STABILIZATION OF PERIODS OF EISENSTEIN SERIES

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1. Introduction

This is a study note for the paper by Lapid and Rogawski on the stabilization of periods of Eisenstein series [LR00].

2. Fourier inversion and stabilization

For E/F a quadratic extension of local fields or number fields, for $E = F \oplus F$ we set

$$A(F) = T'(F)/Z'(F)\mathrm{Nm}(T(E))$$

by lemma $A(F)$ parametrizes the B-orbits in $\iota^{-1}(\omega)$, note that $A(F) \cong (F^*/NE^*)^2$. If $E = F \oplus F$, then $A(F)$ is trivial. In the global case, we define $A(\mathbb{A}_F)$ as the direct sum of corresponding local groups

$$A(\mathbb{A}_F) = \bigoplus_v A(F_v)$$

where v ranges over all places of F. View $A(F)$ as a subgroup of $A(\mathbb{A}_F)$ embedded diagonally, note that $[A(\mathbb{A}_F : A(F))] = 4$.

For an absolutely summable function g on $A(\mathbb{A}_F)$, we may define the Fourier transform

$$\hat{g}(\kappa) = \sum_{x \in A(\mathbb{A}_F)} \kappa(x)\ g(x)$$

for any character κ of $A(\mathbb{A}_F)$. Let X be the set of four characters of $A(\mathbb{A}_F)$ trivial on $A(F)$. Then the following Fourier inversion formula holds

$$\sum_{x \in A(F)} g(x) = \frac{1}{4} \sum_{\kappa \in X} \hat{g}(\kappa)$$

suppose in addition that g is of the form

$$g(x) = \prod_v g_v(x_v)$$

where g_v is a function on $A(F_v)$ for all v and the infinite product converges absolutely. Define the local Fourier transform

$$\hat{g_v}(\kappa) = \sum_{x_v \in A(F_v)} \kappa(x_v)\ g_v(x_v)$$

for any character κ of $A(F_v)$. We shall write κ_v the restriction of κ to $A(F_v)$, then we have the following

Lemma 2.1. *We have* $\hat{g}(\kappa) = \prod_v \hat{g}_v(\kappa_v)$.

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3. Stable local period

Let's consider the local case, fix $\eta \in G(E)$ such that $\eta\theta(\eta)^{-1} = t\omega$ for some $t \in T(F)$. To define the stable local period, we assume that the inducing character $\chi = (\chi_1, \chi_2, \chi_3)$ satisfies $\chi_j|_{E^1} = 1$, it is shown in [LR] that the integral

$$J(\eta, \varphi, \lambda) = \int_{H_\eta(F)\backslash H(F)} e^{\langle \lambda, H(\eta h)\rangle} \varphi(\eta h)\ dh$$

where $\varphi \in I(\chi, \lambda)$ converges for Re λ positive enough. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a character of T' such that $\chi_j = \nu_j \circ \mathrm{Nm}$ for $j = 1,\ 2,\ 3$. Set

$$\Delta_{\nu,\lambda}(\eta) = \nu(t)\omega(t_1 t_3)\ e^{\frac{1}{2}\langle \lambda + \rho, H(t)\rangle}$$

the stable local period is defined to be the distribution

$$J^{st}(\nu, \varphi, \lambda) = \sum_{\iota(\eta)=\omega} \Delta_{\nu,\lambda}(\eta)^{-1} \int_{H_\eta(F)\backslash H(F)} e^{\langle \lambda, H(\eta h)\rangle}\ \varphi(\eta h)\ dh$$

4. Main result

We turn to the global situation, let

$$c = \mathrm{vol}(E^1\backslash E^1(\mathbb{A}))^2$$

from the result in the previous section, we have the following identity

$$\Pi^H(E(\varphi, \lambda)) = c \sum_{\iota(\eta)=\omega} J(\eta, \varphi, \lambda)$$

is proved in [LR], it is valid whenever Re λ is positive enough. From now on, we assume that the Haar measure on $H_\eta(\mathbb{A})$ is the Tamagawa measure, this measure has the property that $\mathrm{vol}(H_\eta(F_v)) = 1$ for almost all v and furthermore $c = 4$.

Fix a character $\nu_0 \in \mathcal{B}(\chi)$ to serve as a base point. Recall that the set of double cosets over ω is parametrized by the group $A(F)$ both locally and globally, for $a \in A(F)$, let η_a be a representative for the double coset corresponding to a such that $\eta_a\theta(\eta_a)^{-1} = t\omega$ for $t \in T'(F)$, fix λ and $\varphi = \otimes\varphi_v \in I(\chi, \lambda)$ let g_v be the function on $A(F_v)$ defined as

$$g_v(a) = \Delta_{\nu_0,\lambda}(\eta_a)^{-1} \int_{H_{\eta_a(F_v)}\backslash H(F_v)} e^{\langle \lambda, H_v(\eta_a h)\rangle}\ \varphi(\eta_a h)\ dh$$

the product function $g = \prod_v g_v$ on $A(\mathbb{A})$ is integrable over $A(\mathbb{A}_F)$ for Re λ positive enough. We may define the global stable intertwining period for Re λ positive enough as the absolutely convergent product

$$J^{st}(\nu_0, \varphi, \lambda) = \prod_v J_v^{st}(\nu_{0v}, \varphi_v, \lambda)$$

for any character κ of $A(\mathbb{A})$ trivial on $A(F)$, we have

$$\hat{g}(\kappa) = J^{st}(\kappa\nu_0, \varphi, \lambda)$$

observe that the characters $\nu = \kappa\nu_0$ comprise $\mathcal{B}(\chi)$.

Theorem 4.1. *We have*

$$\Pi^H(E(\varphi, \lambda)) = \sum_{\nu \in \mathcal{B}(\chi)} J^{st}(\nu, \varphi, \lambda)$$

in particular, this expresses the left-hand side as a sum of factorizable distributions.

References

[LR00] Erez Lapid and Jonathan Rogawski. Stabilization of periods of Eisenstein series and Bessel distributions on $GL(3)$ relative to $U(3)$. *Documenta Mathematica*, 5:317–350, 2000.