

A SPLITTING LEMMA FOR COVERING GROUPS

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1. INTRODUCTION

This is a study note on the Langlands-Weissman program, which is a conjectural extension of the Langlands program for a class of nonlinear covering groups of algebraic origin studied by Brylinski and Deligne.

2. BRYLINSKI-DELIGNE EXTENSIONS

Let k be a field and $\mathbf{G}$ a connected reductive linear group over k , we assume that $\mathbf{G}$ is split over k . Fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Let $Y = \text{Hom}(\mathbb{G}_m, T)$ be the cocharacter lattice and $X = \text{Hom}(T, \mathbb{G}_m)$. Then one has the set of roots $\Phi \subset X$ and the set of coroots $\Phi^\vee \subset Y$ of $(\mathbf{G}, \mathbf{T})$ respectively. The Borel subgroup $\mathbf{B}$ determines the set of simple roots $\Delta \subset \Phi$ and the set of simple coroots $\Delta^\vee \in \Phi^\vee$. For each $\alpha \in \Phi$, we have the associated root subgroup $\mathbf{U}_\alpha \subset \mathbf{G}$ which is normalized by $\mathbf{T}$. We shall fix a pinning for $(\mathbf{G}, \mathbf{T})$, so that for every $\alpha \in \Phi$, we have an isomorphism $x_\alpha : \mathbb{G}_a \rightarrow \mathbf{U}_\alpha$.

The algebraic group $\mathbf{G}$ defines a sheaf of groups on the big Zariski site on $\text{Spec}(k)$. Let K_2 denote the sheaf on $\text{Spec}(k)$ associated to the K_2 -group in Quillen's K-theory. Then a multiplicative K_2 -torsor is an extension,

$$1 \longrightarrow K_2 \longrightarrow \overline{\mathbf{G}} \longrightarrow \mathbf{G} \longrightarrow 1$$

of sheaves of groups on $\text{Spec}(k)$. We consider the category $CExt(\mathbf{G}, K_2)$ of such extensions where the morphisms between objects are given by morphisms of extensions. Given two such central extensions, one can form their Baer sum, this equips $CExt(\mathbf{G}, K_2)$ with the structure of a commutative Picard category.

Suppose $\mathbf{T}$ is a split torus, with cocharacter lattice $Y = \text{Hom}(\mathbb{G}_m, T)$ and character lattice $X = \text{Hom}(T, \mathbb{G}_m)$. Then we have:

Proposition 2.0.1. *Let $\mathbf{T}$ be a split torus over k . The category $CExt(\mathbf{T}, K_2)$ is equivalent as a commutative Picard category to the category whose objects are pairs $(Q, \mathcal{E})$, where*

- Q is a $\mathbb{Z}$ -valued form on Y with associated symmetric bilinear form $B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2)$.
- $\mathcal{E}$ is a central extension of groups

$$1 \longrightarrow k^\times \longrightarrow \mathcal{E} \longrightarrow Y \longrightarrow 1$$

whose associated commutator map $[-, -] : Y \times Y \rightarrow F^\times$ is given by

$$[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}$$

Now suppose that $\mathbf{G}$ is a split simply-connected semisimple group over k and recall that we have fixed the epinglage $(\mathbf{T}, \mathbf{B}, x_\alpha)$. Let $W = N(\mathbf{T})/\mathbf{T}$ be the corresponding Weyl group. Since $\mathbf{G}$ is simply-connected, the coroot lattice is equal to Y , so that the set of simple coroots $\Delta^\vee$ is a basis for Y .

Proposition 2.0.2. *The category $CExt(\mathbf{G}, K_2)$ is equivalent to the category whose objects are W -invariant $\mathbb{Z}$ -valued quadratic form Q on Y , and whose only morphisms are the identity morphisms on each object.*

Let $\mathbf{G}$ be a split connected reductive group over k , with fixed epinglage $(\mathbf{T}, \mathbf{B}, x_\alpha)$. Let

$$i_{sc} : Y^{sc} := \mathbb{Z}[\Delta^\vee] \subset Y$$

be the inclusion of the coroot lattice Y^{sc} into Y .

