THE SYMPLECTIC REPRESENTATION

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1. Introduction

This is a study note for Leslie's paper [Les24], where he defined a Galois action on the dual hyperspherical varieties for certain classes of symmetric varieties using the theory of geometric cocycle.

2. G-inner class for spherical varieties

For a characteristic zero field k and X a smooth affine algebraic variety over k with G a connected reductive algebraic group over k acting algebraically on X.

Assume that X is G-homogeneous with $X(k) \neq \emptyset$. Fix a base point $x_0 \in X(k)$ and let $\mathrm{Stab}_G(x_0) = H$, this induces an isomorphism $X \cong H\backslash G$ identifying $H \cdot 1 = x_0$. A standard calculation shows that

$$X(k) = \bigsqcup_{s \in \ker^1(H,G;k)} x_s \cdot G(k)$$

where $\ker^1(H,G;k) = \ker[H^1(k,H) \to H^1(k,G)]$ and $x_s \in X(k)$ satisfying $H_s = \mathrm{Stab}_G(x_s)$ is obtaining by twsiting H by a 1-cocycle in the class s. In particular, all stabilizers are pure inner twists of H. We may define

$$\mathrm{Aut}^G(X) \longrightarrow \mathrm{Out}(X) \subset \mathrm{Out}(H)$$

denote by $\mathcal{A}_X^b$ the kernel of this map.

Definition 2.1. Suppose that $X = H\backslash G$ and let $X' = H'\backslash G$ be a G-form of X, we say X' is a G-inner form if there exists a choice of ψ such that the cocycle $c_{X,X'} \in Z^1(k, \mathrm{Aut}^G(X))$ takes value in $\mathcal{A}_X^b(\overline{k})$. A G-form of X which is not a G-inner form is called a G-outer form of $X = H\backslash G$.

Remark 2.2. To obtain a more useful definition of the inner forms of X, it is necessary to include homogeneous spherical varieties for the inner forms of G.

Lemma 2.3. *Two G-forms $X' = H'\backslash G$ and $X = H\backslash G$ are G-equivariant isomorphic if and only if H and H' are pure inner forms with H' corresponding to a class in $\ker^1(H,G;k)$.*

Lemma 2.4. *Suppose that G is a reductive k-group and suppose that $X = H\backslash G$ is a homogeneous k-variety and let X' be a G-form of X, then the cocycle $c_{X,X'}$ induces the trivial cohomology class in $H^1(k, Out_X(H))$ if and only if X and X' are G-inner forms.*

Proposition 2.5. *Suppose that G is quasisplit and that $\overline{X} = \overline{H}\backslash G_{\overline{k}}$ is a spherical homogeneous space of $G_{\overline{k}}$. Assume that the corresponding $*$-action of $\{\sigma_*\}_{\sigma\in\Gamma}$ determined by G preserves Ω_X. Then for any lift $\alpha_{\mathcal{D}}$ of the Γ-action from Ω to a continuous action on $\mathcal{D}(X)$, there exists a G-equivariant k-model X of $\overline{X}$ inducing $\alpha_{\mathcal{D}}$. Moreover, $X(k) \neq \emptyset$, so that $X = H\backslash G$ for a k-rational subgroup $H \subset G$.*

The following is an important example which will motivate our later definition of the geometric cocycle.

Example 2.6. Let E/k be a quadratic extension of fields and let $(V, \langle\cdot,\cdot\rangle)$ be a two dimensional Hermitian E-vector space containing an isotropic line. Let $G = U(V)$ denote the corresponding quasi-split unitary group. We assume that the Hermitian form is represented by the matrix

$$J = \epsilon \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \epsilon \in E_{tr=0}$$

