SPHERICAL FUNCTIONS ON UNITARY MATRICES

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1. Introduction

This is a study note for the paper [HK14].

2. The space X

Let k' be an unramified extension of a p-adic field k of odd residue characteristic and consider Hermitian and unitary matrices with respect to k'/k, and denote by a^* the conjugate transpose of $a \in M_{mn}(k')$. Let π be a prime element of k and q the cardinality of the residue field $\mathcal{O}_k/(\pi)$. We fix a unit $\epsilon \in \mathcal{O}_k^\times$ for which $k' = k(\sqrt{\epsilon})$.

We consider the unitary group

$$G = G_n = \{g \in GL_{2n+1}(k') \mid g^* j_{2n+1} g = j_{2n+1}\}, \quad j_{2n+1} = \begin{pmatrix} 0 & \cdots & 1 \\ & \cdots & \\ 1 & \cdots & 0 \end{pmatrix}$$

and the space X of unitary matrices in G

$$X = X_n = \{x \in G \mid x^* = x,\ \Phi_{xj_{2n+1}}(t) = (t^2-1)^n(t-1)\}$$

where $\Phi_y(t)$ is the characteristic polynomial of the matrix y. We note that X is a single $G(\overline{k})$-orbit containing 1_{2n+1} over the algebraic closure $\overline{k}$ of k. The group G acts on X by

$$g \cdot x = gxg^* = gxj_{2n+1}g^{-1}j_{2n+1},\ g \in G,\ x \in X$$

we fix a maximal compact subgroup K of G by

$$K = K_n = G \cap M_{2n+1}(\mathcal{O}_{k'})$$

Proposition 2.1. *There are precisely two G-orbits in X_n:*

$$G \cdot x_0 = \bigsqcup_{\lambda \in \Lambda_n^+,\ |\lambda| \text{ is even}} K \cdot x_\lambda,\ G \cdot x_1 = \bigsqcup_{\lambda \in \Lambda_n^+,\ |\lambda| \text{ is odd}} K \cdot x_\lambda$$

where $|\lambda| = \sum_{i=1}^n \lambda_i$, $x_0 = 1_{2n+1}$ *and* $x_1 = diag(\pi, 1, \cdots, 1, \pi^{-1})$.

Proof. First we know that there are at most two G-orbits in X_n. We extend the k-automrophism $*$ of k' to an element of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ and the action of G on X to $G(\overline{k})$ on $X(\overline{k})$. We recall $X(\overline{k})$ is a single $G(\overline{k})$-orbit and set

$$H(\overline{k}) = \{h \in G(\overline{k}) \mid h \cdot 1_{2n+1} = 1_{2n+1}\}$$

then we can obtain

$$H(\overline{k}) \cong U(1_n)(\overline{k}) \times U(1_{2n+1})(\overline{k})$$

By the exact sequence of Γ-sets

$$1 \longrightarrow H(\overline{k}) \longrightarrow G(\overline{k}) \longrightarrow X_n(\overline{k}) \longrightarrow 1$$
$$g \longmapsto g \cdot 1_{2n+1}$$

we have an exact sequence of pointed sets

$$1 \longrightarrow G \cdot 1_{2n+1} \longrightarrow X_n \longrightarrow H^1(\Gamma, H(\overline{k})) \longrightarrow H^1(\Gamma, G(\overline{k}))$$

since $H^1(\Gamma, H(\overline{k})) \to H^1(\Gamma, G(\overline{k}))$ is a map from $C_2 \times C_2$ to C_2, it cannot be trivial, hence $G \cdot 1_{2n+1} \neq X_n$, hence there are least two G-orbits in X_n and thus exactly two G-orbits. $\square$

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3. Main result

Theorem 3.1. *A set of complete representations of* $K\backslash X$ *can be taken as*

$$\{x_\lambda \mid \lambda \in \Lambda_n^+\}$$

where

$$x_\lambda = \begin{cases} Diag(\pi^{\lambda_1}, \cdots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \cdots, \pi^{-\lambda_1}) & if \;\; m = 2n \\ Diag(\pi^{\lambda_1}, \cdots, \pi^{\lambda_n}, 1, \pi^{-\lambda_n}, \cdots, \pi^{-\lambda_1}) & if \;\; m = 2n+1 \end{cases}$$

$\Lambda_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$

A spherical function on X is a K-invariant function on X which is a common eigenfunction with respect to the common convolutive action of the Hecke algebra $\mathcal{H}(G, K)$, and a typical one is constructed by Poisson transform from the relative invariants of a parabolic subgroup. We take the Borel subgroup B consisting of upper triangular matrices in G.

