

K-FORMS OF BASIC VARIETIES

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1. INTRODUCTION

We let k be a p -adic local field of characteristic zero with odd residue characteristic. In this note, I will study all the k -forms for the basic varieties, this is a collaborated computation with ChatGpt.

Definition 1.0.1. We introduce the list of basic spherical varieties for $\text{Res}_{\ell/k} \text{SL}_2$ where ℓ/k is an unramified extension of p -adic fields.

- $\text{Res}_{\ell/k} \text{SL}_2 \backslash \text{Res}_{\ell/k} \text{SL}_2$.
- $\text{Res}_{\ell/k} T \backslash \text{Res}_{\ell/k} \text{SL}_2$, T a maximal torus of $\text{SL}_{2,\ell}$.
- $N(\text{Res}_{\ell/k} T) \backslash \text{Res}_{\ell/k} \text{SL}_2$.
- $U \backslash \text{Res}_{\ell/k} \text{SL}_2$, U the unipotent radical of the Borel.
- $\text{Res}_{k'/k} T_1 U_1 \backslash \text{Res}_{\ell/k} \text{SL}_2$, here $T_1 U_1$ is a k' -spherical subgroup of $\text{Res}_{\ell/k'} \text{SL}_2$ and k'/k is an unramified extension of degree m , over algebraic closure it has a spherical system of type (a-A-2-n)^m.
- $\text{Res}_{k'/k} \text{SL}_2 \backslash \text{Res}_{\ell/k} \text{SL}_2$, for ℓ/k' an unramified quadratic extension and k'/k a finite unramified extension.
- $\text{Res}_{k'/k} \text{SL}_2 \backslash \text{Res}_{\ell/k} \text{SL}_2$, for ℓ/k' an unramified cubic extension and k'/k a finite unramified extension.

We have the following list of basic spherical varieties for $\text{Res}_{\ell/k} \text{SU}_3$ where ℓ/k is an unramified extension of p -adic fields, SU_3 is the unramified unitary group associated with an ℓ' -Hermitian space for ℓ'/ℓ an unramified quadratic extension.

- $\text{Res}_{\ell/k} \text{SU}_3 \backslash \text{Res}_{\ell/k} \text{SU}_3$.
- $\text{Res}_{\ell/k} \text{SO}_3 \backslash \text{Res}_{\ell/k} \text{SU}_3$.
- $U \backslash \text{Res}_{\ell/k} \text{SU}_3$, U the unipotent radical of the Borel.
- $\text{Res}_{\ell/k} \text{SU}_2 \backslash \text{Res}_{\ell/k} \text{SU}_3$.
- $\text{Res}_{k'/k} T_1 U_1 \backslash \text{Res}_{\ell/k} \text{SU}_3$, here $T_1 U_1$ is a k' -spherical subgroup of $\text{Res}_{\ell/k'} \text{SU}_3$, over algebraic closure it has a spherical system of type (a-A-4-n)^m.
- $\text{Res}_{k'/k} T_2 U_2 \backslash \text{Res}_{\ell/k} \text{SU}_3$, here $T_2 U_2$ is a k' -spherical subgroup of $\text{Res}_{\ell/k'} \text{SU}_3$, over algebraic closure it has a spherical system of (a-A-5-n)^m with n odd.

For spherical subgroups H appearing in this list, we will call them basic spherical subgroups.

2. $T \backslash \text{SL}_2$

For $X = T \backslash \text{SL}_2$, here T is the maximal diagonal torus of SL_2 . We have

$$N_G(H)/H \cong \mu_2$$

the k -forms are classified by

$$H^1(k, \mu_2) = k^\times / k^{\times 2} = \{[1], [u], [\varpi], [u\varpi]\}$$

where u is a nonsquare unit and ϖ is a uniformizer.

For each $d \in k^\times$, an explicit model is given by

$$X_d = \{(x, y, z) \in \mathbb{A}_k^3 : x^2 + yz = d\}$$

we have

$$T_d \backslash G \cong X_d$$

with

$$T_d \cong \text{Res}_{K_d/k}^1 \mathbb{G}_m$$

here $K_d = k[t]/(t^2 - d)$. So the class [1] corresponds to the diagonal split torus T and the other three classes correspond to the nonsplit norm torus associated with the nontrivial quadratic extensions of k .

3. GALOIS CASE

For $X = SL_2 \backslash \text{Res}_{\ell/k} SL_2 = H \backslash G$, we have

$$N_G(H)/H \cong \mu_2$$

as k -schemes. Hence

$$H^1(k, \text{Aut}^G(X)) = H^1(k, \mu_2) = k^\times / k^{\times 2}$$

we can parametrize

$$k^\times / k^{\times 2} = \{[1], [u], [\varpi], [u\varpi]\}$$

Write $\ell = k(\delta)$ with $\delta^2 = u$ and denote the nontrivial automorphism of ℓ/k by a bar, consider the k -vector space of skew-Hermitian matrices

$$V = \{A \in M_2(\ell) : \bar{A}^t = -A\}$$

with the right G -action

$$A \cdot g = \bar{g}^t A g$$

For $a \in k^\times$, define

$$X_a = \{A \in V : \det A = a\}$$

introduce $H_a = \text{SU}(h_a)$, $h_a = \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{u} \end{pmatrix}$, then we have $X_a = H_a \backslash G$.

Then the four represented forms are precisely,

$$X_1, X_u, X_\varpi, X_{u\varpi}$$

we have

$$X_1 = SL_2 \backslash \text{Res}_{\ell/k} SL_2, X_u = SL_2 \backslash \text{Res}_{\ell/k} SL_2$$

but X_1 and X_u are not k -isomorphic.

