THETA CORRESPONDENCE IN THE STABLE RANGE

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1. Introduction

This is my study note on the unitarity and irreducibility of theta correspondence at the stable range, this is used by Gan-Gomez [GG14] to establish certain cases of the Sakellaridis-Venkatesh conjecture for spherical varieties for classical groups.

2. Classical dual pairs

Let k be a local field and let $|\cdot|$ denote its absolute value, let $D = k$, or a quadratic field extension of k or the quaternion division k-algebra, and let $x \mapsto \overline{x}$ be its canonical involution. We have the *reduced trace map* $Tr: \ D \to k$ and the *reduced norm map* $Q : D \to k$, if $D \neq k$ then $\mathrm{Tr}(x) = x + \overline{x} \in k$ and $Q(x) = x \cdot \overline{x} \in k$.

Let V, W be two right D-modules, we will denote the set of right D-modules between V and W by $\mathrm{Hom}_D(V, W)$ which consists of

$$\{T : V \to W \mid T(v_1 a + v_2 b) = T(v_1)a + T(v_2)b \text{ for all } v_1,\ v_2 \in V,\ a, b\ \in D\}$$

we can define $\mathrm{Hom}_D(V, W)$ for two left D-modules in the same way. From now on, we will focus on the right D-modules. Set

$$GL(V, D) = \{\ T \in \mathrm{End}_D(V) \mid T \text{ is invertible}\}$$

Let V' be the set of right D-linear functionals on V, there is a natural left D-module structure on V' given by

$$(a\lambda)(v) = a\lambda(v)$$

for all $a \in D,\ v \in V$ and $\lambda \in V'$. Given $T \in \mathrm{Hom}_D(V, W)$, we can define an element in $\mathrm{Hom}_D(W', V')$, we will still denote it by T. This correspondence give natural isomorphisms between $\mathrm{End}_D(V)$ and $\mathrm{End}_D(V')$.

Definition 2.1. Let $\epsilon = \pm 1$, we say that (V, B) is a right ϵ-Hermitian D-module if V is a right D-module and B is an ϵ-Hermitian form, that is $B : V \times V \to D$ is a map such that

- B is sequilinear: for all $v_1, v_2, v_3 \in V,\ a, b \in D$
$$B(v_1, v_2 a + v_3 b) = B(v_1, v_2)a + B(v_1, v_3)b$$
$$B(v_1 a + v_2 b, v_3) = \overline{a}B(v_1, v_3) + \overline{b}B(v_2, v_3)$$
- B is ϵ-Hermitian:
$$B(v, w) = \epsilon\overline{B(w, v)}$$
- B is nondegenerate.

1-Hermitian D-modules will be called Hermitian, while the -1-Hermitian D-modules are called skew-Hermitian. Given a right ϵ-Hermitian D-module (V, B), we will define

$$G(V, B) = \{g \in GL(V) \mid B(gv, gw) = B(v, w) \text{ for all } v, w \in V\}$$

Given a right ϵ-Hermitian D-module (V, B), we can construct a left ϵ-Hermitian D-module (V^*, B^*) as follows: as a set V^* is $\{v^* \mid v \in V\}$, we give V^* a left D-module structure by setting

$$v^* + w^* = (v + w)^*,\ av^* = (v\overline{a})^*$$

for all $v, w \in V$ and $a \in D$. And we set

$$B^*(v^*, w^*) = \overline{B(w, v)}$$

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for all $v, w \in V$.

A D-submodule $X \subset V$ is said to be *totally isotropic* if $B|_{X\times X} = 0$, if X is a totally isotropic submodule then there exists a totally isotropic submodule $Y \subset V$ such that $B|_{X\oplus Y\times X\oplus Y}$ is nondegenerate.

Let (V, B_V) be a right ϵ_V-Hermitian D-module and let (W, B_W) be a right ϵ_W-Hermitian D-module such that $\epsilon_V\epsilon_W = -1$, then on the k-vector space $V \otimes_D W^*$ we can define a symplectic form B by

$$B(v_1 \otimes \lambda_1, v_2 \otimes \lambda_2) = \mathrm{Tr}(B_W(w_1, w_2)B_V^*(\lambda_2, \lambda_1))$$

for all $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in V^*$, we let

$$Sp(V \otimes W^*) = \{g \in GL(V \otimes W^*) \mid B(gv, gw) = B(v, w) \text{ for all } v, w \in V \otimes W^*\}$$

there is a natural map $G(V) \times G(W) \to Sp(V \otimes W^*)$ given by

$$(g_1, g_2) \cdot v \otimes \lambda = g_1 v \otimes \lambda g_2^*$$

we will use this map to identify $G(V)$ and $G(W)$ as subgroups of $Sp(V \otimes W^*)$ and these two subgroups are commutants of each other, this is an example of *reductive dual pair*.