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Thus $G = \{g \in GL(V) \ : \ J\overline{g}^{-t}J^{-1} = g\}$, now we consider the two involutions

$$\theta_t = \mathrm{Ad}\begin{pmatrix} & t^{-1} \\ t & \end{pmatrix}, \ t \in \{1, \epsilon\}$$

we compute

$$H_t := G^{\theta_t} = \{\begin{pmatrix} a & b \\ t^2 b & a \end{pmatrix} \ : \ \mathrm{Nm}(a) - t^2\mathrm{Nm}(b) = 1, \quad a\bar{b} = \bar{a}b\}$$

so that $H_1 \cong \mathrm{Res}_{E/k}(\mathbb{G}_m)$ while $H_\epsilon \cong \mathrm{Nm}^1_{E/k}(\mathbb{G}_m)^2$.

We have

$$X_t = H_t\backslash G = \{\begin{pmatrix} x & y \\ -t^2 y & z \end{pmatrix} \ : \ x, y, z \in k, \ xz + t^2y^2 = 1\}$$

then $A_X \cong A$ with the canonical map $A \to A_X$ being the squaring map. In particular, $\Delta^n_X = \{2\alpha\}$. Since $\{-I\} \in X(k)$, we see that $\alpha \in \chi$. The simple calculation shows that the divisor $\{z = 0\}$ is stable under the upper triangular Borel subgroup with cocharacter $\check{\omega} = \frac{\check{\alpha}}{2} \in X_*(A_X)$, geometrically with two irreducible components $\{\mathcal{D}_1, \ \mathcal{D}_2\} = \mathcal{D}(X_t)$.

The two symmetric k-varieties X_t become isomorphic upon base changing to E, so that they have isomorphic homogeneous spherical data

$$\Omega_X = (2X^*(A), \{\alpha\}, \emptyset, \{(\check{\omega}, \{\alpha\})\})$$

and the fiber $(\rho \times \zeta)^{-1}(\check{\omega}, \{\alpha\}) = \{D_1, \ D_2\}$ consists of two colors in $X_{\overline{k}}$, note that

$$X_{t,\{z=0\}}(k) = \begin{cases} \{\pm\begin{pmatrix} x & 1 \\ -1 & \end{pmatrix}\} & : \ t = 1 \\ \emptyset & : \ t = \alpha, \ \text{since } \epsilon^2 \notin (k^\times)^2 \end{cases}$$

thus the Γ-action on $\{D_1, \ D_2\}$ is trivial when $t = 1$ and acts non-trivially through the quotient $\mathrm{Gal}(E/k)$ when $t = \epsilon$.

We can define the doubling autormophism group $\mathrm{Aut}_d(X)$ of X: for H geometrically connected

$$\mathrm{Aut}_d(X) = \prod_{i \in I_X} \mathrm{Res}_{k_i/k}(\mu_2)$$

In general, we can define the doubling automorphism group as

$$\mathrm{Aut}_d(X) := \mathrm{Aut}_d(X^\circ)/\mathrm{Aut}_d(\pi_0(H))$$

Lemma 2.7. *Suppose that $X = H\backslash G$ is spherical and that H is geometrically connected. In the long short exact sequence on cohomology, the map*

$$H^1(k, \mathcal{A}_X) \longrightarrow H^1(k, Aut_d(X))$$

is surjective.

Conjecture 2.8. *Suppose that G is a quasisplit reductive k-group and $X = H\backslash G$ is a homogeneous spherical G-variety, then the map*

$$H^1(k, \mathcal{A}_X) \longrightarrow H^1(k, Aut_d(X))$$

is surjective.

To two G-inner forms X and X' of $\overline{X}$, we can associate to them μ^d_X and $\mu^d_{X'}$ two geometric cocycles, if $\pi_0(H)$ is trivial, then lemma 2.7 implies that there exist G-forms of X associated to each geometric class as soon as G-form exists, which occurs as soon as the $*$-action of (G, A) preserves the combinatorial data $\Omega_{\overline{X}}$ by proposition 2.5. Conjecture 2.8 asserts this should hold for general H.