We have the following result for a general connected split reductive group $\mathbf{G}$,

Theorem 2.0.3. *The category $CExt(\mathbf{G}, K_2)$ is equivalent to the category $BD_{\mathbf{G}}$ whose objects are triples $(Q, \mathcal{E}, f)$, where:*

- $Q : Y \rightarrow \mathbb{Z}$ is a W -invariant quadratic form.
- $\mathcal{E}$ is an extension of Y by $k^\times$ with commutator map $[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}$.
- $f : \mathcal{E}_{Q^{sc}} \cong q^*(\mathcal{E})$ is an isomorphism of extensions of Y^{sc} by $k^\times$.

The set of morphisms from $(Q, \mathcal{E}, f)$ to $(Q', \mathcal{E}', f')$ is empty unless $Q = Q'$, in which case it consists of isomorphisms of extensions $\phi : \mathcal{E} \rightarrow \mathcal{E}'$ such that $f = f' \circ q^*(\phi)$.

2.1. Bisectors and Incarnation. While the above Theorem 2.0.3 give a nice classification of multiplicative K_2 -torsors over split reductive group $\mathbf{G}$ over k , it is sometimes useful and even necessary to work with explicit cocycles for computation.

We consider $\mathbb{Z}$ -valued bilinear forms D on Y satisfying: $D(y, y) = Q(y)$ for all $y \in Y$ so that

$$B_Q(y_1, y_2) = D(y_1, y_2) + D(y_2, y_1)$$

such a D is called a bisector of Q . For any Q , there exists an associated bisector.

We consider a category

$$\text{Bis}_G = \bigcup_Q \text{Bis}_{G, Q}$$

where the full subcategory $\text{Bis}_{G, Q}$ consists of pairs (D, η) where D is a bisector of Q and

$$\eta : Y^{sc} \rightarrow k^\times$$

is a group homomorphism. Given two pairs (D_1, η_1) and (D_2, η_2) , the set of morphisms is the set of functions $\xi : Y \rightarrow k^\times$ such that

- $\xi(y_1 + y_2) \cdot \xi(y_1)^{-1} \xi(y_2)^{-1} = (-1)^{D_1(y_1, y_2) - D_2(y_1, y_2)}$.
- $\xi(\alpha^\vee) = \eta_2(\alpha^\vee) / \eta_1(\alpha^\vee)$ for all $\alpha^\vee \in \Delta^\vee$.

and where the composition of morphisms is given by multiplication: $\xi_1 \circ \xi_2(y) = \xi_1(y) \cdot \xi_2(y)$.

There is an incarnation functor

$$\text{Inc}_G : \text{Bis}_G \rightarrow \text{BD}_G \rightarrow CExt(\mathbf{G}, K_2)$$

the second functor is quasi-inverse to the functor in Theorem 2.0.3. On the level of objects, the first functor sends the pair (D, η) in $\text{Bis}_{G, Q}$ to the triple $(Q, \mathcal{E}, f)$ defined as follows:

- $\mathcal{E} = Y \times k^\times$ with group law

$$(y_1, a_1)(y_2, a_2) = (y_1 + y_2, a_1 a_2 (-1)^{D(y_1, y_2)})$$

- $f : \mathcal{E}_{Q^{sc}} \rightarrow \mathcal{E}$ is given by

$$f(s_{Q^{sc}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee))$$

for every $\alpha \in \Delta$.

3. TOPOLOGICAL COVERING GROUPS

3.1. BD covering groups. Start with a multiplicative K_2 -torsor $K_2 \rightarrow \overline{\mathbf{G}} \rightarrow \mathbf{G}$ with associated BD data $(Q, \mathcal{E}, f)$. By taking k -points, we obtain a short exact sequence of abstract groups:

$$1 \rightarrow K_2(k) \rightarrow \overline{\mathbf{G}}(k) \rightarrow G = \mathbf{G}(k) \rightarrow 1$$

The Hilbert symbol gives a map,

$$(-, -)_k : K_2(k) \rightarrow \mu(k)$$

For any n dividing $\#\mu(k)$, one has the n -th Hilbert symbol,

$$(-, -)_n : K_2(k) \rightarrow \mu_n(k)$$

Pushing the above exact sequence out by the n -th Hilbert symbol, one obtains a short sequence of locally compact topological groups

$$1 \rightarrow \mu_n(k) \rightarrow \overline{G} \rightarrow G \rightarrow 1$$

we call this the BD covering group associated with to the BD data $(Q, \mathcal{E}, f, n)$.

