PERIODS OF AUTOMORPHIC FORMS ASSOCIATED TO STRONGLY TEMPERED SPHERICAL VARIETIES

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This is a study note on the Wan-Zhang paper, they computed the unramified relative characters for all homogeneous affine strongly tempered spherical varieties and the strongly tempered spherical varieties that can be obtained by Whittaker induction of strongly tempered spherical varieties.

Let (G, H) be a reductive strongly tempered spherical pair with $G(F)$ split, assume that it does not have type N spherical root. Let $B = TN$ be a Borel subgroup of G defined over F, T the maximal split torus in B and N the unipotent radical of B, and $\overline{B} = T\overline{N}$ be its opposite. There exists a unique open Borel orbit $B(F)\eta H(F)$. For all the homogeneous affine strongly tempered models, we have verify that $H(F) \cap \eta^{-1}B(F)\eta = Z_{G,H}(F)$, the stabilizer of the open Borel orbit belongs to the center of G.

Our goal is to compute the local relative character

$$I(\phi_\theta) = \int_{H(F)/Z_{G,H}(F)} \phi_\theta(h)\ dh$$

where ϕ_θ is the unramified matrix coefficient of $I_B^G(\theta)$ normalized by $\phi_\theta(1) = 1$, θ is a unitary unramified character of $T(F)$.

Let f_θ be the unramified vector in $I_B^G(\theta)$, then the normalzied unramified matrix coefficient ϕ_θ is given by $\phi_\theta(g) = \int_K\ f_\theta(kg)\ dk$ this implies that

$$\begin{aligned} I(\phi_\theta) &= \int_{H(F)/Z_{G,H}(F)} \phi_\theta(h)\ dh = \int_{H(F)/Z_{G,H}(F)} \int_K\ f_\theta(kh)\ dkdh \\ &= \int_K \int_{H(F)/Z_{G,H}(F)}\ f_\theta(kh)\ dh\ dk \end{aligned}$$

Note since the integral is convergent if we replace θ by its absolute value, the above double integral is absolutely convergent. In particular, the integral

$$\int_{H(F)/Z_{G,F}(F)}\ f_\theta(kh)\ dh$$

is absolutely convergent for almost all $k \in K$. As a function on $k \in G$, this integral is right $H(F)$-invariant and left $(B(F), \delta_B^{1/2}\theta)$-invariant, where δ_B is the modular character of B.

Consider the function $\mathcal{Y}_\theta$ on $G(F)$ satisfying the conditions

- $\mathcal{Y}_\theta$ is supported on the orbit $B(F)\eta H(F)$ with $\mathcal{Y}_\theta(\eta) = 1$.
- $\mathcal{Y}_\theta$ is right $H(F)$-invariant and left $(B(F), \theta^{-1}\delta_B^{1/2})$-invariant.

For $g \in B(F)\eta H(F)$, $\mathcal{Y}_{\theta^{-1}}(g)$ is proportional to $\int_{H(F)/Z_{G,F}(F)}\ f_\theta(kh)\ dh$ and then

$$\int_{H(F)/Z_{G,F}(F)}\ f_\theta(gh)\ dh = \int_{H(F)/Z_{G,H}(F)}\ f_\theta(\eta h)dh \cdot \mathcal{Y}_{\theta^{-1}}(g)$$

Lemma 0.1. *Under the above notation, for $f \in C_c^\infty(G(F))$, we have*

$$\int_G\ f(g)\ dg = \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)} \int_{H(F)/Z_{G,H}(F)} \int_{B(F)}\ f(b\eta h)\ dbdh$$

where $rk(G)$ is the F-rank of G.

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By lemma 0.1, we have

$$\int_K \mathcal{Y}_\theta(k)\ dk = \int_{G(F)} 1_K(g)\mathcal{Y}_\theta(g)\ dg = \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}\int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\ dh$$

Proposition 0.2. *The local relative character* $I(\phi_\theta)$ *is equal to*

$$\begin{aligned}&\int_K \mathcal{Y}_{\theta^{-1}}(k)\ dk \times \int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\ dh\\ =&\frac{\Delta_{H/Z_{G,H}}(1)}{\Delta_G(1)}\zeta(1)^{rk(G)}\int_K \mathcal{Y}_{\theta^{-1}}(k)\ dk \times \int_K \mathcal{Y}_\theta(k)\ dk\end{aligned}$$

We have the following computation of $\int_K \mathcal{Y}_\theta(k)\ dk$

Proposition 0.3. *Let* Φ^+ *be the set of positive roots of* G*, then there is a decomposition of the weights of a representation* ρ_X *of* $\hat{G}$*, denoted by* $\Theta = \Theta^+ \cup \Theta^-$ *such that*

$$\int_K \mathcal{Y}_\theta(k)\ dk = \frac{\Delta G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}\cdot\beta(\theta)$$

where

$$\beta(\theta) = \frac{\prod_{\alpha\in\Phi^+} 1 - q^{-1}e^{\alpha^\vee}}{\prod_{\gamma^\vee\in\Theta^+} 1 - q^{-\frac{1}{2}}e^{\gamma^\vee}}(\theta)$$

here for $\alpha \in \Phi^+$*, we use* $e^{\alpha^\vee}(\theta)$ *to denote* $\theta(e^{\alpha^\vee}(\varpi))$.

Combining propositions 0.2 and 0.3 we have

$$\begin{aligned}I(\phi_\theta) &= \frac{\Delta_{H/Z_{G,H}}(1)}{\Delta_G(1)}\zeta(1)^{rk(G)}\int_K \mathcal{Y}_{\theta^{-1}}(k)\ dk \times \int_K \mathcal{Y}_\theta(k)\ dk\\ &= \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}\cdot\beta(\theta)\cdot\beta(\theta^{-1}) = \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\cdot\frac{L(1/2,\pi,\rho_X)}{L(1,\pi,\mathrm{Ad})}\end{aligned}$$

The L-function $\frac{L(1/2,\pi,\rho_X)}{L(1,\pi,\mathrm{Ad})}$ is L_X, the L-function of the spherical variety $X = G/H$.