L-FUNCTIONS FOR U(3)

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1. Introduction

Following [GPS06] and [GR91] we describe two different zeta integral representations of the degree 6 L-function attached to a cuspidal representation π of $U(3)$, the first is a Rankin-Selberg integral valid only for generic π and the second is a more complicated "Shimura-type" integral valid for arbitrary π.

2. L-functions for $U(3)$

2.1. **Eisenstein series on $U(1,1)$.** Recall that V is the three dimensional vector space over E with skew-Hermitian form given by the matrix

$$\begin{pmatrix} & & 1 \\ & \xi & \\ -1 & & \end{pmatrix}$$

denote by $\ell_{-1},\ \ell_0,\ \ell_1$ the corresponding basis for V, let W be the subspace spanned by ℓ_{-1} and ℓ_1, then W is a skew-Hermitian space with unitary group $H = U(W)$ may be identified with the subgroup of G stabilizing ℓ_0. Let B_H denote the Borel subgroup

$$\{\begin{pmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\alpha}^{-1} \end{pmatrix}\} \subset H$$

with maximal torus $\cong E^\times$. Fixing any character ξ of the idele class group of E and any $s \in \mathbb{C}$ we define a character ω_ξ^s of B_H via

$$\omega_\xi^s(\begin{pmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\alpha}^{-1} \end{pmatrix}) = \xi(\alpha)|\alpha|_E^s \quad \alpha \in \mathbb{A}_E^\times$$

we denote $F_s^* :\ H(\mathbb{A}) \to \mathbb{C}$ a smooth function satisfying

$$F_s^*(bh) = \omega_\xi^{s+1}(b)F_s^*(h)$$

where $b \in B_H(\mathbb{A})$ and $h \in H(\mathbb{A})$, we define the Eisenstein series by

$$\sum_{\gamma \in B_H \backslash H(F)} F_s^*(\gamma h) = E_\xi(h, F^*, s)$$

this is known to be convergent only in some right half plane $\mathrm{Re}(s) > s_0$. The only possible pole of E_ξ in the right half plane is at $s = 1$ and its residue is propotional to the character $\xi(\det h)$. In general E_ξ defines an autormophic forms on $H(\mathbb{A})$, we shall assume that $F^*(g)$ is decomposable

$$F^*(g) = \prod F_v^*(g_v)$$

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2.2. **The Rankin-Selberg integral.** Fix an arbitrary automorphic cuspidal representation π of $G(\mathbb{A})$ acting on the space V_π, to each $\varphi \in V_\pi$ and Eisenstein series data ξ and F^* as above, we may form the zeta-integral

$$L(\varphi, F^*, \xi, s) = \int_{H(k)\backslash H(\mathbb{A})} \varphi(h)\ E_\xi(h, F^*, s)\ dh$$

because E_ξ is an automorphic form on $H(F)\backslash H(\mathbb{A})$ and the restriction of φ to $H(F)\backslash H(\mathbb{A})$ is still rapidly decreasing, it follows that the integral converges and defines a meromorphic function in all of $\mathbb{C}$. Moreover $L(\varphi, F^*, \xi, s)$ has a functional equation. The only possible pole of $L(\varphi, F^*, \xi, s)$ is in $Re(s) > 0$ is at $s = 1$ and it is proportional to the period integral

$$\int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\xi(h)\ dh \tag{2.1}$$

Proposition 2.1. *Let S denote the finite set of places of F outside of which v is finite and the data φ_v, F_v^*, ψ_v and ξ_v are unramified. Then for $Re(s)$ sufficiently large, we have*

$$\begin{aligned} L(\varphi, F^*, \xi, s) &= \int_{U_H(\mathbb{A})\backslash H(\mathbb{A})} W_\varphi^\psi(h) F^*(h)\ dh \\ &= (\prod_{v\in S} L(W_v, F_v^*, s)) L_S(s, \pi\times\xi) \end{aligned}$$

here W_φ^ψ denotes the ψ-Whittaker function of φ but restricted to $H \subset G$ and $L(W_v, F_v^*, s)$ is the local zeta integral

$$\int_{U_v\backslash H_v} W_{\pi_v}(h) F_v(h)\ dh$$

and $L_S(s, \pi\times\xi)$ is the Langlands L-function, as a product outside the places v outside S. U_H the unipotent subgroup of H.

Remark 2.2. Because $H(F)\backslash H(\mathbb{A})$ may be regarded as an algebraic cycle in $G(F)\backslash G(\mathbb{A})$ we may interpret (2.1) as a period integral and conclude the existence of a pole for $L(s, \pi\times\xi)$ is related to the non-vanishing of this period.