3.2. **Covering tori.** We consider the pullback of $\overline{G}$ to the maximal split torus T :

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{T} \longrightarrow T \longrightarrow 1$$

$\overline{T} = T \times_D \mu_n(F)$ is generated by elements $\zeta \in \mu_n(F)$ and $y(a)$ for $y \in Y$ and $a \in F^\times$ subject to the relations:

- the elements ζ are central.
- $[y_1(a), y_2(b)] = (a, b)_n^{B_Q(y_1, y_2)}$ for all $y_1, y_2 \in Y$ and $a, b \in F^\times$.
- $y_1(a) \cdot y_2(a) = (y_1 + y_2)(a) \cdot (a, a)_n^{D(y_1, y_2)}$.
- $y(a) \cdot y(b) = y(ab) \cdot (a, b)_n^{Q(y)}$.

Proposition 3.2.1. *We have $\overline{T}$ is abelian if and only if*

$$B_Q(Y, Y) \subseteq n\mathbb{Z}$$

Proof. The covering torus $\overline{T}$ is generated by its central subgroup μ_n and the elements $\tilde{y}(a)$ with $y \in Y$ and $a \in k^\times$. Their commutators satisfy

$$[\tilde{y}(a), \tilde{z}(b)] = (a, b)_n^{B_Q(y, z)}$$

If $B_Q(Y, Y) \subseteq n\mathbb{Z}$, then all these commutators are trivial, because $(a, b)_n \in \mu_n$. Thus the generators of $\overline{T}$ commute and $\overline{T}$ is abelian.

Conversely, suppose $\overline{T}$ is abelian. Fix $y, z \in Y$ and put $m = B_Q(y, z)$, the commutator formula gives

$$(a, b)_n^m = 1, \quad \text{for all } a, b \in k^\times$$

Let ϖ be a uniformizer of k , by bilinearity

$$(\varpi^m, b)_n = (\varpi, b)_n^m = 1$$

for every $b \in k^\times$.

The nondegeneracy of the Hilbert pairing

$$k^\times / (k^\times)^n \times k^\times / (k^\times)^n \longrightarrow \mu_n$$

implies that $\varpi^m \in (k^\times)^n$, taking the valuation gives $m \in n\mathbb{Z}$. Since y, z are arbitrary, we conclude

$$B_Q(Y, Y) \subseteq n\mathbb{Z}$$

□

Choose a basis $e_1, \dots, e_r$ of $Y = X_*(T)$ with dual characters $x_1, \dots, x_r$. Write

$$D_{ij} = D(e_i, e_j)$$

We introduce

$$c_D(t, u) = \prod_{i,j} (x_i(t), x_j(u))_n^{D_{ij}}$$

If we identify the underlying set of $\overline{T}$ with $T \times \mu_n$, then the multiplication is given by

$$(t, \zeta)(u, \xi) = (tu, \zeta \xi c_D(t, u))$$

4. ON A SPLITTING LEMMA FOR k -SUBGROUPS

In this section, let k be a p -adic local field of characteristic 0 with odd residue characteristic with $\mu_n \subseteq k^\times$, let G be a split connected reductive group over k , and $\overline{G}$ be its BD cover.

We want to find a combinatorial criterion for the existence of a splitting $s : H \rightarrow \overline{G}$.

Proposition 4.0.1. *Let k be a p -adic local field of odd residue characteristic. Then we have*

$$B_Q(Y, Y) \subseteq n\mathbb{Z} \text{ if and only if } \overline{T} \longrightarrow T \text{ admits a continuous splitting}$$

Proof. If $\bar{T} \longrightarrow T$ admits a continuous splitting, then we have $\bar{T}$ is abelian and from Proposition 3.2.1 we get

$$B_Q(Y, Y) \subseteq n\mathbb{Z}$$

Now if $B_Q(Y, Y) \subseteq n\mathbb{Z}$. Choose a basis $e_1, \dots, e_r$ of Y and an upper-triangular bisector D with

$$D(e_i, e_i) = Q(e_i), \quad D(e_i, e_j) = B_Q(e_i, e_j) \quad (i < j)$$

because the off-diagonal entries are divisible by n , the torus cocycle reduces to

$$c_D(t, u) = \prod_i (t_i, u_i)_n^{Q(e_i)}$$

if n is odd, then $n|Q(e_i)$, thus $c_D = 1$ giving a splitting.

