A COUNTEREXAMPLE TO STABILITY FOR UNRAMIFIED MULTIPLICITY-ONE SYMMETRIC PAIRS

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1. Introduction

Let k be a p-adic local field of characteristic 0 and $\mathfrak{o}$ its integer ring. For a symmetric pair (G, H, θ), fix a hyperspecial compact subgroup $K = G(\mathfrak{o}) \subset G(k)$. We call an irreducible smooth representation π unramified with respect to K if $\pi^K \neq 0$.

We say that the pair has unramified multiplicity one property if

$$\dim \mathrm{Hom}_{G(k)}\big(\pi, C^\infty(H\backslash G(k))\big) \leq 1$$

for all irreducible unramified representation π.

Aizenbud and Gourevitch introduced stablity of a symmetric pair (Definition 3.2.2) in their study of Gelfand properties [AG09]. Carmeli introduced p-stability and proved that it follows from stability [Car19, Theorem 5.27]. We consider the following question about the relation between unramified multiplicity one and stability.

$$\text{Does unramified multiplicity one imply stability of a symmetric pair?} \tag{1.1}$$

We take

$$k = \mathbb{Q}_{17}, \qquad (n_1, \ldots, n_6) = (4, 4, 8, 8, 4, 8), \qquad h_j = n_j/2.$$

Let $T = \mathbb{G}_m^6$, with coordinates $t = (s_1, t_1, s_2, t_2, u, v)$, and put

$$\tau(t) = (t_1, s_1, t_2, s_2, u, v).$$

Write $p_i : T \to \mathbb{G}_m$ for the i-th coordinate projection. Define $\chi_j = c_j^{h_j}$ using the characters

$$c_1 = p_1 p_2 p_3 p_6, \tag{1.2}$$
$$c_2 = p_3, \tag{1.3}$$
$$c_3 = p_1 p_6, \tag{1.4}$$
$$c_4 = p_1, \tag{1.5}$$
$$c_5 = p_1 p_3 p_5 p_6, \tag{1.6}$$
$$c_6 = p_1 p_3 p_5. \tag{1.7}$$

We introduce the group G as

$$G = \left\{ ((A_j, B_j)_j, t) \in \prod_{j=1}^{6} \mathrm{GL}_{n_j}^2 \times T \;\middle|\; \begin{matrix} \det A_j = \chi_j(t), \\ \det B_j = \chi_j(\tau t) \quad (1 \leq j \leq 6) \end{matrix} \right\}. \tag{1.8}$$

The involution

$$\theta((A_j, B_j)_j, t) = ((B_j, A_j)_j, \tau t) \tag{1.9}$$

has full fixed subgroup H.

Theorem 1.0.1. *The symmetric pair defined by* (1.8)–(1.9) *has unramified multiplicity one (Proposition 2.0.4) and fails stability (Proposition 3.6.1).*

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2. Unramified multiplicity one spherical pairs

In this section, let G be a connected split reductive group over k, B its Borel subgroup over k and A a maximal split torus of B. Let X be a homogeneous affine spherical G-variety. Write $\mathring{X}$ for the open orbit of a Borel subgroup B in X and $P(X)$ for the parabolic subgroup stabilizing $\mathring{X}$. Let A_X be the maximal torus of X. Let A^* and A_X^* be their complex dual tori, $S = \overline{A_X^*}$ the image of $A_X^* \to A^*$, and denote W and W_X the Weyl group of G and the little Weyl group of X. The map onto S can have finite kernel.

We recall the generic multiplicity formula of Sakellaridis.

Theorem 2.0.1 ([Sak13, Theorem 3.2.2])**.** *Under the standing hypotheses of [Sak13, §1.1 and §1.7], if an irreducible unramified representation π of $G(k)$ is X-distinguished, it is the unramified subquotient of $I_{P(X)}(\chi)$ for some $\chi \in S$, with induction as in the cited theorem. For generic $\chi \in S$, this subquotient satisfies*

$$\dim \mathrm{Hom}_{G(k)}\big(\pi, C^\infty(X(k))\big) = g \cdot a,$$

where g is the generic degree onto the image of $S/W_X \to A^*/W$ and a is the number of $B(k)$-orbits in $\mathring{X}(k)$.

Definition 2.0.2. We say that a spherical pair (G, H) has generic multiplicity one if $g = 1$ and $a = 1$ for $X = H\backslash G$.

