ψ we can associate an infinitesimal character χ_Ψ and O_Ψ a unipotent orbit: the image of $\mathbb{C}^*$ is contained in ${}^L T^0$ of ${}^L G^0$, we write $\Psi(z) = z^\mu \bar{z}^\nu \in X_*({}^L T^0)$, we let χ_Ψ be the infinitesimal character of G corresponding to $\lambda + \mu$ via the Harish-Chandra homomorphism. Next for $\Psi|_{SL_2(\mathbb{C})}$ by the Jacobson-Morozov theorem, it corresponds to a unipotent orbit ${}^L O_\Psi$ of ${}^L G^0$ (by orbit we will always mean coadjoint orbit in the dual of a Lie algebra or conjugacy class in a Lie group). We note that $\lambda = d\Psi|_{SL_2(\mathbb{C})}$ is integral unless ${}^L O_\Psi$ is the principal unipotent orbit of ${}^L G^0$, in which case it is the infinitesimal character of the trivial representation. Now ${}^L O_\Psi$ corresponds to a special unipotent orbit O_Ψ of G.

Definition 2.1. Suppose $\Psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L G$ and the image of $\mathbb{C}^*$ is contained in the center of ${}^L G^0$, then the weak Arthur packet $\Pi(\Psi)$ is the finite set of irreducible representation π of G such that

- The infinitesimal character of π is χ_Ψ.
- The wave-front set of π is equal to the closure of O_Ψ.

recall that the wave-front set of an irreducible representation π is a finite union of coadjoint G-orbits.

Connection with the ABV definition.

we will see how to construct the packets $\Pi(\Psi)$ later.

Lemma 2.2. *In the setting of previous definition, Ann(π) is the same for all $\pi \in \Pi(\Psi)$.*

More generally, if the image of C^* is not necessarily contained in the center of ${}^L G^0$, let ${}^L C^0$ denote the identity component of the centralizer of the image of C^* in ${}^L G^0$. Now let $\Psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^E C \to {}^L G$, the image of $\mathbb{C}^*$ is contained in the center of ${}^L C^0$. By the construction above, we get the Arthur packet $\Pi(\Psi)$ of representations of a covering group $\tilde{C}$ determined by ${}^E C$.

Conjugacy classes of Levi subgroups of $G(\mathbb{C})$ are in bijection with conjugacy classes of Levi subgroups of ${}^L G^0$, suppose L is a θ-stable Levi subgroup of G such that the conjugacy class of $L(\mathbb{C})$ corresponds to ${}^L C^0$. Furthermore, assume L is an inner form of C, let $\Pi^L(\Psi)$ be the Arthur packet constructed in the previous paragraph taking $L = C$. This is a finite set of representations of $\tilde{L}$, which are special unipotent when restricted to the derived subgroup. Given $\pi_L \in \Pi^L(\Psi)$, choose a parabolic subgroup $Q(\mathbb{C}) = L(\mathbb{C})U(\mathbb{C}) \subset G(\mathbb{C})$, we assume $Q(\mathbb{C})$ is weakly non-negative, this is a condition on the imaginary roots of $\mathfrak{u}$.

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Let $\mathcal{R}(\pi_L)$ be the derived functor module of π_L, let $\mathbb{C}_{\rho(\mathfrak{u})}$ denote the one-dimensional representation of $\tilde{L}_{\rho(\mathfrak{u})}$ with weight $\rho(\mathfrak{u})$, here $\tilde{\rho}_{\rho(\mathfrak{u})}$ is the metaplectic cover of L defined by element $\rho(\mathfrak{u})$. Then $\tilde{L} \cong \tilde{L}_{\rho(\mathfrak{u})}$ and $\pi_L \otimes \mathbb{C}_{\rho(\mathfrak{u})}$ is naturally a representation of L, set $S = \frac{1}{2}\mathfrak{k}/\mathfrak{k} \cap \mathfrak{l}$, and let $\mathcal{R}(\pi_L) = \Gamma^S \circ \mathrm{pro}(\pi_L \otimes \mathbb{C}_{\rho(\mathfrak{u})})$, this has the same infinitesimal character as π_L.

The construction of $\mathcal{R}(\pi_L)$ can be broken up into two steps: there exists $L \subset L_\theta \subset G$ with the following properties: there is a real parabolic subgroup P of L_θ containing L as its reductive part and ℓ_θ is the Levi factor of a θ-stable parabolic subalgebra of $\mathfrak{g}$. Furthermore $\mathcal{R}(\pi_L) \cong \mathcal{R}_\theta \circ \mathrm{Ind}(\pi_L)$ where Ind is the ordinary parabolic induction from P to L_θ, and $\mathcal{R}_\theta$ is the cohomological parabolic induction from the θ-stable parabolic subalgebra $\mathfrak{q}$ (up to one-dimensional twists).