Denote D the unique quaternion algebra over k , then

$$X_\varpi = SL_1(D) \backslash \text{Res}_{\ell/k} SL_2, X_{u\varpi} = SL_1(D) \backslash \text{Res}_{\ell/k} SL_2$$

Will different embeddings cause a difference to the branching problem?

4. CUBIC CASE

For $X = SL_2 \backslash \text{Res}_{\ell/k} SL_2 = H \backslash G$ with ℓ/k an unramified cubic extension. We have

$$N_G(H)/H = \text{Res}_{\ell/k} \mu_2 / \mu_2 = M$$

The short exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \text{Res}_{\ell/k} \mu_2 \longrightarrow M \longrightarrow 1$$

we have

$$k^\times / k^{\times 2} \xrightarrow{\text{res}} \ell^\times / \ell^{\times 2} \rightarrow H^1(k, M) \rightarrow \text{Br}(k)[2] \xrightarrow{\text{res}} \text{Br}(\ell)[2]$$

both the restriction maps are isomorphisms. It follows that

$$H^1(k, \text{Aut}^G(X)) = 0$$

hence X has a unique k -form.

5. $T_1U_1 \backslash \text{Res}_{\ell/k}SL_2$

For $X = T_1U_1 \backslash \text{Res}_{\ell/k}SL_2$, denote $n = [\ell : k]$. We have

$$H^1(k, \text{Aut}^G(X)) = \begin{cases} 0, & n \text{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^2, & n \text{ even} \end{cases}$$

Choose a unit $\nu \in \mathcal{O}_\ell^\times$ that is nonsquare in ℓ . For $b \in \ell^\times$, define

$$U_b = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : \text{Tr}_{\ell/k}(bz) = 0 \right\}, \quad H_b = T_1U_b$$

Then the two k -forms can be described as

$$X_1 = H_1 \backslash G, \quad X_\nu = H_\nu \backslash G$$

the other two k -forms don't have k -points.

6. $SO_3 \backslash SU_3$

We let ℓ/k be an unramified quadratic extension which is used to define SU_3 . For $X = SO_3 \backslash SU_3$, we have

$$H^1(k, \text{Aut}^G(X)) = S(k)/S(k)^3$$

here $S = \text{Res}_{\ell/k}^1 \mathbb{G}_m$.

We have

$$S(k)/S(k)^3 = \begin{cases} 0, & q \equiv 1 \pmod{3} \\ \mathbb{Z}/3\mathbb{Z}, & q \equiv -1 \pmod{3} \end{cases}$$

Choose $\xi \in S(k) \backslash S(k)^3$, we get

$$X_\xi = H_\xi \backslash G$$

here $H_\xi = d_\xi^{-1}SO(h)d_\xi$ with $h = \text{diag}(1, -1, 1)$ and $d_\xi = \text{diag}(\xi, 1, 1)$.

7. $SU_2 \backslash SU_3$

Let ℓ/k be an unramified quadratic extension and

$$G = \text{SU}(h), \quad h = \text{diag}(1, -1, 1)$$

For $X = SU_2 \backslash SU_3$, then we have

$$\text{Aut}^G(X) = \text{Res}_{\ell/k}^1 \mathbb{G}_m = S, \quad H^1(k, S) = k^\times / \text{N}_{\ell/k}(\ell^\times) = \{[1], [\varpi]\}$$

we have

$$X_1 = SU_2 \backslash SU_3, \quad X_\varpi = SL_1(D) \backslash SU_3$$

an explicit model for each form is the Hermitian norm sphere

$$X_t = \{v \in \ell^3, \quad \bar{v}^t h v = t\}$$

8. $T_1U_1 \backslash \text{Res}_{\ell/k}SU_3$

Let SU_3 be a unitary group defined over ℓ associated with the unramified extension L/ℓ . For $X = T_1U_1 \backslash \text{Res}_{\ell/k}SU_3$, denote $\text{Res}_{L/\ell}^1 \mathbb{G}_m = S$, then we have

$$\text{Aut}^G(X) = \text{Res}_{\ell/k} S[3]$$

and

$$H^1(k, \text{Aut}^G(X)) = S(\ell)/S(\ell)^3$$

Choose $d_a = \text{diag}(1, a, 1) \in U_3$, then we have

$$X_a = H_a \backslash G, \quad H_a = d_a^{-1} H d_a$$

we have X_a is G -isomorphic to X_b if and only if $a/b \in S(\ell)^3$. We have

$$H_a \cong H_1 = T_1U_1$$

9. $T_2U_2 \backslash \text{Res}_{\ell/k} SU_3$

Let $n = [\ell : k]$ be an odd integer and let K/k be an unramified quadratic extension and put $E = \ell K$, so that E/ℓ is the quadratic extension defining SU_3 .

Let SU_3 be a unitary group defined over ℓ associated with the unramified extension L/ℓ . For $X = T_2U_2 \backslash \text{Res}_{\ell/k} SU_3$, denote $\text{Res}_{L/\ell}^1 \mathbb{G}_m = S$, then we have

$$\text{Aut}^G(X) = \frac{Z(G)}{H \cap Z(G)} = \frac{\text{Res}_{\ell/k} S_\ell[3]}{S_k[3]}$$

we have

$$H^1(k, \text{Aut}^G(X)) = \frac{S_\ell(\ell)}{S_\ell(\ell)^3 S_k(k)}$$

For each norm 1 element $a \in S_\ell(\ell)$, we put $d_a = \text{diag}(1, a, 1) \in U_3$. Then we have

$$X_a = H_a \backslash G, \quad H_a = d_a^{-1}(T_2U_2)d_a = T_2U_{2,a}$$