A CONJECTURE ON WHITTAKER-FOURIER COEFFICIENTS OF CUSP FORMS

RUI CHEN

1. Introduction

This is a study note for the Lapid-Map paper, it sharpens the conjectures of Sakellaridis-Venkatesh in the case at hand [LM15].

2. The conjecture

Let G be a quasisplit reductive group over a number field F and $\mathbb{A}$ the ring of adeles of F, let B be a Borel subgroup of G defined over F and N the unipotent radical of B and fix a non-degenerate character ψ_N of $N(\mathbb{A})$, trivial on $N(F)$, for a cusp form φ of $G(F)\backslash G(\mathbb{A})$, we consider the Whittaker-Fourier coefficient

$$\mathcal{W}^{\psi_N}(\varphi) := \int_{N(F)\backslash N(\mathbb{A})} \varphi(n)\psi_N(n)^{-1}\, dn$$

If π is an irreducible cuspidal representation then $\mathcal{W}$ if non-zero gives a realization of π in the space of Whittaker functions on $G(\mathbb{A})$ which by local multiplicity one depends only on π as abstract representation.

For the general linear group, the theory of Rankin-Selberg integrals, developed in higher rank expresses among other things, the Pettersson inner product in terms of a canonical linear product on the local Whittaker model of π.

The integral

$$\int_{N(\mathbb{A})} (\pi(n)\varphi, \varphi^\vee)\ \psi_N(n)^{-1}\, dn$$

doesn't converge, even the local integral

$$I_v(\varphi, \varphi^\vee) = \int_{N(F_v)} (\pi_v(n_v)\varphi_v, \varphi_v^\vee)_v\ \psi_N(n_v)^{-1}\, dn_v$$

doesn't converge unless π_v is square integrable. However, it is possible to regularize I_v and by the Casselman-Shalika formula, almost everywhere we have

$$I_v(\varphi_v, \varphi_v^\vee) = \frac{\Delta_{G,v}(1)}{L(1, \pi_v, \mathrm{Ad})}$$

if $\varphi_v,\ \varphi_v^\vee$ are unramified vectors with $(\varphi_v, \varphi_v^\vee) = 1$. By local multiplicity one there exists a constant $c_\pi^{\psi_N}$ depending on π such that

$$\mathcal{W}^{\psi_N}(\varphi) W^{\psi_N^{-1}}(\varphi^\vee) = (c_\pi^{\psi_N} \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1))^{-1} \frac{\Delta_G^S(s)}{L^S(s, \pi, \mathrm{Ad})}|_{s=1} \prod_{v\in S} I_v(\varphi_v, \varphi_v^\vee)$$

for all $\varphi = \otimes\varphi_v \in \pi$ and $\varphi^\vee = \otimes_v\ \varphi_v^\vee \in \pi^\vee$ and all S sufficiently large.

The Rankin-Selberg theory for GL_n alluded to above shows that $c_N^{\psi_N} = 1$ for any irreducible cuspidal representations of GL_n. It is desirable to extend this relation to other quasi-split groups. The first proble is that $c_\pi^{\psi_N}$ depends on the autormophic realization of π.

There is a notion of ψ_N-generic spectrum as defined by Piatetski-Shapiro, it is orthogonal complement to the L^2 autormophic forms with vanishing Whittaker functions. We denote this space by $L^2_{disc,\psi_N}(G(F)\backslash G(\mathbb{A})^1)$, this space is multiplicity free and is meaningful to study $c_\pi^{\psi_N}$ for irreducible constituents of the ψ_N-generic spectrum.

Date: September 2025.

Another more speculative way is to admit Arthur conjectures, namely a canonical decomposition

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})^1) = \hat{\oplus}_\phi \ \overline{\mathcal{H}_\phi}$$

according to elliptic Arthur parameters.

For the group $G = GL_n$ the space $\mathcal{H}_\phi$ are always irreducible and $\mathcal{H}_\phi$ is cuspidal if and only if it is generic and happens if and only if ϕ has trivial SL_2-type. For other groups, the reducibility of $\mathcal{H}_\phi$ is measured to a large extent by a certain finite group $\mathcal{S}_\phi$ attached to ϕ. In general, we expect that if ϕ is of Ramanujan type then $\mathcal{H}_\phi \cap L^2_{cusp,\psi_N}(G(F)\backslash G(\mathbb{A})^1)$ is irreducible. We denote the representation on this space by $\pi^{\psi_N}(\phi)$, if ϕ is not of Ramanujan type then $\mathcal{W}^{\psi_N}$ vanishes on $\mathcal{H}_\phi$.