Definition 2.3. The Arthur packet $\Pi^G(\Psi)$ associated to Ψ is the set of irreducible constituents of the modules $\mathcal{R}(\pi_L)$ as L, π_L run over all possible choices given above.

Given G, suppose $\{G_i\}$ is a set of groups which are inner forms of G with $G = G_0$, we identify the E-groups for each G_i with those for G, given $\Psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L G$, we obtain $\Pi^{G_i}(\Psi)$ as above, and we will write $\Pi^{\{G_i\}}(\Psi) = \cup_i \Pi^{G_i}(\Psi)$.

A similar procedure can be used to define $\Pi(\Psi)$ when $\Psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^E G$.

3. Conjectures A, B and C

In this section, we state the three conjectures from Adams' paper.

Proposition 3.1. *Let (G, G') be an irreducible reductive dual pair with G the smaller group, we consider (G, G') as a subgroup of $Mp_{2n}(\mathbb{R})$, we can define a homomorphism between E-groups $\gamma : {}^E G \to {}^E G'$.*

Let ${}^L H'^{\circ}$ be the identity component of the centralizer of $\gamma({}^L G^{\circ})$ in ${}^L G'^{\circ}$, then we denote $T : SL_2(\mathbb{C}) \to {}^L H'^{\circ}$ be the homomorphism corresponding to the principal unipotent orbit in ${}^L H'^{\circ}$.

Example 3.2. For the dual pair $(O(p,q)(\mathbb{R}), \mathrm{Sp}_{2m}(\mathbb{R}))$ inside $\mathrm{Sp}_{2m(p+q)}(\mathbb{R})$: let $n = p+q$, then the oscillator representation factors to $\mathrm{Sp}_{2m}(\mathbb{R})$ if and only if $n \in 2\mathbb{Z}$. For the L-group of Sp_{2m} we take as $SO(2m+1, \mathbb{C}) \times \Gamma$, for the L-group of $O(p,q)$, we take as $O(2n, \mathbb{C}) \rtimes \Gamma$.

a.<u>$n \leq m$</u>, we have

- $\gamma(g \times 1) = \mathrm{diag}(g, \det(g) I_{2(m-n)+1}) \times 1,\ g \in O(2n, \mathbb{C})$.
- $$\gamma(1 \times \sigma) = \begin{cases} I_{2m+1} \times \sigma, & p - q \equiv 0 \pmod 4 \\ \mathrm{diag}(\epsilon, -I_{2(n-m)+1}) \times \sigma, & p - q \equiv 2 \pmod 4 \end{cases}$$

  here $\epsilon = \mathrm{diag}(1, 1, \cdots, 1, -1)$.

b.<u>$n > m$</u> Let $G = \mathrm{Sp}_{2m}(\mathbb{R})$, $G' = O(p,q)$, we define $\gamma : {}^L G \to {}^L G'$ as

- $\gamma(g \times 1) = \mathrm{diag}(g, I_{2(n-m)-1}) \times 1,\ g \in SO(2m+1, \mathbb{C})$.
- $\gamma(1 \times \sigma) = \begin{cases} I_{2n} \times \sigma, & p - q \equiv 0 \pmod 4 \\ \epsilon \times \sigma, & p - q \equiv 2 \pmod 4 \end{cases}$

Conjecture 3.3. *(conjecture A) Let (G, G') be a reductive dual pair inside the group $Sp_{2n}(\mathbb{R})$, suppose π is an irreducible representation of G occuring in the representation correspondence for the dual pair, let π' be the corresponding irreducible representation of π'. Suppose $\Psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^E G$ is an admissible homomorphism, such that π is contained in the corresponding A-packet $\Pi(\Psi)$. Let $\Psi' : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^E G$ be defined by*

$$\begin{aligned} \Psi'(\omega \times 1) &= \gamma \circ \Psi(\omega) && \omega \in W_{\mathbb{R}} \\ \Psi'(1 \times g) &= \gamma \circ \Psi(g) T(g) && g \in SL_2(\mathbb{C}) \\ \Psi'(\omega \times g) &= \Psi'(\omega \times 1) \Psi'(1 \times g) && \omega \in W_{\mathbb{R}},\ g \in SL_2(\mathbb{C}) \end{aligned}$$

Then Ψ' is an admissible homomorphism, let $\Pi(\Psi')$ denote the corresponding Arthur packet, then $\pi' \in \Pi(\Psi')$.

The next conjecture also takes the various inner forms of G and G' into consideration. Given (G, G') a reductive dual pair, let $\{(G_i, G')\}$ $(i = 0, 1, \cdots, k)$ be a set of representatives for the equivalence of dual pairs such that G_i is an inner form of G, with $G_0 = G$, for example, if $(G, G') = (O(2m), \mathrm{Sp}_{2n}(\mathbb{R}))$, then

$(G_i, G') = (O(2m-2i, 2i), \mathrm{Sp}_{2n}(\mathbb{R}))$ $(i = 1, 2, \cdots, m)$. We note that all (G_i, G') have the same L-group and maps γ, given $\Psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L G$, we let $\Pi^{\{G_i\}}(\Psi)$ be a set of irreducible representations of some G_i: $\Pi^{\{G_i\}}(\Psi) = \cup_i \Pi^{G_i}(\Psi)$.