Definition 2.9. Suppose that G is a quasisplit reductive k-group and X is a homogeneous spherical G-variety, we say that X is **well adapted** if the natural morphism

$$\mathrm{Out}(X) \longrightarrow \mathrm{Aut}_d(X)$$

is an isomorphism.

In particular, when X is well-adapted and G is quasisplit, the data

$$(\Omega_X, [\mu_X^d]) = (\chi, \Delta_X, \Omega^{(1)}, \Omega^{(2)}, [\mu_X^d])$$

with $[\mu_X^d] \in H^1(k, \mathrm{Aut}_d(X))$ determines a k-form X up to G-inner form.

Definition 2.10. Suppose that G is k-simple, simply-connected group over k and that θ is an involution on G. We say that θ is an *involution of Chevalley type* if the spherical root system satisfying that $\mathrm{rk}(A_X) > 1$ and there is at most one Galois orbit of spherical roots $\Gamma \cdot \alpha_X \subset \Delta_X$ not of type N.

More generally, suppose G is a connected reductive group and θ is an involution, and X is a symmetric variety associated to θ, we say that X has a *factor of Chevalley type* if X_{sc} has a k-simple factor of Chevalley type.

Theorem 2.11. *Suppose that G is quasisplit over k and that $X = H\backslash G$ is a symmetric G-variety that has no factors of Chevalley type, then X is well adapted.*

Theorem 2.12. *Suppose that k is perfect and G is quasisplit over k and that $X = H\backslash G$ is a symmetric variety. There exists a G-inner form X_{qs} of X and $x \in X_{qs}(k)$ such that the connected component of the identity of $Stab_G(x) = H_{qs}$ is quasisplit over k.*

Let G be a quasisplit reductive group over k with k-rational involution θ, on the basis of theorems 2.11 and 2.12, we impose the following restriction

Assumption 2.13. Assume that the symmetric G_{sc}-variety $X_{sc} = H_{sc}\backslash G_{sc}$ contains no simple factors of Chevalley type and the conjecture 2.8 holds for X.

3. The dual group of a spherical variety

Let G be a connected reductive group over k, let $X = H\backslash G$ be a spherical variety and let $\Omega_X = (\chi, \Delta_X, \Omega^{(1)}, \Omega^{(2)})$ be the homogeneous spherical datum determined by a choice of a Borel pair (A, B), passing to the normalized root system, Sakellaridis and Venkatesh showed that $(\check{\chi}^{SV}, \Delta_X^{SV}, \chi^{SV}, \Delta_X^{SV})$ is a based root datum.

Definition 3.1. The dual group of the G-variety X is the connected complex group $\check{G}_X$ associated to the dual based root datum $(\check{\chi}^{SV}, \check{\Delta}_X^{SV}, \chi^{SV}, \Delta_X^{SV})$.

Now the surjection $A \to A_X$ produces a canonical morphism $\check{A}_X \to \check{A}$, considering the inclusions of lattices

$$\chi \subset \chi^{SV} = \chi + \mathbb{Z}\Delta_X^{SV} \subset X^*(T)$$

we obtain a sequence

$$\check{A}_X \to \check{A}_X \to \check{A}$$

Equip the dual group $\check{G}$ of G with a pinning $e_{\alpha^\vee}$ of $\mathfrak{g}_{\alpha^\vee}^\vee$ for each $\alpha \in \Delta$. For each $\sigma \in \Delta_X$, Knop defines a one dimensional subspace $\mathfrak{g}_{\check{\sigma}}^\vee$ of $\mathfrak{g}^\vee$ by

