WEYLGRUPPE UND MOMENTABBILDUNG

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1. Introduction

This is my study note for Knop's paper[Kno90], he assigns to any G-variety X a finite cristallographic reflection group W_X by means of the moment map on the cotangent bundle. He also determines the closure of the image of the moment map and the generic isotropy group of the action of G on the cotangent bundle.

In this note, I will fix k a characteristic zero algebraically closed field and G a connected reductive group, $B \subseteq G$ a Borel subgroup, $T \subseteq B$ a maximal torus $U \subseteq B$ its maximal unipotent subgroup.

2. Horospherical subgroup

Definition 2.1. A subgroup S of G is called horospherical if it contains a maximal unipotent subgroup of G. A G-variety X is called horospherical if all the stabilizer subgroup are horospherical.

For a parabolic subgroup P of G we will denote P' its commutator subgroup.

Proposition 2.2. *For $S \subseteq G$ a horospherical subgroup, $P := N_G(S)$ is a parabolic subgroup and S contains P', also P/S is a torus.*

Proposition 2.3. *For X an G-variety, there is a nonempty B-stable open subset $V \subseteq X$ with the property that all the stabilizer subgroup B_x for $x \in V$ are conjugated to a subgroup B_0 of B via conjugation in B.*

Furthermore, there is a parabolic subgroup P, such that $B \subseteq P$ and a Levi subgroup L with $(L, L) \cap B \subseteq B_0 \subseteq L \cap B$.

For P, L and B_0 as in the previous proposition, and P^- the opposite parabolic of P, then there exists exactly one horospherical subgroup S with $S \cap B = B_0$ and $N_G(S) = P^-$.

Definition 2.4. The conjugacy class $\mathfrak{S}_X$ of S will be called the *horospherical type* of the G-variety X, and rank of X is defined to be rg $X := \dim P/S$.

Question: Given a spherical variety X how to calculate its horospherical type?

3. Moment image

Let X be a smooth G-variety, $T^*X := Spec\ S^*\Omega_X^\vee$ with canonical projection $\pi : T^*X \to X$, we denote

$$\tilde{\Phi} : T^*X \longrightarrow \mathfrak{g}^*, \quad \alpha \mapsto [\xi \mapsto \alpha(\xi_{\pi(\alpha)})]$$

we will introduce a refined moment map with better properties later. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra and $\mathscr{D}(X)$ the algebra of linear differential operators on X. There is a natural homomorphism

$$\psi_X : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathscr{D}(X)$$

both algebras are equipped with a canonical filtration so that $\psi(\mathfrak{U}_n) \subseteq \mathscr{D}_n(X)$ holds. We will denote the kernel of ψ_X by I_X.

For the associated graded algebras

$$gr\ \mathfrak{U} = S(\mathfrak{g}) = k[\mathfrak{g}^*]$$

If X is homogeneous $X = G/H$, then $T^*X = G \times^H \mathfrak{h}^\perp$, and $\tilde{\Phi}$ is

$$\tilde{\Phi} : G \times^H \mathfrak{h}^\perp \longrightarrow \mathfrak{g}^*, \ [g, \alpha] \mapsto g\alpha$$

Set $\mathscr{M}_X := \mathfrak{U}/I_X = \psi_X(\mathfrak{U}) \subseteq \mathscr{D}(X)$, $\mathscr{M}_X$ has two canonical filtrations, one is the G-filtration and the other is the X-filtration induced from $\mathscr{D}(X)$.

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Proposition 3.1. *For $S \in \mathfrak{S}_X$, $I_X = I_{G/S}$.*

from this proposition, we know there is a third filtration on $\mathscr{M}_X$ namely the G/S-filtration via $\mathscr{M}_X = \mathscr{M}_{G/S}$.

Definition 3.2. A filtration of a $\mathfrak{U}$-module is said to be good if $gr\ \mathscr{M}$ as a $gr\ \mathfrak{U}$-module is finitely generated.

The G-filtration is trivially good, but we also have

Corollary 3.3. *The X-filtration on $\mathscr{M}_X$ is good and it is consistent with the G/S-filtration.*

Proposition 3.4. *For G a connected reductive group and X a smooth G-variety, $S \in \mathfrak{S}_X$ with Lie algebra $\mathfrak{s}$, then the closures of the image of the moment map of T^*X and $T^*(G/S)$ are equal*

$$\overline{\tilde{\Phi}(T^*X)} = G \cdot \mathfrak{s}^{\perp} \subseteq \mathfrak{g}^*$$

4. The factorization of the moment image

We denote $\tilde{M}_X := \overline{\tilde{\Phi}(T^*X)} = G \cdot \mathfrak{s}^{\perp}$, in general the fiber of $\tilde{\Phi}: \ T^*X \to \tilde{M}_X$ is not irreducible, this means that $k[\tilde{M}_X]$ is not algebraically closed in $k[T^*X]$, we will denote M_X the spectrum of the algebraical closure of $k[\tilde{M}_X]$ in $k[T^*X]$, we will denote the morphism $T^*X \to M_X$ by Φ. We will denote $\tilde{L}_X := \mathrm{Im}\ \tilde{M}_X \subseteq \mathfrak{g}^*//G := \mathrm{Spec}\ k[\mathfrak{g}^*]^G$ and $L_X := \mathrm{Spec}\ k[M_X]^G$. Denote the morphism $T^*X \to L_X$ by Ψ and $\Pi: \ M_X \to L_X$ the quotient morphism. We have the following commutative diagram

$$\begin{array}{ccccccc} T^*X & \xrightarrow{\Phi} & M_X & \longrightarrow & \tilde{M}_X & \longrightarrow & \mathfrak{g}^* \\ & & \downarrow{\scriptstyle \Pi} & & \downarrow & & \downarrow \\ & & L_X & \longrightarrow & \tilde{L}_X & \longrightarrow & \mathfrak{g}^*//G \end{array}$$