Conjecture 3.4. *(conjecture B) Suppose we are in the situation of the previous conjecture, with (G, G') in the stable range, thus we are given $\pi \in \Pi^G(\psi)$, π occuring in the representation correspondence for G*

- *Suppose σ is an irreducible representation of G_i for some i, and σ is contained in $\Pi^{G_i}(\Psi)$, then σ occurs in the representation correspondence for the dual pair (G_i, G'), let σ' denote the corresponding representation, then $\sigma' \in \Pi^{G'}(\Psi')$.*
- *$\sigma \to \sigma'$ is a bijection between $\Pi^{\{G_i\}}(\Psi)$ and $\Pi^{G'}(\Psi')$.*

This conjecture is not true without the stable range assumption: the trivial representation of $O(4)$ occurs for the dual pair $(O(4), \mathrm{Sp}_4(\mathbb{R}))$, but the sign representation does not. It is not clear what the correct range of validity of this conjecture should be.

We can also extend the conjecture B to one about endoscopic groups and stable distributions, we will call this conjecture C. Suppose we are in the setting of conjecture B, and as in that conjecture, there is a bijection $\Pi^{\{G_i\}}(\Psi) \to \Pi^{G'}(\Psi')$, where $\Pi^{\{G_i\}}(\Psi)$ is considered as a set of representations of $\{G_i\}$. By a virtual character of a group we mean a complex linear combination of irreducible characters.

Definition 3.5. We make the following definition

- A virtual character of $\{G_i\}$ is a formal sum $\Theta = \Theta_1 + \Theta_2 + \cdots \Theta_k$ where Θ_i is a virtual character of G_i.
- The virtual character Θ corresponds to the virtual character $\Theta' = \sum_i a_i \Theta'_i$ of G' if Θ_i corresponds to Θ'_i for all i.
- The virtual character Θ is super-stable only if Θ_i is stable for all i.

Super-stable characters in the ABV perspective.

Example 3.6. An example of the super-stable distribution is the following: let $\{G_i\}$ be a complete set of representatives for the isomorphism classes of inner forms of G, let $\Psi : W_{\mathbb{R}} \to {}^L G$ be the parameter corresponding to an L-packet $\Pi^G(\Psi)$ of discrete series representations, let $\Theta = \sum_i \theta_{\pi_i}$ be the stable sum of virtual character of G_i, we let Θ_i be defined similarly, then $\sum_i \Theta_i$ is a super-stable virtual character.

Theorem 3.7. *Suppose we are in the stable range, then there is a distinguished super-stable virtual character Θ_0 (resp. Θ'_0) in $\Pi(\Psi)$ (resp. $\Pi(\Psi')$).*

For H an endoscopic subgroup of G, we have a map Lift_H^G taking super-stable virtual characters in $\Pi^H(\Psi)$ to virtual characters in $\Pi(\Psi)$, where these Arthur packets are considered as spaces of virtual characters of the inner forms of G and H.

Conjecture 3.8. *(conjecture C) Suppose we are in the setting of conjecture B, with $\gamma : \Pi(\Psi) \to \Pi(\Psi')$, we suppose H is an endoscopic group for G, and $\Psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L H \to {}^L G$, thus ${}^L H^0$ is the centralizer of a semisimple element h of ${}^L G^0$. We have a corresponding map $\gamma : {}^L H \to {}^L H'$ for γ restricts to ${}^L H$ and H' is an endoscopic subgroup of G', then the following should hold:*

- $\gamma(\Theta_0) = \Theta'_0$.
- *Suppose $\Theta = \mathit{Lift}_H^G(\Theta_H) \in \Pi(\Psi)$, for $\Theta_H \in \Pi^H(\Psi)$ a super-stable virtual character. Then there is a super-stable virtual character $\Theta_{H'} \in \Pi^{H'}(\Psi')$ such that $\gamma(\Theta) = \mathit{Lift}_{H'}^{G'}(\Theta_{H'}) \in \Pi(\Psi')$.*

4. Proof of conjecture A and B in special cases

In this section, we will describe the Adams' proof of the conjecture A and B in some special cases.

As Adams pointed out, the results of his paper can be extended in a number of ways, however, the methods are not intended as a method of proof in general, so pushing these results as far as possible is probably not worth the effort. For example the results on the discrete series can certainly be strengthened quite a bit to representations with singular infinitesimal character and to groups outside of the stable range.

References

[Ada89] Jeffrey Adams. L-functoriality for dual pairs. *Astérisque*, 171(172):85–129, 1989.