We introduce the restriction map as in [Sak13, (8.2)]

$$\mathrm{Res} : \mathbb{C}[A^*]^W \longrightarrow \mathbb{C}[\delta_{(X)}^{1/2} A_X^*]^{W_X}, \tag{2.1}$$

Theorem 2.0.3 ([Sak13, Corollary 8.0.4])**.** *Assume the standing hypotheses of [Sak13, §1.1 and §1.7], including the integral orbit axioms, and suppose that $\mathring{X}(k)$ is a single $B(k)$-orbit. Assume Statement 6.3.1 of [Sak13] for every simple spherical root of X and Statement 7.1.5 for its set Θ. If* (2.1) *is surjective, then every irreducible unramified representation π satisfies*

$$\dim \mathrm{Hom}_{G(k)}\big(\pi, C^\infty(X(k))\big) \leq 1.$$

Proposition 2.0.4. *The symmetric pair (G, H)* (1.8)–(1.9) *has unramified multiplicity one with respect to $K = G(\mathbb{Z}_{17})$.*

Proof. We can check that the assumptions in Theorem 2.0.3 are satisfied for $X = H\backslash G$, hence this proposition follows from Theorem 2.0.3. □

3. Stable symmetric pairs

In this section we recall some results of [AG09] on Gelfand properties for symmetric pairs over k.

3.1. Distributions.

Definition 3.1.1. A B-analytic manifold is an analytic manifold over a non-Archimedean field k.

Let V be a finite-dimensional rational representation of a reductive k-group G. We put $Q_G(V) = (V/V^G)(k)$ and identify it with the canonical invariant complement to $V^G(k)$. For the map $\pi : V(k) \to (V/G)(k)$ to the categorical quotient $V/G = \mathrm{Spec}\, k[V]^G$, put

$$\Gamma_G(V) = \pi^{-1}(\pi(0)), \qquad R_G(V) = Q_G(V) \setminus \Gamma_G(V).$$

Invariant linear forms vanish on $\Gamma_G(V)$, so $\Gamma_G(V) \subset Q_G(V)$.

Definition 3.1.2. For a B-analytic manifold M, let $\mathcal{S}(M) = C_c^\infty(M)$ be the space of compactly supported locally constant complex functions. The space of Schwartz distributions $\mathcal{S}^*(M)$ is its algebraic dual.

3.2. **Symmetric pairs.**

Definition 3.2.1. A **symmetric pair** is a triple (G, H, θ), where G and $H \subset G$ are reductive groups, θ is an involution of G, and $H = G^\theta$. We call the pair **connected** if $H\backslash G$ is connected as an algebraic variety.

For a symmetric pair (G, H, θ), define the anti-involution $\sigma(g) = \theta(g^{-1})$ and the symmetrization map

$$s : G \longrightarrow G^\sigma, \qquad s(g) = g\sigma(g).$$

For $\mathfrak{g} = \mathrm{Lie}(G)$, we write $\mathfrak{g}^\sigma = \ker(d\theta + 1)$, since $d\sigma = -d\theta$. The adjoint action of H preserves this space.

Definition 3.2.2. A symmetric pair (G, H, θ) is **stable** if $\sigma(\mathcal{O}) = \mathcal{O}$ for every double orbit $\mathcal{O} = H(k)gH(k)$ that is closed in the local-field topology on $G(k)$.

Definition 3.2.3 ([AG09, Definition 7.1.8])**.** A symmetric pair (G, H, θ) is a **GK pair** if

$$\mathcal{S}^*(G(k))^{H(k)\times H(k)} \subset \mathcal{S}^*(G(k))^\sigma.$$

3.3. **Descendants of symmetric pairs.**

Definition 3.3.1 ([AG09, Definition 7.2.2])**.** Let (G, H, θ) be a symmetric pair. An element $g \in G(k)$ is $H \times H$-semisimple if its orbit $H(k)gH(k)$ is closed in $G(k)$ in the local-field topology. For such a g, put $x = s(g)$, $G_x = Z_G(x)$ and $H_x = H \cap G_x$. The pair $(G_x, H_x, \theta|_{G_x})$ is called a **descendant**.

By [AG09, Proposition 7.2.1], x is semisimple. Its reductive centralizer is θ-stable because $\theta(x) = x^{-1}$, and H_x is its full fixed subgroup. Taking $g = 1$ gives the original pair.

3.4. **Regular symmetric pairs.**

Definition 3.4.1. Let (G, H, θ) be a symmetric pair. An element $g \in G(k)$ is **admissible** if $\mathrm{Ad}(g)$ commutes with θ on G and its action on $\mathfrak{g}^\sigma$ is H-admissible.

Definition 3.4.2. A symmetric pair (G, H, θ) is **regular** if for every admissible $g \in G(k)$ the inclusion

$$\mathcal{S}^*(R_H(\mathfrak{g}^\sigma))^{H(k)} \subset \mathcal{S}^*(R_H(\mathfrak{g}^\sigma))^{\mathrm{Ad}(g)}$$

implies

$$\mathcal{S}^*(Q_H(\mathfrak{g}^\sigma))^{H(k)} \subset \mathcal{S}^*(Q_H(\mathfrak{g}^\sigma))^{\mathrm{Ad}(g)}.$$

Theorem 3.4.3 ([AG09, Theorem 7.4.5])**.** *Let (G, H, θ) be a stable symmetric pair such that all its descendants are regular. Then it is a GK pair.*

We recall the conjecture on regularity from [AG09].