The group $Sp(V \otimes W^*)$ has an S^1-cover $Mp(V \otimes W^*)$ which is called a metaplectic group, it is known that this S^1-cover splits over the subgroups $G(V)$ and $G(W)$, except when V is an odd-dimensional quadratic space, in which case it doesn't split over $G(W)$. The splitting are not necessarily unique.

Assume that there is a complete polarization $W = E \oplus F$ where E, F are complementary isotropic subspaces of W, let

$$P = \{p \in G(W) \mid p \cdot E = E\}$$

be the Siegel parabolic subgroup of $G(W)$ and let $P = MN$ be its Langlands decomposition. Given $\epsilon = \pm 1$, we set

$$\mathrm{Hom}_D(F, E)_\epsilon = \{T \in \mathrm{Hom}_D(F, E) \mid T^* = \epsilon T\}$$

then we have

$$\mathrm{Hom}_D(F, E) = \mathrm{Hom}_D(F, E)_1 \oplus \mathrm{Hom}_D(F, E)_{-1}$$

now we have

$$M = \{\begin{pmatrix} A & \\ & (A^*)^{-1} \end{pmatrix} \mid A \in GL(E)\}$$

and

$$N = \{\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \mid X^* = -\epsilon_W X\} \cong \mathrm{Hom}_D(F, E)_{-\epsilon_W}$$

3. Oscillator representation

Fix a nontrivial unitary character χ of k, associated with this character there exists a very special representation of the metaplectic group, the oscillator representation Π of $Mp(V \otimes W^*)$, restricting to $G(V) \times G(W)$, we get

$$\Pi|_{G(V)\times G(W)} = \int_{G(W)^\wedge} \pi \otimes \Theta(\pi)\, d\mu_\theta\ (\pi) \tag{3.1}$$

here $\Theta(\pi)$ is a possibly zero, possibly reducible unitary representation of $G(V)$, we will call the map Θ the L^2-theta correspondence.

By studying the relation between the smooth and unitary theta correspondence, one can show that the L^2-theta correspondence induces a map

$$\Theta : G(W)^\wedge \longrightarrow R_{\geq 0}(G(V)^\wedge)$$

where $R_{\geq 0}(G(V)^\wedge)$ is the Grothendieck semigroup of unitary representations of $G(V)$ of finite length. If k is not 2-adic, Θ takes value in $G(V)^\wedge \cup \{0\}$.

Let (V, B_V) and (W, B_W) as defore, if we assume now that there is a totally isotropic D-module $X \subset V$ such that $\dim_D(X) = \dim_D(W)$, in other words, the pair $(G(V), G(W))$ is in the *stable range*, then the map

$$\Theta : \hat{G}(W) \longrightarrow \hat{G}(V)$$

can be understood in terms of the result of Jianshu Li, the measure μ_θ is also known in this case, it is precisely the Plancherel measure of the group $G(W)$.

4. Unitarity

First note we have the following classification of irreducible dual pairs: let W be a vector space endowed with a non-degenerate symplectic form $\langle , \rangle$, and $\mathrm{Tr}: D \to F$ the trace map

Theorem 4.1. *([How89]) Let G, G' be an irreducible dual pair in $Sp(W)$, then either*

- *There are a division algebra D with involution ι, vector spaces V and V', and semi-linear forms $(,)$ and $(,)'$, one hermitian and one skew-hermitian, such that $W = V \otimes_D V'$, the symplectic form is given by*
$$\langle v_1 \otimes v_1', v_2 \otimes v_2' \rangle = tr((v_1, v_2) \cdot (v_2', v_1'))$$
and G is identified with the isometry group of $(,)$, G' is identified with the isometry group of $(,)'$.
- *There is a division algebra D and a complete polarization $W = U \oplus U^*$ preserved by $G \cdot G'$, and isomorphisms $U \cong V \otimes_D V'$, $U^* \cong V^* \otimes_D V'^*$, where V, V' are vector spaces over D, and we can identify G with $GL_D(V)$ and G' with $GL_D(V')$.*

We will call the first case the type I dual pairs and the second case type II dual pairs. In [], Howe actually showed that every dual pair in Sp_{2n} is the direct sum of irreducible ones in an essentially way and the irreducible dual pairs are given above.