$$\mathfrak{g}_{\check{\sigma}}^\vee = \begin{cases} \mathfrak{g}_{\check{\sigma}}^\vee & : \sigma \in R^+ \\ [\mathfrak{g}_{\beta^\vee}^\vee, e_{\delta_1^\vee} - e_{\delta_2^\vee}] & : \sigma \text{ is of type } D_{n\geq 3} \\ [\mathfrak{g}_{\beta^\vee}^\vee, 2e_{\delta_1^\vee} - e_{\delta_2^\vee}] & : \sigma \text{ is of type } B_3'' \\ \mathbb{C}(e_{\delta_1^\vee} - e_{\delta_2^\vee}) & : \sigma \text{ is of type } D_2 \end{cases} \tag{3.1}$$

here $\beta^\vee := \gamma_1^\vee - \delta_1^\vee = \gamma_2^\vee - \delta_2^\vee$ when σ is of type G. We remark that a choice is made in the case of type D_2 roots.

Definition 3.2. A homomorphism $\varphi_X : \check{G}_X \to \check{G}$ is distinguished if $\varphi_X|_{\check{A}_X} = \varphi_A$ and $\varphi(\mathfrak{g}_{X,\check{\sigma}}^\vee) = \mathfrak{g}_{\check{\sigma}}^\vee$ as defined before.

Theorem 3.3. *Suppose that X is a spherical G-variety, then distinguished morphisms φ_X exist, moreover, they are unique up to $\check{A}_X$-conjugacy, the image*

$$\check{G}_X^* := \varphi_X(\check{G}_X)$$

is a well-defined subgroup of $\check{G}$ independent of φ_X.

We record the following property of the dual group

Proposition 3.4. *Suppose that X and Y are spherical G-varieties and assume that there exists a dominant morphism $X \to Y$, there exists a unique homomorphism with finite kernel $\check{G}_Y \to \check{G}_X$ which is compatible with the homomorphisms to $\check{G}$.*

The set of parabolic roots Δ_X^p corresponds to a Levi subgroup $A \subset L_X \subset G$ and a dual Levi subgroup $\check{L}_X \subset \check{G}$, our choice of a pinning of $\check{G}$ induces one on $\check{L}_X$, in particular, the pinning determines a principal homomorphism $\iota_X :\ SL_2 \longrightarrow \check{L}_X$. In particular, the pinning determines a principal homomorphism $\iota_X : SL_2 \to \check{L}_X$. It was shown that the images of φ_X and ι_X commute with each other in $\check{G}$, we thus obtain a morphism

$$\xi_X :\ \check{G}_X \times SL_2 \longrightarrow \check{G}$$

The results of Knop also indicate how to extend this notion to incorporate the Γ-action on $\check{G}_X$, viewed as a subgroup of $\check{G}$, there is a unique Γ-action on $\hat{G}_X$ such that the inclusion $\hat{G}_X$ intertwines with the L-action on $\check{G}$, we denote the corresponding semi-direct product by

$${}^L\hat{G}_X \hookrightarrow {}^LG$$

though this is arguably inappropriate as the Γ-action can fail to preserve the pinning in some cases. The additional requirement that φ_X preserves the embeddings forces a unique Γ-action on $\check{G}_X$ with the property that the morphism ξ_X intertwines with the action with the above action on $\hat{G}_X$. We can thus define the Galois form of the L-group to be

$${}^LX := \check{G}_X \rtimes \Gamma$$

with respect to this unqiue action. The map ξ_X extends uniquely to a morphism

$${}^L\xi_X :\ {}^LX \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

Remark 3.5. As noted above, Knop fixes the choice of image $\mathfrak{g}_{\check{\sigma}}$ for all $\sigma \in \Delta_X$ of type D_2, which we adopt here. It is worth mentioning that other choices relate to replacing H-periods with (H, χ)-periods for certain characters χ of H.

4. The symplectic representation S_X

We fix a choice of a distinguished morphism $\varphi_X :\ G_X^\vee \to \hat{G}_X$ compatible with $\check{A}_X \to \check{A}$ and pinning induced of $\check{G}$. This induces Γ-stable Borel subgroups for $\check{G}$, $\hat{G}_X$ and $G_X^\vee$ denoted by $\check{B}$, $\hat{B}_X$ and $\check{B}_X$. Moreover $\hat{B}_X$ is θ-stable and we have $\varphi_X(\check{B}) \subset \hat{B}_X$.