Lemma 4.1. *Let $P \subseteq G$ be a parabolic subgroup with Levi factor L, suppose $\alpha_1, \ \alpha_2 \in \mathfrak{p}_u^{\perp}$ satisfies $\alpha_1|_{\mathfrak{l}} = \alpha_2|_{\mathfrak{l}}$, then α_1 and α_2 have the same image in $\mathfrak{g}^*//G$.*

For $S \in \mathfrak{S}_X, \ P = N_G(S), \ A = P/S$, lemma 4.1 factorizes the morphsim $\mathfrak{s}^{\perp} \to \mathfrak{g}^*//G$ through $\mathfrak{a}^*$, we have the following commutative diagram

$$\begin{array}{ccc} T^*X & & \mathfrak{a}^* \\ \downarrow & & \downarrow \\ L_X & \longrightarrow & \mathfrak{g}^*//G \end{array}$$

$\mathfrak{a}^*$ and L_X have the same image in $\mathfrak{g}^*//G$.

Lemma 4.2. *There exists a morphism $\mathfrak{a}^* \to T^*X$ such that the following diagram is commutative*

$$\begin{array}{ccc} T^*X & \xleftarrow[\sigma]{} & \mathfrak{a}^* \\ \downarrow{\scriptstyle \Psi} & & \downarrow \\ L_X & \longrightarrow & \mathfrak{g}^*//G \end{array}$$

The subgroup $W = W(\mathfrak{t}^*)$ is the Weyl group of G, and we can identify $\mathfrak{t}^*/W$ with $\mathfrak{g}^*//G$, we set $W_1 = W(\mathfrak{a}^*)$, since L_X is normal, we have the following inclusions

$$k[\mathfrak{a}^*]^{W_1} \subseteq k[L_X] \subseteq k[\mathfrak{a}^*]$$

From the Galois theory, we know that there is a subgroup $W_X \subseteq W_1$ such that $k[L_X] = k[\mathfrak{a}^*]^{W_X}$ and we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{a}^* & \longrightarrow & T^*X \\ \downarrow & & \downarrow{\scriptstyle \Psi} \\ \mathfrak{a}^*/W_X & \longrightarrow & L_X \end{array}$$

Definition 4.3. The group W_X is defined to be the *Weyl group* of X. For singular X we define $W_X := W_{X^{reg}}$.

5. Geometry of moment map

Corollary 5.1. *Let G/H be a homogeneous spherical variety, then $k[\mathfrak{h}^\perp]^H$ is a polynomial ring and it is flat over $k[\mathfrak{h}^\perp]$.*

Proof. We have

$$k[\mathfrak{h}^\perp]^H = k[T^*(G/H)]^G = k[L_{G/H}]$$

□

Example 5.2. For $X = G/H$ a symmetric variety, $\mathfrak{a} \subseteq \mathfrak{h}^\perp \subseteq \mathfrak{g}$ is a maximal commutative subalgebra, let's denote $W = N_H(\mathfrak{a})/Z_H(\mathfrak{a})$ the so called small Weyl group, the restriction map gives an isomorphism

$$k[\mathfrak{h}^\perp]^H \cong k[\mathfrak{a}]^W$$

the previous corollary 5.1 gives $W_X = W$.

Example 5.3. For $G = Sp_4$, $H = \mathbb{G}_m \times SL_2 \subseteq Sp_2 \times Sp_2$, $X = G/H$, then X is spherical of rank two, and $k[\mathfrak{h}^\perp]^H$ is generated by two quadratic polynomials, and $W_X = (\mathbb{Z}/2\mathbb{Z})^2$, there is no Cartan subspace as in the previous example.

6. The generic stabilizer group

We consider the open set T^0X of T^*X of all points $x \in T^*X$ such that

- The stabilizer G_x has minimal dimension.
- The stabilizer $G_{\Phi(x)}$ has minimal dimension.
- The morphism $\tilde{\Psi}:\ T^*X \to \tilde{L}_X$ is smooth at x.
- $\mathfrak{a}_X \to \tilde{L}_X$ is smooth above $\tilde{\Psi}_{(X)}$.

Now look at the following diagram

$$\begin{array}{ccccc} T^*X & & T^*(G/S) & \xrightarrow{\pi_0} & G/S \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} & & \\ M_X & \longleftarrow & M_{G/S} & & \end{array}$$

Definition 6.1. A horospherical subgroup S_0 is called a polarization at x if there exists $y_0 \in T^*(G/S)$ and $y_0 \mapsto y$, $S_0 = G_{\pi_0(y_0)}$.

Proposition 6.2. *For $x \in T^0X$, $y = \Phi(x) \in M_X$ and S a polarization of x, then $G_y \cap S$ is a subgroup of G_x of finite index and for generic x, equality even holds, in particular G_x is normal in G_y and the quotient is a torus, there is a canonical surjection*

$$P/S = G_y/G_y \cap S \longrightarrow G_y/G_x$$

References

[Kno90] Friedrich Knop. Weylgruppe und Momentabbildung. *Inventiones mathematicae*, 99(1):1–23, 1990.