If n is even, put

$$\epsilon_i = \frac{2Q(e_i)}{n} \pmod{2}$$

Compatibility of the Hilbert symbols gives

$$c_D(t, u) = \prod_i (t_i, u_i)_2^{\epsilon_i}$$

Choose a uniformizer ϖ and let

$$\lambda : \mathcal{O}_k \longrightarrow \{\pm 1\}$$

be the quadratic character of the residue field. For $a \in \varpi^r u$, define

$$b(a) = \lambda(-1)^{r(r-1)/2} \lambda(u)^r$$

The Hilbert symbol formula gives

$$\frac{b(a)b(a')}{b(aa')} = (a, a')_2$$

Therefore for

$$B(t) = \prod_i b(t_i)^{\epsilon_i}$$

we have

$$B(t)B(u) = c_D(t, u)B(tu)$$

in the cocycle model

$$(t, \zeta)(u, \xi) = (tu, \zeta \xi c_D(t, u))$$

the map

$$s(t) = (t, B(t)^{-1})$$

gives a continuous group-theoretic splitting. □

Proposition 4.0.2 (Splitting criterion for simply connected groups). *Let $n \geq 1$ satisfy $\mu_n \subseteq k^\times$. Let G be a split simply connected semisimple group over k , with a fixed pinning. Write*

$$Y = X_*(T) = Y^{\text{sc}}$$

for the cocharacter lattice of its maximal split torus T . Let

$$1 \longrightarrow \mu_n \longrightarrow \bar{G} \xrightarrow{\pi} G(k) \longrightarrow 1$$

be the topological cover obtained from a BD extension with quadratic invariant $Q : Y \rightarrow \mathbb{Z}$ by pushout along the n -th Hilbert symbol.

Then $\bar{G} \rightarrow G(k)$ admits a continuous group-theoretic splitting if and only if

$$Q(Y) \subseteq n\mathbb{Z}.$$

Equivalently, it splits if and only if

$$n \mid Q(\alpha^\vee) \quad \text{for every simple coroot } \alpha^\vee.$$

When a splitting exists, it is unique.

Proof. Suppose first that a splitting $s : G \rightarrow \overline{G}$ exists. Let $\tilde{h}_\alpha(a)$ denote the canonical coroot lift, constructed from these root-group splittings. Since $\alpha^\vee(a)$ is expressed as a product of root-group elements, we have

$$s(\alpha^\vee(a)) = \tilde{h}_\alpha(a)$$

The canonical lift satisfy

$$\tilde{h}_\alpha(a)\tilde{h}_\alpha(b) = \tilde{h}_\alpha(ab)(a, b)_n^{Q(\alpha^\vee)}$$

multiplicativity of s gives

$$\tilde{h}_\alpha(a)\tilde{h}_\alpha(b) = \tilde{h}_\alpha(ab)$$

Consequently

$$(a, b)_n^{Q(\alpha^\vee)} = 1$$

for all $a, b \in k^\times$.

Choose a uniformizer ϖ of k . By bilinearity of the Hilbert symbol,

$$(\varpi^{Q(\alpha^\vee)}, b)_n = 1$$

for every $b \in k^\times$. Nondegeneracy of the Hilbert pairing gives

$$\varpi^{Q(\alpha^\vee)} \in (k^\times)^n$$

Taking the valuation gives

$$n|Q(\alpha^\vee)$$

Since G is simply connected, its simple coroots $\alpha_1^\vee, \dots, \alpha_r^\vee$ form a basis of Y . Weyl invariance of Q gives

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \langle \alpha_i, \alpha_j^\vee \rangle Q(\alpha_j^\vee)$$

For $y = \sum m_i \alpha_i^\vee$, we have

$$Q(y) = \sum_i m_i^2 Q(\alpha_i^\vee) + \sum_{i < j} m_i m_j B_Q(\alpha_i^\vee, \alpha_j^\vee) \in n\mathbb{Z}$$

This proves necessity and also shows that the divisibility on simple coroots is equivalent to $Q(Y) \subseteq n\mathbb{Z}$.