2.3. **Unramified computation.** In this section, we will assume that everything is unramified. Thus we suppose that F is a local non-archimedean field of odd characteristic, and E is an unramified quadratic extension of F. Let $\mathcal{O}_F$ (resp. $\mathcal{O}_E$) denote the ring of integers of F (resp. E), ϖ a generator of the prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ and ψ a character of F of conductor 1.

Let K be the standard maximal compact subgroup of G, because E is unramified, we have

$$G = NAK$$

where $A = \mathrm{diag}(t, 1, t^{-1})\ \ t \in F^\times$ is the maximal F-split torus of G.

Suppose π is an unramified representation of G with respect to K, then π is of the form $\pi = \mathrm{Ind}_B^G \nu$, where ν is an unramified character of $E^\times$. The function W is uniquely characterized by the following properties

- $W(nak) = \psi_N(a) W(a)$ for all $n \in N,\ a \in A,\ k \in K$.
- $W(\begin{pmatrix} \delta & & \\ & 1 & \\ & & \delta^{-1} \end{pmatrix}) = 0$ if $|\delta|_F > 1$.
- for all $n \geq 0$
$$W(\begin{pmatrix} \varpi^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi^{-n} \end{pmatrix}) = |\varpi|^{2n} \frac{\nu(\varpi)^{n+1} - \nu(\varpi)^{-(n+1)}}{\nu(\varpi) - \nu(\varpi)^{-1}}$$

We now compute $L^\mu(W, F_\Phi, s)$ with μ an unramified character of $E^\times$ and Φ the characteristic function of the $\mathcal{O}_E$-module in $E\ell_{-1} \oplus E\ell_1$ generated by ℓ_{-1} and ℓ_1.

Let $K_H = K \cap H$, since $Z = N \cap H$, we have $H = ZAK_H$ with corresponding integration formula

$$\int_{Z\backslash H} f'(h)\ dh = \int_{K_H} \int_{F^\times} f'(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} k) |a|^{-2} d^\times a\ dk$$

here f' is a function of $Z\backslash H$ and Haar measure $d^\times a$ on $F^\times$ is normalized so that $m(\mathcal{O}_F^\times) = 1$. Note that

$$F_\Phi(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} k) = \mu(a)|a|_E^s F_\Phi(1)$$

we have

$$L^\mu(W, F_\Phi, s) = F_\Phi(1) \int_{F^\times} W(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix})\mu(a)|a|_F^{2s-2} d^\times \ a$$

we have $F_\Phi(1) = L_E(s, \mu)$.

It remains to calculate the integral with the unramified Whittaker function, we have

$$\begin{aligned} &\int_{F^\times} W(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix})\mu(a)|a|^{2s-2} \ d^\times a \\ &= \sum_{n=0}^{\infty} \mu(\varpi^n)|\varpi^n|^{2s} \sum_{i+j} \nu(\varpi)^i \nu(\varpi^{-1})^j \\ &= \frac{1}{1-\mu(\varpi)\nu(\varpi)|\varpi|^{2s}} \frac{1}{1-\mu(\varpi)\nu^{-1}(\varpi)|\varpi|^{2s}} \\ &= L_F(2s, \mu\nu) L_F(2s, \mu\nu^{-1}) \end{aligned}$$

all together we have $L(s, \pi, \mu) = Q_0(q^{-s})^{-1}$ with Q_0 a polynomial of degree 6 in q^{-s}.

3. On periods of cusp forms and algebraic cycles for $U(3)$

The following is a summary of the results of the paper [GRS93].

3.1. The relation between $P(\pi, c, \chi)$ and theta-lifting. We fix $U(1,1)$ to act on the Hermitian space $W = E\omega_1 \oplus E\omega_2$ with corresponding Hermitian form

$$\Phi' = \begin{pmatrix} 0 & \xi^{-1} \\ -\xi^{-1} & 0 \end{pmatrix}$$

then $U(W) \cong U(1,1) \cong H_1$, its derived subgroup is $SL_2(F)$, $U(V) \times U(W)$ embeds into the symplectic group and embeds into the metaplectic group of $V \otimes W$ for each choice of splitting data $(\psi, \gamma, \chi_1, \chi_2)$

Theorem 3.1. *Let π be a cuspidal representation of G, then the following are equivalent:*

- *$P(\pi, c, \chi) \neq 0$ for some c and χ.*
- *π has a non-zero theta-lift to H_1.*
- *π is a theta-lift to some cuspidal representation σ of H_1.*

Furthermore, suppose σ is the theta-lift of π to H_1 relative the specific lifting data $(\psi, \gamma, \chi_1, \chi_2)$ then $P(\pi, c, \gamma^1\chi_2) \neq \{0\}$ if and only if σ has a non-zero Whittaker model $\mathcal{W}(\sigma, \psi_c)$ relative to the additive character ψ_c.