Let $\mathcal{D}(X)$ be the set of B-colors of X and let

$$\rho \times \zeta :\ \mathcal{D}(X) \longrightarrow X_*(A_X) \times \mathcal{P}(\Delta)$$

be the cocharacter map, Γ acts on Δ and hence on $\mathcal{P}(\check{\Delta})$.

Recall Δ_X^{dist} the set of distinguished roots, so that by theorem 6.1

$$\mathrm{Out}_X(H) \cong \prod_{i \in I_X} \mathrm{Res}_{k_i/k}\ \mu_i$$

where I_X is the set of Γ-orbits in Δ_X^{dist}. Using the proposition B.7, there us a corresponding subset $\check{\Sigma}_X^{dist} \subset \rho(\mathcal{D}(X))$ such that for every $\alpha \in \Delta_X^{dist}$, $\frac{1}{2}\check{\alpha} \in \check{\Sigma}_X^{dist}$.

Definition 4.1. We define $\check{\Delta}_X^{dist} = \{\frac{c_\alpha}{2}\check{\alpha} \in \Delta_X^{dist}\}$ by rescaling elements $\frac{1}{2}\check{\alpha}$ to the unique minimal multiple $\frac{c_\alpha}{2}\check{\alpha} \in \check{X}^{SV}$, here we note that $X^{SV} := X^* + \mathbb{Z}\Delta_X^{SV}$.

We define S_X as the unique representation of $G_X^\vee$ satisfying

$$S_X = \oplus_{\lambda \in \mathfrak{s}_X}\ V(\check{\lambda}) \otimes M(\check{\lambda})$$

where

- $\mathfrak{s}_X = \{$highest weights contained in $W_X \cdot \check{\Delta}_X^{dist}\}$.
- the multiplicity space $M(\check{\lambda})$ has a basis indexed by the colors $D \in \mathcal{D}(X)$ satisfying $\rho(D) = \frac{1}{2}\check{\alpha}$ for any $\check{\alpha}$ lying in the W_X-orbit of $\frac{2}{c_\alpha}\check{\lambda}$.

We will construct an action of ${}^L X$ on S_X. There is a canonical decomposition

$$\Delta_X^{dist} = \bigsqcup_{i \in I_X} \mathcal{O}_i$$

and for each $i \in I_X$, there is a canonical quadratic character $\mu_i: \ \Gamma_i \to \{\pm 1\}$. For any $\alpha \in \mathcal{O}_i$, we let $\mu_\alpha: \ \Gamma_\alpha \to \{\pm 1\}$ be the corresponding quadratic character on the stabilizer. Set $\mathcal{D}(X)^{dist} \subset \mathcal{D}(X)$ be the set of colors D such that $\rho(D) \in \check{\Sigma}_X^{dist}$.

Proposition 4.2. *Suppose $X = H\backslash G$ is a symmetric variety with H geometrically connected. Then there exists a unique ${}^L X$ representation on S_X extending the algebraic action of $\check{G}_X$ such that*

- *there is an isomorphism of Γ-representations*

$$S_X^{\check{B}_X} \cong \mathbb{C}[\mathcal{D}(X)^{dist}]$$

  where for $\alpha \in \Delta_X^{dist} \backslash \Delta_X^{(2)}$ and $\sigma \in \Gamma_\alpha$ we have $\sigma \cdot D_\alpha = \mu_\alpha(\sigma) D_\alpha$ where D_α is the unique color satisfying $\rho(D_\alpha) = \frac{1}{2}\check{\alpha}$.
- *there is a symplectic structure on S_X such that ${}^L X$ acts by symplectic morphisms.*

The proof relies on the classification of symmetric varieties. The proof of this proposition relies on the classification of symmetric varieties on a crucial way, in particular we use the calculation of colors of symmetric varieties in section B.2 to reduce to the case of type C root systems in lemma 4.3 since these are the only cases where a distinguished root occurs for symmetric varieties.