Let G, G' be an irreducible type I dual pair in $Sp(W)$ with $W = V \otimes V'$, let ω be the oscillator representation attached to some character ψ realized on the Hilbert space $\mathcal{H}$. We want to understand when is the unitarity preserved under the duality correspondence.

If $\pi \otimes \pi'$ occurs in the discrete spectrum, then π and π' are unitary, it can be shown that if $\pi \otimes \pi'$ occurs in the discrete spectrum and $\dim_D V \leq \dim_D V'$ then π must belong to the discrete series of $\tilde{G}$.

Let ω_t be associated to the character ψ_t and realized on $\mathcal{H}_t$, let σ be an irreducible admissible representation $\tilde{G}'$ realized on V_σ. If σ occurs in the duality correspondence, we let $\theta(\sigma)$ be the corresponding representation of $\tilde{G}$. Otherwise, we set $\theta(\sigma) = 0$. Consider the algebraic tensor product $\mathcal{H}_t^\infty \otimes V_\sigma$ and its dual $\mathcal{H}_{-t}^\infty \otimes V_{\sigma^*}$. The canonical pairing between $\mathcal{H}^\infty$, V_σ and $\mathcal{H}_{-t}^\infty$, V_{σ^*} give rise to a pairing $\{ , \}$ between $\mathcal{H}_t^\infty \otimes V_\sigma$ and $\mathcal{H}_{-t}^\infty \otimes V_{\sigma^*}$, now consider the form

$$(\Phi, \Phi')_\sigma = \int_{\tilde{G}'} \{\omega(x)\Phi,\ \sigma^*(x)\Phi'\}\, dx \tag{4.1}$$

suppose it converges, then it factors through

$$[(\mathcal{H}_t^\infty \otimes V_\sigma)/R] \times [(\mathcal{H}_{-t}^\infty \otimes V_{\sigma^*})/R^*]$$

and defines a non-degenerate, $\tilde{G}$-invariant form. If σ is unitary, then there will be a conjugate linear isomorphism from $\mathcal{H}_t^\infty \otimes V_\sigma$ to its dual, and hence a non-degenerate $\tilde{G}$-invariant hermitian form, we denote $\pi(\sigma)$ the action of $\tilde{G}$ on the quotient space $\mathcal{H}_t^\infty \otimes V_\sigma / R$.

Now we assume that for G' the smaller member of (G, G'), the Witt index of $V \geq \dim_D V'$. We assume the local field $F \neq \mathbb{C}$, and ϵ the unique non-trivial character of the kernel of the covering map $\tilde{G} \to G$. We deonte $(\tilde{G}_\epsilon)^\wedge$ the set of irreducible *genuine* unitary representations of $\tilde{G}$.

Theorem 4.2. *([Li89]) Suppose G, G' is in the stable range with G' the smaller number. Exclude the case $G = O(2n, 2n)$, $G' = Sp_{2n}$ and σ the trivial representation, we have*

- *$\pi(\sigma) = \theta(\sigma) \neq 0$ for all $\sigma \in (\tilde{G}_\epsilon)^\wedge$.*
- *duality correspondence gives an injection $(\tilde{G}_\epsilon')^\wedge \hookrightarrow (\tilde{G}_\epsilon)^\wedge$.*
- *Let G_1' be the isometry group of another form $(V_1', (,)_1')$ and G, G_1' also forms a reductive dual pair in the stable range, then the image of $(\tilde{G}_\epsilon')^\wedge$ is disjoint from the image of $(\tilde{G}_1')^\wedge$ unless V' and V_1' are isometric.*

The main idea in the proof of this theorem is to show that the hermitian form defined by (4.1) is semi-positive definite and non-zero under the stated conditions.

Przebinda has used similar ideas to establish the unitarity of $\theta(\sigma)$ for a few cases beyond the stable range. It is interesting to make a deeper study of the form (4.1) and see whether it always defines a non-negative form whenever σ is unitary and the integrals converge.

References

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