The proof is based on the computation of the Fourier coefficient of the theta-lift $\varphi \in \pi$ to H_1

$$f_{\psi_c}(e) = \int_{G_c(\mathbb{A})\backslash G(\mathbb{A})} \gamma^1\chi_2(\det\ g)\Phi(v_c g \otimes \omega_2) P(\varphi^g, c, \gamma^1\chi_2) \ dg$$

where $\varphi^g(x) = \varphi(xg)$. Now since Φ is an arbitray Schwartz function, we can conclude that $f_{\psi_c}(e) = 0$ if and only if $P(\varphi^g, c, \gamma^1\chi_1) = 0$ for all $g \in G(\mathbb{A})$. In particular we conclude $P(\pi, c, \chi) \neq 0$ for some c and χ if and only if π has a non-zero theta-lift to H_1.

3.2. **Periods of stable π.** We say that a cuspidal π is *stable* if and only if any theta-series lift of π to any unitary group in two variables is zero. In particular any theta lift of π to $H_1 \cong U(1,1)$ is zero and hence for stable π

$$P(\pi, c, \chi)$$

for all c and χ. This fact is philosophically consistent with Tate's conjecture relating the poles for $L(s, \pi \times \xi)$ to the existence of non-trivial periods of π since $L(s, \pi \times \xi)$ is always entire for stable cuspidal π.

Therefore we may restrict our discussion to the periods to the case of *endoscopic* π.

3.3. **Periods of exceptional π.** Suppose π is cuspidal but in an A-packet $\Pi(\rho)$, if the theta lift of π to $U(1,1)$ were non-trivial, it would automatically be cuspidal and π itself would then be a theta-lift of this cuspidal σ on $U(1,1)$, but then 3.1 will imply that $\pi \in \Pi(\rho)$ with ρ cuspidal, this is a contradiction to our assumption that π is exceptional. Thus we conclude that the periods of such exceptional π must be all vanish.

3.4. **The case of generic cuspidal endoscopic π.** Suppose π is a generic element of a cuspidal endoscopic packet $\Pi(\rho)$, in this case the fact $L(s, \pi \times \xi)$ has a pole at $s = 1$ for some $\xi = \xi_0$ is equivalent to the fact that an appropriate theta-series lift of π to $U(1,1)$ is non-zero, and the residue $L(s, \pi \times \xi_0)$ can be expressed directly in terms of the period $P(\pi_0, 1, \xi_0)$. Using 3.1, we can conclude that for generic cuspidal π, the following are equivalent

(1) $L(s, \pi \times \xi)$ has a pole at $s = 1$ for some fixed $\xi = \xi_0$.
($2^{'}$) $P(\pi, 1, \xi_0) \neq 0$.
($3^{'}$) an appropriate theta-series lift of π to $U(1,1)$ is non-zero.

Now what is the situation for arbitrary π? By the theory of [GR91], the conditions (1) and ($3^{'}$) are still equivalent, provided the phrase "to $U(1,1)$" is replaced by the phrase "to some $U(\Phi')$".

3.5. **Compact periods of hypercuspidal π.** Let's call a cuspidal representation σ fo $U(Y)$ is theta-stable if any theta-lift of σ to $U(1) = U(Ev_c)$ is zero.

Proposition 3.2. *There exist hypercuspidal endoscopic cuspidal π on $U(3)$ with the property that*

$$P(\pi, c, \chi) = 0$$

for all c and χ. Indeed if σ is a theta-stable cuspidal representation of an anisotropic $U(Y)$ and the lifting is choosen so that the lift of σ to $U(3)$ is zero, then any irreducible component of $\pi = \Theta_{\psi,\gamma}^{\chi_1,\chi_2}(\sigma)$ is a cuspidal representation on $U(3)$.

Proposition 3.3. *There exists a hypercuspidal endoscopic π such that*

$$P(\pi, c, \chi) \neq 0$$

for some c and χ. Namely: take a cuspidal σ on $U(W) \cong U(1,1)$, a character ψ of $\mathbb{A}/F$ such that $\mathcal{W}(\sigma, \psi) = \{0\}$, and the lifting data $(\psi, \gamma, \chi_1, \chi_2)$ such that $\Theta_{\psi,\gamma}^{\chi_1,\chi_2}(\sigma) \neq \{0\}$ on $U(3)$, then each irreducible component of $\Theta_{\psi,\gamma}(\sigma)$ is an irreducible hypercuspidal endoscopic π of the above type.

First we want to show $\pi = \Theta_{\psi,\gamma}(\sigma)$ is hypercuspidal, hence $\Theta(\sigma)$ generates an irreducible hypercuspidal endoscopic representation π of $U(3)$, moreover π itself is a theta-lift from σ cuspidal on $U(1,1)$, it must have a non-zero lift back to $U(1,1)$, hence by 3.1, some $P(\pi, c, \chi)$ is non-zero.

References

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