Lemma 4.3. *Suppose that G is quasisplit and $X = H\backslash G$ is a symmetric variety. Fix $\alpha \in \Delta_X^{dist}$ and let $\Gamma_\alpha \subset \Gamma$ denote its stabilizer, set k_α/k be the associated field extension, then there exists a unique k_α-rational reductive normal subgroup $G_\alpha \subset G$, stabilized by θ such that*

- *If $H_\alpha := H \cap G_\alpha$ and $X_\alpha := H_\alpha \backslash G_\alpha$, there exist surjective morphisms π_α, $\pi_{\alpha,X}$ fitting into a commutative diagram*

$$\begin{array}{ccc} \check{G}_X & \longrightarrow & \check{G} \\ \downarrow & & \downarrow \\ \check{G}_{X_\alpha} & \longrightarrow & \check{G}_\alpha \end{array}$$

- *let $\check{\lambda}_\alpha \in \mathfrak{s}_X$ be the unique dominant weight of $\check{G}_X$ associated to α, then the $\check{G}_X$-action on the highest weight module $V(\check{\lambda}_\alpha)$ factors through $\pi_{\alpha,X}$.*
- *Assume that X satisfies assumption 6.21. The derived subgroup of $\check{G}_\alpha$ is of type C. The corresponding highest weight representation of $\check{G}_X$ is minuscule and symplectic.*

We impose the assumption 6.21, fix $\alpha \in \Delta_X^{dist}$ as in the previous lemma, since (G_α, X_α) is k_α-rational, the morphism $\check{\pi}_\alpha$ is Γ_α-equivariant with respect to the given action on $\check{G}_X$ and a unique Γ_α-action on $\check{G}_{X_\alpha}$. We now extend the Γ_α-action on $\check{G}_{X_\alpha}$ to the representation V_α of $\check{G}_{X_\alpha}$. We may assume that $k = k_\alpha$ and $G = G_\alpha$, therefore assuming G_{der} is absolutely simple. In particular, $\Delta_X^{dist} = \{\alpha\}$.

By lemma 4.3, $\check{G}_{X,der}$ is simple of type C and acts on the standard representation so that we must have $\check{G}_{X,der} = \mathrm{Sp}_{2n}(\mathbb{C})$. In this case the Γ-action on $\check{A}_{X,ad}$ is trivial as $\mathrm{Sp}_{2n}(\mathbb{C})$ has trivial outer automorphism group, the $*$-action on Φ_X is trivial. In particular, the Γ-action on $\check{G}_X$ preserves our fixed pinning $\{x_{\check{\gamma}}\}_{\check{\gamma} \in \check{\Delta}_X^{SV}}$ induced by $\check{B}_X$ up to sign, so that the action is completely determined by a unique set of characters

$$\chi_\gamma: \ \Gamma_\gamma \longrightarrow \{\pm 1\}$$

such that $\sigma x_{\check{\gamma}} = \chi_\gamma(\sigma) x_{\check{\gamma}}$ for $\sigma \in \Gamma_\gamma = \mathrm{Stab}_\Gamma(\check{\gamma})$ and $\check{\gamma} \in \check{\Delta}_X^{SV}$.

