WEYL INTEGRATION FORMULA

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1. Introduction

This is my study note for Weyl's integration formula following Knapp's book [KK96].

2. Compact Lie groups

We have the following theorem on analytic aspect of the abstract representation of compact groups

Theorem 2.1. *(Peter-Weyl theorem) If G is a compact group, then the linear span of all matrix coefficients for all finite dimensional irreducible representations of G is dense in $L^2(G)$.*

Example 2.2. For $G = S^1$ which is abelian, every irreducible representation is 1-dimensional, the matrix coefficients are the function $e^{in\theta}$, the Peter-Weyl theorem in this case says that the finite linear combination of these functions are dense in $L^2(S^1)$. An equivalent formulation is that $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ is an orthonormal basis of $L^2(S^1)$.

We have the following formula for characters of finite dimensional representations of complex semisimple Lie algebra

Theorem 2.3. *(Weyl character formula) Let V be an irreducible finite dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight λ, then*

$$char(V) = d^{-1} \sum_{\omega \in W} \epsilon(\omega) e^{\omega(\lambda+\delta)}$$

here

$$d = e^{\delta} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$$

Example 2.4. For the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, for $\lambda \in \mathfrak{h}^*$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\lambda(zh) = zn$, V_λ the highest weight λ representation, the Weyl character formula takes the form

$$\mathrm{char}(V_\lambda) = \frac{e^{\lambda+\delta} - e^{-(\lambda+\delta)}}{e^{\delta} - e^{-\delta}}$$

Theorem 2.5. *(Weyl character formula) Let G be a compact connected Lie group, let T be a maximal torus and $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ a positive system of simple roots. Let $\lambda \in \mathfrak{t}^*$ be analytically dominant and integral, then the character χ_{Φ_λ} of the highest weight λ representation Φ_λ is given by*

$$\chi_{\Phi_\lambda}(t) = \frac{\sum_{\omega \in W} \epsilon(\omega) \xi_{\omega(\lambda+\delta)-\delta}(t)}{\prod_{\alpha \in \Delta^+} (1 - \xi_{-\alpha}(t))}$$

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3. Weyl integration formula

We let G be a connected compact Lie group and T a maximal torus, we let G' and T' be the set of regular elements in G and T. We let $\mathfrak{g}_0$ and $\mathfrak{t}_0$ be the real Lie algebra of G and T.

Lemma 3.1. *We have a surjective map $\psi:\ G/T \times T \to G$ by $\psi(\overline{g}, t) = gtg^{-1}$ and each member of G' has exactly $|W(G,T)|$ preimages under ψ.*

Then lemma 3.1, together with

$$|\det(d\ \psi)_{(g,t)}| = |\det(\mathrm{Ad}(t^{-1}) - 1)|_{\mathfrak{t}_0^{\perp}})| = \prod_{\alpha\in\Delta^+} |\xi_\alpha(t^{-1}) - 1|^2$$

gives us the following theorem

Theorem 3.2. *(Weyl integration formula) Let T be a maximal torus of a compact connected Lie group G and let invariant measures are normalized on $G, T, G/T$ so that*

$$\int_G f(x)\ dx = \int_{G/T} (\int_T f(xt)\ dt)\ d(xT)$$

for all continuous function f on G. Then for every Borel function $F \geq 0$ on G we have

$$\int_G F(x)\ dx = \frac{1}{|W(G,T)|} \int_T (\int_{G/T} F(gtg^{-1}) d(gT))\ |D(t)|^2\ dt$$

where $|D(t)|^2 = \prod_{\alpha\in\Delta^+} |1 - \xi_\alpha(t^{-1})|^2$.

This integration formula is the starting point for the analytic part of the representation theory for compact connected Lie groups.