We introduce a spherical function $\omega(x; s)$ on X by Poisson transform from relative B-invariants, for a matrix $g \in G$, denote $d_i(g)$ the determinant of lower right i by i block of g. Then $d_i(x),\ 1 \leq i \leq n$ are relative B-invariants on X associated with rational characters ψ_i of B where

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)) \; (x \in X,\ p \in B)$$

For $x \in X$ and $s \in \mathbb{C}^n$, we consider the integral

$$\omega(x; s) = \int_K |d(k \cdot x)|^s \, dk, \quad |d(y)|^s = \prod_{i=1}^n |d_i(y)|^{s_i}$$

Then the right hand side is absolutely convergent for $\mathrm{Res}(s_i) \geq 0,\ 1 \leq i \leq n$ and continued to be a rational function of $q^{s_1}, \cdots, q^{s_n}$. Since $d_i(x)$ are relative B-invariants on X such that

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)) \quad (p \in B,\ x \in X,\ 1 \leq i \leq n)$$

we see $\omega(x; s)$ is a spherical function X which satisfies

$$f * \omega(x; s) = \lambda_s(f) \omega(x; s), \ f \in \mathcal{H}(G, K)$$
$$\lambda_s(f) = \int_B f(p) \Pi_{i=1}^n |\psi_i(p)|^{-s_i} \delta(p) \, dp$$

The Weyl group W of G relative to B acts on the rational characters of B, hence on z and s also. The group W is generated by S_n which acts on z by permutation of indices and by τ such that

$$\tau(z_1, \cdots, z_n) = (z_1, \cdots, z_{n-1}, -z_n)$$

To describe the results, we prepare some notation, we set

$$\Sigma^+ = \Sigma_s^+ \sqcup \Sigma_\ell^+$$
$$\Sigma_s^+ = \{e_i + e_j,\ e_i - e_j \mid 1 \leq i < j \leq n\}, \ \Sigma_\ell^+ = \{2e_i \mid 1 \leq i \leq n\}$$

where $e_i \in \mathbb{Z}^n$ is the i-th unit vector, we define a pairing

$$\langle\ ,\ \rangle \; : \; \mathbb{Z}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}, \quad \langle \alpha, z \rangle = \sum_{i=1}^n \alpha_i z_i$$

Theorem 3.2. *The function* $G(z) \cdot \omega(x; z)$ *is holomorphic and* W*-invariant, hence belong to* $\mathbb{C}[q^{\pm z_1}, \cdots, q^{\pm z_n}]^W$ *where*

$$G(z) = \prod_\alpha \frac{1 + q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle} - 1}$$

where α *runs over the set* Σ_s^+ *for* $m = 2n$ *and* Σ^+ *for* $m = 2n+1$.

Theorem 3.3. *(Explicit formula) For each $\lambda \in \Lambda_n^+$, one has*

$$\omega(x_\lambda; z) = \frac{c_n}{G(z)} \cdot q^{\langle \lambda, z_0 \rangle} \cdot Q_\lambda(z; t)$$

where $G(z)$ is given as in 3.2, z_0 is the value in z-variable corresponding to $s = 0$, c_n is some explicit constants.

$$Q_\lambda(z; t) = \sum_{\sigma \in W} \sigma(q^{-\langle \lambda, z \rangle} c(z; t)), \quad c(z; t) = \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle}}$$

here

$$t_\alpha = \begin{cases} t_s \text{ if } \alpha \in \Sigma_s^+ \\ t_\ell \text{ if } \alpha \in \Sigma_\ell^+ \end{cases}$$

where $t_s = -q^{-1}$, and $t_\ell = \begin{cases} q^{-1} & \text{if } m = 2n \\ -q^{-2} & \text{if } m = 2n + 1 \end{cases}$

We see that $Q_\lambda(z; t) \in \mathcal{R}$ by 3.2. It is known that $Q_\lambda(z; t) = W_\lambda(t) P_\lambda(z; t)$ with Hall-Littlewood polynomial $P_\lambda(z; t)$ and Poincare polynomial $W_\lambda(t)$ and the set $\{P_\lambda(z; t) \mid \lambda \in \Lambda_n^+\}$ forms an orthogonal $\mathbb{C}$-basis for $\mathcal{R}$ for each $t_\alpha \in \mathbb{R}$, $|t_\alpha| < 1$.

In particular, we have

$$\omega(x_0; z) = \frac{(1 - q^{-1})^n \omega_n(-q^{-1}) \omega_{m'}(-q^{-1})}{\omega_m(-q^{-1})} \cdot G(z)^{-1}, \quad m' = [\frac{m+1}{2}]$$

We modify spherical functions as follows

$$\Psi(x; z) = \frac{\omega(x; z)}{\omega(1_{2n}; z)} \in \mathcal{R} = \mathbb{C}[q^{\pm z_1}, \cdots, q^{\pm z_n}]^W$$

and define the spherical Fourier transform on the Schwartz space by

$$\begin{aligned} F : \ & \mathcal{S}(K \backslash X) \longrightarrow \mathcal{R} \\ & \varphi \longmapsto F(\varphi)(z) = \int_X \varphi(x) \ \Psi(x; z) \ dx \end{aligned}$$

Theorem 3.4. *The spherical transform F gives an $\mathcal{H}(G, K)$-module isomorphism*

$$\mathcal{S}(K \backslash X) \cong \mathbb{C}[q^{\pm z_1}, \cdots, q^{\pm z_n}]^W$$