Lemma 4.4. *Let $\chi: \Gamma \to \check{A}_{X,ad}$ denote the character uniquely determined by*

$$\sigma x_{\check{\gamma}} = \chi_\gamma(\sigma) x_{\check{\gamma}} = Ad(\chi(\sigma)) x_{\check{\gamma}}$$

for all $\check{\gamma} \in \check{\Delta}_X^{SV}$. For any quadratic character $\epsilon: \ \Gamma \to \{\pm 1\}$, there exists a natural lift

$$\tilde{\chi}_\epsilon \in Hom(\Gamma, \check{A}_X[2]) \cong H^1(\Gamma, \check{A}_X[2])$$

Continuing the assumptions of the lemma, let $\check{\lambda}_\alpha \in X^*(\check{A}_X)$ be the dominant weight associated to $\alpha \in \Delta_X^{dist}$, we define $\mathcal{D}(\alpha) \subset \mathcal{D}(X)$ to be those colors satisfying $\rho(D) = \frac{1}{2}\check{\alpha}$, we define the Γ-representation

$$M(\alpha) := \mathbb{C}[\mathcal{D}(\alpha)]$$

where

- If $\alpha \in \Delta_X^{(2)}$, $M(\alpha)$ is determined by the Γ-action on the basis $\mathcal{D}(\alpha)$, in this case, we take $\epsilon \equiv 1$ in the lemma and use $\tilde{\chi}_1$ to give a $\check{G}_X \rtimes \Gamma$-action on $V_\alpha \otimes_{\mathbb{C}} M_\alpha$ by letting $\sigma \in \Gamma$ acts by

$$\sigma(v \otimes m) = \tilde{\chi}_1(\sigma)v \otimes \sigma(m)$$

- If $\alpha \in \Delta_X^{dist} \backslash \Delta_X^{(2)}$, then $\mathcal{D}(\alpha) = 1$ and $M(\alpha) = \mathbb{C}$, we can take $\epsilon = \mu_\alpha$ so that the $\check{G}_X \rtimes \Gamma$ acts on $V_\alpha \otimes_{\mathbb{C}} M(\alpha) = V_\alpha$ by letting $\sigma \in \Gamma$ act by

$$\sigma(v) = \tilde{\chi}_{\mu_\alpha}(\sigma)(v)$$

  this representation is symplectic.

We now return to the general setting of (G, X) satisfying assumption 2.13 and H-connected, for each Γ-orbit $\mathcal{O} \subset \Delta_X^{dist}$ of distinguished roots and $\alpha \in \mathcal{O}$ let $\check{G}_{X,\mathcal{O}}$ be the induced group

$$\check{G}_{X,\mathcal{O}} = \mathrm{Ind}_{\Gamma/\Gamma_\alpha}(\check{G}_{X_\alpha}) \cong \prod_{\Gamma/\Gamma_\alpha} \check{G}_{X_\alpha}$$

such that $\check{G}_{X_\alpha,der} = [\check{G}_{X_\alpha}, \check{G}_{X_\alpha}] = \mathrm{Sp}(V_\alpha)$ where $V_\alpha = V(\check{\lambda}_\alpha)$ is the vector space of the associated representation. It follows from lemma 9.6 there exists a Γ-equivariant quotient map

$$\check{G}_X \to \prod_{\mathcal{O} \subset \Delta_X^{dist}} \check{G}_{X,\mathcal{O}}$$

for each Γ-orbit $\mathcal{O}$, we thus obtain a $\check{G}_{X,\mathcal{O}} \rtimes \Gamma$ representation

$$S_{\mathcal{O}} := \mathrm{Ind}_{\Gamma_\alpha}^{\Gamma}(V_\alpha \otimes M(\alpha))$$

and we have a ${}^L G_X$-representaiton on

$$S_X = \bigoplus_{\mathcal{O} \subset \Delta_X^{dist}} S_{\mathcal{O}}$$

via pull back along

$$\check{G}_X \rtimes \Gamma \to \prod_{\mathcal{O}} \check{G}_{X,\mathcal{O}} \rtimes \Gamma$$

4.1. **Application to rationality.** The theorem below compares the two k-forms X and X' of a given symmetric $G_{\overline{k}}$-variety $\overline{X}$, we say that X and X' are normally related to (A, B) if the rational involutions θ and θ' associated to X and X' are both normally related to (A, B). Since we may conjugate θ and θ' by elements of $G(k)$ to ensure this, it leads to no loss in generality.