Let's define $D(t) = \xi_\delta(t) \prod_{\alpha\in\Delta^+}(1 - \xi_{-\alpha}(t))$, then for any Borel function f constant on conjugacy classes we get

$$\int_G f(x)dx = \frac{1}{|W(G,T)|}\ f(t)|D(t)|^2\ dt \tag{3.1}$$

here we take $dx,\ dt,\ d(gT)$ to have total mass one.

For every $\lambda \in \mathfrak{t}^*$ dominant and analitically integral, we can define

$$\chi_\lambda(t) = \frac{\Sigma_{s\in W(G,T)}\ \epsilon(s)\xi_{s(\lambda+\delta)}(t)}{D(t)}$$

then χ_λ is $W(G,T)$ invariant, and $\chi_\lambda(t)$ extends to a function χ_λ on G constant on conjugacy classes. Applying (3.1) with $f = |\chi_\lambda|^2$, we get

$$\int_G |\chi_\lambda|^2\ dx = \ 1$$

Applying (3.1) with $f = \chi_\lambda \overline{\chi_{\lambda'}}$, we get

$$\int_G \chi_\lambda(x)\overline{\chi_{\lambda'}(x)}\ dx = 0$$

for $\lambda \neq \lambda'$.

Let χ be the character of a finite dimensional representation on G. On T, χ must be of the form $\sum_\mu\ \xi_\mu$, since $D(t)\chi(t)$ must be of the form $\sum_v\ n_v\xi_v(t)$, we can show further $\chi(t) = \sum_\lambda\ a_\lambda\chi_\lambda(t)$ with $a_\lambda \in \mathbb{Z}$, now from the integration result for χ_λ, we obtain

$$\int_G |\chi(x)|^2\ dx = \sum_\lambda\ |a_\lambda|^2$$

for an irreducible character, from the Schur orthogonality relation, we can show the left hand side must be 1, so one a_λ is ± 1 and others are 0. Since χ is of the form ξ_μ, we must have $a_\lambda = 1$ for some λ. Hence every irreducible characters is of the form $\chi = \chi_\lambda$ for some λ.

This gives an *analytic proof* of the Weyl character formula 2.5. Using the Peter-Weyl theorem 2.1, we can see that no L^2 function on G that is constant on the conjugacy classes can be orthogonal to all irreducible characters, this proves the existence of an irreducible representation corresponding to a given dominant analytically integral form as highest weight.

4. Harish-Chandra's work

Let G be a reductive Lie group, based on the result that the regular elements G' in G is contained in the G-conjugates of the Cartan subgroups of G, Harish-Chandra proves the following generalization of the Weyl integration formula for general reductive Lie groups

Theorem 4.1. *Let G be a reductive Lie group and $(\mathfrak{h}_1)_0, \cdots, (\mathfrak{h}_r)_0$ a maxial set of nonconjugate θ stable Cartan subalgebras of $\mathfrak{g}_0$, let H_i be the corresponding Cartan subgroups, let the invariant measure on H_j and G/H_j normalized so that*

$$\int_G f(x)\ dx = \int_{G/H_j} \int_{H_j} f(gh)\ dh\ d(gH_j)$$

for all f compactly supported function on G, then for every Borel function $F \geq 0$ on G we have

$$\int_G F(x)\ dx = \sum_{j=1}^{r} \frac{1}{|W(G,H_j)|} \int_{H_j} \int_{G/H_j} F(ghg^{-1})\ d(gH_j) |D_{H_j}(h)|^2\ dh$$

where $|D_{H_j}(h)|^2 = \prod_{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}_j)} |1 - \xi_\alpha(h^{-1})|$.

Harish-Chandra proves a generalization of the Peter-Weyl theorem 2.1 for general semisimple Lie groups. Harish-Chandra also proves a generalization of the Weyl character formula for discrete series representations for general semisimple Lie groups, the construction of the discrete series with given infinitesimal character is much harder.

References

[KK96] Anthony W Knapp and Anthony William Knapp. *Lie groups beyond an introduction*, volume 140. Springer, 1996.