We introduce the inner product $\langle \ , \ \rangle_{\mathcal{R}}$ on $\mathcal{R}$ by

$$\langle P, Q \rangle_{\mathcal{R}} = \int_{\mathfrak{a}^*} P(z) \overline{Q(z)} \ d\mu(z), \ \ P, Q \in \mathcal{R}$$

here $\mathfrak{a}^* = \{\sqrt{-1}(\mathbb{R}/\frac{2\pi}{\log q}\mathbb{Z})\}^n$ and

$$d\mu = \frac{1}{n! 2^n} \cdot \frac{\omega_n(-q^{-1}) \omega_{n+1}(-q^{-1})}{(1 + q^{-1})^{n+1}} \cdot \frac{1}{|c(z)|^2} dz \tag{3.1}$$

Theorem 3.5. *Let $d\mu$ be the measure defined by (3.1), then by the normalization of G-invariant measure dx such that*

$$v(K \cdot x_\lambda) = q^{-2\langle \lambda, z_0 \rangle} \frac{\widetilde{\omega_0}(-q^{-1})}{\widetilde{\omega_\lambda}(-q^{-1})}$$

for any $\varphi, \psi \in \mathcal{S}(K \backslash X)$, we have

$$\int_X \varphi(x) \overline{\psi(x)} \ dx = \int_{\mathfrak{a}^*} F(\varphi)(z) \overline{F(\psi)(z)} \ d\mu(z)$$

Proposition 3.6. *Assume $n = 1$, for $x_\ell = diag(\pi^\ell, 1, \pi^{-\ell})$, $\ell \geq 0$, it holds that*

$$\omega(x_\ell; s) = \frac{1 + q^{-3-2s}}{(1 + q^{-3})(1 - q^{-4-4s})} \{q^{\ell s}(1 - q^{-4-2s}) - q^{-2(\ell+1) - \ell s}(1 - q^{-2s})\}$$

where $s = -z - 1 - \frac{\pi\sqrt{-1}}{2log\ q}$ *and for any* $x \in X_1$

$$\omega(x; z) = \frac{1 - q^{-1+2z}}{q^{2z} - q^{-1}}\omega(x; -z)$$

Let's discuss the proof of this proposition.

Lemma 3.7. *We have* $K_1 = K_{1,1} \sqcup K_{1,2}$.

Theorem 3.8. *There are precisely two* G*-orbits in* X_1

$$G \cdot x_0 = \bigsqcup_{\lambda\in\Lambda_n^+,\ |\lambda|\ even} K \cdot x_\lambda,\ G \cdot x_1 = \bigsqcup_{\lambda\in\Lambda_n^+,\ |\lambda|\ odd} K \cdot x_\lambda$$

It is easy to see that $vol(K_{1,1}) : vol(K_{1,2}) = 1 : q^{-3}$. For $k \in K_{1,2}$, we have

$$|d_1(k \cdot x_\ell)| = |\pi|^{-\ell}$$

and

$$\int_{K_{1,2}} |d_1(k \cdot x_\ell)|^s\ dk = \frac{q^{-3+\ell s}}{1 + q^{-3}}$$

Assume ℓ is even and positive, then

$$(1-q^{-1-2s})(1+q^{-3})q^{-\ell s} \int_{K_{1,1}} |d_1(k{\cdot}x_\ell)|^s\ dk = \frac{1 - q^{-1+2s}}{1 - q^{-4-4s}}(1-q^{-3}+q^{-3-2s}-q^{-4-2s}-q^{-2(\ell+1)-2\ell s}(1-q^{-2s})(1+q^{-3-2s}))$$

Hence we obtain for even ℓ

$$\begin{aligned}&\int_K |d_1(k \cdot x_\ell)|^s\ dk \\ &= \frac{(1 + q^{-3-2s})q^{\ell s}}{(1 + q^{-3})(1 - q^{-4-4s})}\{(1 - q^{-4-2s}) - q^{-2\ell-2\ell s}(q^{-2} - q^{-2-2s})\}\end{aligned}$$

changing the variable from s to z we get

$$\begin{aligned}&\int_K |d_1(k \cdot x_\ell)|^s\ dk \\ &= \frac{\sqrt{-1}^\ell q^{-\ell}(1 - q^{-1+2z})}{(1 + q^{-3})(1 + q^{2z})}\{q^{-\ell z}\frac{1 + q^{-2+2z}}{1 - q^{2z}} + q^{\ell z}\frac{1 + q^{-2-2z}}{1 - q^{-2z}}\}\end{aligned}$$

Assume ℓ is odd, we can calculate similarly

$$\begin{aligned}&\int_K |d_1(k \cdot x_\ell)|^s\ dk \\ &= \frac{\sqrt{-1}^\ell q^{-\ell}(1 - q^{-1+2z})}{(1 + q^{-3})(1 - q^{4z})}\{q^{-\ell z}(1 + q^{2z-2}) - q^{\ell z}(q^{-2} + q^{2z})\}\end{aligned}$$

References

[HK14] Yumiko Hironaka and Yasushi Komori. Spherical functions on the space of p-adic unitary hermitian matrices II, the case of odd size. *arXiv preprint arXiv:1403.3748*, 2014.