Conversely, suppose that $Q(Y) \subseteq n\mathbb{Z}$. Then

$$Q_0 := Q/n : Y \longrightarrow \mathbb{Z}$$

is an integer-valued Weyl-invariant quadratic form. Let $\mathcal{G}_0$ be the BD extension associated with Q_0 , and let

$$1 \longrightarrow \mu_n \longrightarrow \overline{G}_0 \xrightarrow{\pi_0} G(k) \longrightarrow 1$$

be its pushout along the same n -th Hilbert symbol.

By additivity of the BD classification for simply connected semisimple groups, $Q = nQ_0$ implies

$$\overline{G} \simeq \underbrace{\overline{G}_0 \boxplus \dots \boxplus \overline{G}_0}_{n \text{ factors}}$$

as topological central extensions.

To split this Baer sum, form

$$P = \underbrace{\overline{G}_0 \times_{G(k)} \dots \times_{G(k)} \overline{G}_0}_{n \text{ factors}}, \quad K = \left\{ (\zeta_1, \dots, \zeta_n) \in \mu_n^n : \prod_i \zeta_i = 1 \right\}.$$

The Baer sum is P/K . For $g \in G(k)$, choose a lift $\tilde{g} \in \overline{G}_0$ and put

$$s(g) = [\tilde{g}, \dots, \tilde{g}] \in P/K.$$

Replacing $\tilde{g}$ by $\zeta \tilde{g}$ changes the tuple by $(\zeta, \dots, \zeta) \in K$, because $\zeta^n = 1$. Hence s is well defined. Moreover,

$$s(g)s(h) = [\tilde{g}\tilde{h}, \dots, \tilde{g}\tilde{h}] = s(gh),$$

and its projection to $G(k)$ is g .

The diagonal map $\overline{G}_0 \rightarrow P/K$ is continuous and constant on the fibers of the quotient map π_0 . Therefore the induced splitting $s : G(k) \rightarrow P/K$ is continuous. Transporting it through the above isomorphism gives a continuous splitting of $\overline{G}$.

Finally, two splittings differ by a homomorphism $G(k) \rightarrow \mu_n$. Such a homomorphism is trivial on every root subgroup, and $G(k)$ is generated by its root subgroups. Thus the splitting is unique. $\square$

We now give a combinatorial criterion for general split group G .

Proposition 4.0.3. *Let G be a split connected reductive group over k . Fix a maximal split torus T , and write*

$$Y = X_*(T), \quad Y^{\text{sc}} \subseteq Y$$

for its cocharacter lattice and the coroot lattice. Let

$$1 \longrightarrow \mu_n \longrightarrow \overline{G} \longrightarrow G(k) \longrightarrow 1$$

be the topological cover associated to BD data $(Q, \mathcal{E}, f)$.

Assume that $Q(Y) \subseteq n\mathbb{Z}$. Choose an incarnation (D, η) of these data such that

$$D(Y, Y) \subseteq n\mathbb{Z}, \quad \eta : Y^{\text{sc}} \longrightarrow k^\times.$$

Such a choice of D exists under the stated assumption. Then $\overline{G}$ admits a continuous group-theoretic splitting if and only if

$$\eta_n : Y^{\text{sc}} \longrightarrow k^\times / (k^\times)^n, \quad y \longmapsto [\eta(y)],$$

extends to a homomorphism

$$\tilde{\eta}_n : Y \longrightarrow k^\times / (k^\times)^n.$$

Let $\mathbf{H}$ be a split connected reductive k -subgroup of $\overline{\mathbf{G}}$. Suppose for the restricted quadratic form we have

$$Q_H(Y_H) = Q(j(Y_H)) \subseteq n\mathbb{Z}$$

Proposition 4.0.4. *Suppose $Q_H(Y_H) = Q(j(Y_H)) \subseteq n\mathbb{Z}$, then*

$$\overline{G} \text{ splits over } H \iff (\eta_H)_n : Y_H^{\text{sc}} \rightarrow k^\times / (k^\times)^n \text{ extends to } Y_H$$

Proof. This follows from the previous Proposition 4.0.3. $\square$