Conjecture 2.1. *For any elliptic Arthur parameter ϕ of Ramanujan type we have $c^{\psi_N}_{\pi^{\psi_N}(\phi)} = |\mathcal{S}_\phi|$.*

Lemma 2.2. *Suppose that $\tilde{\pi}$ is an irreducible cuspidal ψ_N-generic representation of $\widetilde{G}(\mathbb{A})$ realized on $V_{\tilde{\pi}}$. Assume that the space of $\tilde{\pi} \otimes \omega$ is orthogonal to that of $\tilde{\pi}$ for any $\omega \notin X(\tilde{\pi})$ (this condition is satisfied if the cuspidal multiplicity of $\tilde{\pi}$ is one.) Let $V'_{\tilde{\pi}} = \{\varphi|_{G(\mathbb{A})} \ : \ \varphi \in V_{\tilde{\pi}}\}$. Then*

- *$V'_{\tilde{\pi}}$ is the direct sum of distinct irreducible cuspidal representations of $G(\mathbb{A})$.*
- *There is a unique ψ_N-generic irreducible constituent of $V'_{\tilde{\pi}}$.*
- *If π is the ψ_N-generic irreducible constituent of $V'_{\tilde{\pi}}$ then $c^{\psi_N}_\pi = |X(\tilde{\pi})| c^{\psi_N}_{\tilde{\pi}}$.*

Corollary 2.3. *Suppose that $\tilde{\pi}$ is a cuspidal irreducible autormophic representation of GL_m realized on $V_{\tilde{\pi}}$, let π be the unqiue irreducible constituent of $SL_m(\mathbb{A})$ on*

$$V'_{\tilde{\pi}} = \{\varphi|_{SL_m(\mathbb{A})} \ : \ \varphi \in V_{\tilde{\pi}}\}$$

on which $\mathcal{W}^{\psi_N}$ is nonzero. Then we have $c^{\psi_N}_\pi = |X(\tilde{\pi})|$ where $X(\tilde{\pi})$ is the group of Hecke characters ω such that $\tilde{\pi} \otimes \omega = \tilde{\pi}$.

We remark that every irreducible constituent of $V'_{\tilde{\pi}}$ is generic with respect to some non-degenerate character of $N(\mathbb{A})$, we also remark that in the case SL_2 we have multiplicity one for SL_2, but this is not true for $m > 2$.

For any irreducible cuspidal representation $\tilde{\pi}$ of $\widetilde{G}$ we denote by $X(\tilde{\pi})$ the group of characters of $\widetilde{G}(\mathbb{A})$ trivial $G(\mathbb{A})\widetilde{G}(F)$ such that $\tilde{\pi} \otimes \omega = \tilde{\pi}$. The group $\widetilde{G}$ will be the product of restriction of scalars of general linear groups, in particular

- $\widetilde{G}$ has multiplicity one.
- $c^{\psi_N}_{\tilde{\pi}} = 1$.
- $c^{\psi_N}_\pi = |X(\tilde{\pi})|$ for the ψ_N-generic constituent π of $V'_{\tilde{\pi}}$.

Introduce two equivalence relations on the set of irreducible cuspidal representations of $\widetilde{G}(\mathbb{A})$: $\pi_1 \sim \pi_2$ (resp. $\pi_1 \sim_\omega \pi_2$) if there exists a character ω of $\widetilde{G}(F)G(\mathbb{A})\backslash\widetilde{G}(\mathbb{A})$ (resp. $G(\mathbb{A})\backslash\widetilde{G}(\mathbb{A})$) such that $\pi_2 \cong \pi_1 \otimes \omega$, then G has multiplicity one if and only if the two equivalence relations coincide.

3. Example

Consider the case $G = SL_2$, we take $\widetilde{G} = GL_2$ and $t_{\widetilde{G}}$ to be the adjoint representation. Here $X(\tilde{\pi})$ is as in corollary 2.3, there are three possibilities

- $|X(\tilde{\pi})| = 1$.
- $|X(\tilde{\pi})| = 2$.
- $|X(\tilde{\pi})| = 4$.

Let Π be the adjoint lifting of $\tilde{\pi}$ to GL_3. In the first case, $\tilde{\pi}$ is not dihedral and Π is cuspidal. In the second case, let E be the quadratic extension of F defined by the non-trivial element ω of $X(\tilde{\pi})$ and let θ be the Galois involution of E/F, then $\tilde{\pi} = AI_{E/F}(\mu)$ for some Hecke character μ of $\mathbb{A}^*_E$, $\theta(\mu)/\mu$ is not quadratic and $\Pi = \omega \boxplus AI_{E/F}(\theta(\mu)/\mu)$. Finally in the third case let $X(\tilde{\pi}) = \{1, \omega_1, \omega_2, \ \omega_3\}$ defines a biquadratic extension K of F and $\Pi = \omega_1 \boxplus \omega_2 \boxplus \omega_3$.

In all cases we have $|X(\tilde{\pi})| = |\mathcal{S}_{\tilde{\pi}}|$.

References

[LM15] Erez Lapid and Zhengyu Mao. A conjecture on Whittaker–Fourier coefficients of cusp forms. *Journal of Number Theory*, 146:448–505, 2015.