Theorem 4.5. *Suppose that G is quasisplit over k and suppose that $\overline{X} = \overline{H}\backslash G_{\overline{k}}$ is a symmetric $G_{\overline{k}}$-variety satisfying assumtpion 6.21 and that $\overline{H}$ is connected, consider two k-rational G-forms $X = H\backslash G$ and $X' = H'\backslash G$ of $\overline{X}$, assume X and X' are both noramlly related to (A, B), then we have $\hat{G}_X = \hat{G}_{X'}$. Given a pair of distinguished morphisms φ_X and $\varphi_{X'}$, there is a canonical isomorphism $f_X : {}^L X \cong {}^L X'$ such that $\varphi_X = \varphi_{X'} \circ f_X$.*

Suppose there exists an f_X-equivariant isomorphism $f_S : S_X \longrightarrow S_{X'}$, then X and X' are G-inner forms.

We sketch the proof here: Let θ and θ' be associated involutions for X and X' respectively, we may assume that θ and θ' are both normally related to (A, B), since we have assumed that there is a $G_{\overline{k}}$-equivariant isomorphism $X_{\overline{k}} \cong X'_{\overline{k}}$, this implies

$$\Delta_X = \Delta_{X'},\ \hat{\Delta}_X = \hat{\Delta}_{X'},\ \text{and } \theta|_A = \theta'|_A$$

moreover $A_X \cong A_{X'}$.

The construction of the associated group relies only on the inclusion of root system generated by the coroots of the associated roots of X into the root system of $\check{G}$. Thus we obtain a canonical identification

$\hat{G}_X = \hat{G}_{X'}$. Similarly the construction of $\check{G}_X$ and the pinned action defined in (3.1) are combinatorial and depend only on the k-group structure of G, so there exists $f_X : {}^L X \cong {}^L X'$. Since distinguished morphisms have a finite kernel determined by the kernel of $\check{A}_X \to \check{A}$, we see that for any choice of such distinguished mrphisms there exists a unique f_X such that $\varphi_X = \varphi_{X'} \circ f_X$.

Assume now that we are given an f_X-equivariant isomorphism $f_S : S_X \longrightarrow S_{X'}$. In particular, this restricts to an isomorphism of representations of algebraic groups (proposition 4.2)

$$\bigoplus_{\alpha \in \check{\Delta}_X^{dist}} V(\check{\lambda}_\alpha) \otimes M(\alpha) \cong \bigoplus_{\alpha' \in \check{\Delta}_{X'}^{dist}} V(\check{\lambda}_{\alpha'}) \otimes M(\alpha')$$

which induces a bijection $\Delta_X^{dist} \cong \Delta_{X'}^{dist}$ by highest weight theory, this is uniquely determined by the isomorphism f_X. This is Γ-equivariant and we obtain an identification of Galois orbits $\Gamma \cdot \gamma_i \mapsto \Gamma \cdot \gamma_i'$, recall that there is an isomorphism of Γ-modules

$$S_X^{(\check{B}_X)} \cong \bigoplus_{\mathcal{O}_i \subset \Delta_X^{dist}} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(M(\alpha_i))$$

To recover the character μ_i, note that the Γ_i-action on $M(\alpha_i)$ is uniquely determined by this character and there is a canonical Γ_i-equivariant morphism

$$\begin{aligned} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(M(\alpha_i)) &\longrightarrow M(\alpha_i) \\ [f : \Gamma \to M(\alpha_i)] &\longmapsto f(1) \end{aligned}$$

from which we may compute μ_i, now passing to the weight spaces, we obtain a $\Gamma_i = \Gamma_i'$ equivariant isomorphsim $M(\alpha_i) \cong M(\alpha_i')$, in particular $\mu_i = \mu_i'$, all together we see that X and X' are inner forms.

References

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