Conjecture 3.4.4 ([AG09, Conjecture 4])**.** *Every symmetric pair is regular.*

See [AG09, Remark 7.5.1] for the classical and exceptional cases known there.

3.5. **Gelfand properties.**

Definition 3.5.1 ([AG09, Definition 8.1.2])**.** Let $H \subset G$ be reductive groups. For an irreducible admissible representation π of $G(k)$, write $\widetilde{\pi}$ for its smooth contragredient.

- The pair (G, H) satisfies **GP1** if for every such π,
$$\dim \mathrm{Hom}_{H(k)}(\pi, \mathbb{C}) \leq 1.$$
- The pair (G, H) satisfies **GP2** if for every such π,
$$\dim \mathrm{Hom}_{H(k)}(\pi, \mathbb{C}) \cdot \dim \mathrm{Hom}_{H(k)}(\widetilde{\pi}, \mathbb{C}) \leq 1.$$
- The pair (G, H) satisfies **GP3** if for every irreducible unitary representation of $G(k)$ on a Hilbert space $\mathcal{H}$,
$$\dim \mathrm{Hom}_{H(k)}(\mathcal{H}^\infty, \mathbb{C}) \leq 1,$$
where $\mathcal{H}^\infty$ is the space of smooth vectors.

We have GP1 $\Rightarrow$ GP2 $\Rightarrow$ GP3 by [AG09, Proposition 8.1.3]. The following criterion gives GP2.

Proposition 3.5.2 ([AG09, Theorem 8.1.5]). *Let $H \subset G$ be reductive groups, and let ι be an involutive anti-automorphism of G with $\iota(H) = H$. If every bi-$H(k)$-invariant Schwartz distribution on $G(k)$ is ι-invariant, then (G, H) satisfies GP2.*

Proposition 3.5.3 ([AG09, Corollary 8.1.6]). *Every symmetric GK pair satisfies GP2.*

Indeed, σ restricts to inversion on $H(k)$, so Definition 3.2.3 gives the hypothesis of Proposition 3.5.2 with $\iota = \sigma$.

3.6. **An unstable symmetric pair.** In this section, we let

$$k = \mathbb{Q}_{17}, \qquad (n_1, \dots, n_6) = (4, 4, 8, 8, 4, 8), \qquad (d_1, \dots, d_6) = (3, 3, 17, 17, 51, 51).$$

Choose roots of unity $i, \zeta \in k$ characterized by

$$i^2 = -1, \qquad i \equiv 4 \pmod{17},$$

and

$$\zeta^4 = -1, \qquad \zeta \equiv 2 \pmod{17}.$$

These roots exist uniquely by Hensel's lemma, and $\zeta^2 = i$. Write

$$\zeta_4 = i, \qquad \zeta_8 = \zeta.$$

For $N \in \{4, 8\}$ and $d \in \{3, 17, 51\}$, let

$$R(N, d) = \begin{pmatrix} 0 & 0 & \cdots & 0 & d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Thus $R(N, d)^N = dI_N$.

Define

$$Z(N, d) = R(N, d)^{N/2}\big(I_N + R(N, d)\big)\big(I_N + \zeta_N R(N, d)\big)^{-1}.$$

The six matrices used in the construction are

$$Z_1 = Z_2 = Z(4, 3), \tag{3.1}$$
$$Z_3 = Z_4 = Z(8, 17), \tag{3.2}$$
$$Z_5 = Z(4, 51), \tag{3.3}$$
$$Z_6 = Z(8, 51). \tag{3.4}$$

We set

$$A_1 = \operatorname{diag}(17^2, 1, 1, 1), \qquad A_j = I_{n_j}\ (j = 2, \dots, 6), \qquad B_j = A_j Z_j,$$

and put

$$t_* = (1, 17, 1, 3, 1, 1), \qquad g_* = \big((A_j, B_j)_{j=1}^6, t_*\big).$$

Writing $\chi(t) = (\chi_j(t))_{j=1}^6$ as defined in the introduction, we obtain

$$(\det A_j)_j = (17^2, 1, 1, 1, 1, 1) = \chi(t_*),$$
$$(\det B_j)_j = (51^2, 3^2, 17^4, 17^4, 51^2, 51^4) = \chi(\tau t_*),$$

by the determinant identities and the characters (1.2)–(1.7). Thus $g_* \in G(\mathbb{Q}_{17})$.

Proposition 3.6.1. *The symmetric pair defined by* (1.8)–(1.9) *is not a stable symmetric pair.*

Proof. We have $H(k)g_*H(k)$ is closed and does not contain $\sigma(g_*)$. □

References

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