CLASSIFICATION OF SPHERICAL VARIETIES

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Contents

1. Introduction

This is my study note for the classification of spherical varieties over $\mathbb{C}$ based on the papers [BP05], [BP16], we also present the complete result for type A spherical systems [Lun01].

There is another approach studied by Cupit-Foutou [CF09] by means of a suitable class of invariant Hilbert schemes.

2. Notation

We will fix G a connected reudctive group over $\mathbb{C}$, A a maximal torus of G, B contains A a Borel subgroup, S set of simple roots of G determined by B, the root datum will be denoted by $\mathcal{R} = (\chi^*, \Phi, \chi_*, \Phi^\vee)$ with $\chi^* = X^*(A)$.

X will be a spherical G-variety over $\mathbb{C}$.

3. Invariants of spherical varieties

In this section, we will introduce some invariants for spherical varieties.

We will denote the characters of B-semiinvariant functions on X by $\chi(X)$, the associated parabolic subgroup of X is the standard parabolic subgroup

$$P(X) := \{g \in G | \mathring{X} \cdot g = \mathring{X}\}$$

From the local structure theorem, we have an isomorphism $\mathring{X} \cong A_X \times U_{P(X)}$, and it can be shown that $\chi = X^*(A_X)$.

We will denote

$$\Lambda(X) = \chi(X)^*, \ \mathfrak{a}_X = \Lambda(X) \otimes \mathbb{Q}$$

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we can think $\Lambda(X)$ as the cocharacter lattice of X. An B-invariant, $\mathbb{Q}$-valued valuation on $\mathbb{C}(X)$ which is trivial on $\mathbb{C}^\times$ will induce an element of $\Lambda(X)$ via restriction to $\mathbb{C}(X)^B$ and we will denote $\mathscr{V} \subset \mathfrak{a}_X$ the cone generated by the images of G-invariant valuations. $\mathscr{V}$ contains the image of negative Weyl chamber under the natural map $\mathfrak{a} \to \mathfrak{a}_X$. $\mathscr{V}$ contains the image of the negative Weyl chamber under the natural map $\mathfrak{a} \to \mathfrak{a}_X$. We will denote by $\Lambda(X)^+ = \Lambda(X) \cap \mathscr{V}$. The cone $\mathscr{V} = \mathfrak{a}_X^+$ is the fundamental domain for a finite reflection group $W_X \subset \mathrm{End}(\mathfrak{a}_X)$, called the *little Weyl group* of X.

Consider the strictly convex cone negative dual to $\mathscr{V}$:

$$\mathscr{V}^\perp = \{\chi \in \chi(X) \otimes \mathbb{R} |\ \langle \chi, v\rangle \leq 0 \text{ for every } v \in \mathscr{V}\}$$

The generators of the intersections of the extremal rays with $\chi(X)$ are called the *spherical roots* of X.

The spherical roots are known to form the set of simple roots of a based root system with Weyl group W_X. This root system will be called the *spherical root system* of X, following the notation of [Lun96], we will denote the set of simple roots by Σ_X.

Remark 3.1. There is also a different normalization of spherical roots proposed in [SV17], the normalized spherical roots which is aimed for application to representation theory.

4. Wonderful varieties

Wonderful varieties is a class of spherical varieties which arises in the embedding theory of spherical varieties.

Definition 4.1. An algebraic G-variety X is *wonderful* of rank r if:

- X is smooth and complete.
- G has a dense orbit in X whose complement is the union of r smooth prime divisors $D_i,\ i = 1, \cdots r$ with normal crossings.
- the intersection of the divisors D_i is nonempty and for all $I \subseteq \{1, \cdots, r\}$

$$(\cap_{i\in I} D_i)\backslash(\cup_{i\notin I} D_i)$$

  is a G-orbit.

A wonderful G-variety is always projective and spherical, this is proved in [Lun96].

Definition 4.2. A spherical variety $H\backslash G$ is called *wonderful* if $H\backslash G$ admits an embedding which is a wonderful variety.

Next, we will fix X a wonderful variety for G. The following proposition can be viewed as a localization principle

Proposition 4.3. *Let $z \in X$ be the unique fixed point of B^- and consider the orbit $Z = G \cdot z$ which is the unique closed orbit in X, then the spherical roots are the T-weights appearing in T_zX/T_zZ.*

One can associate to each spherical root γ a G-stable prime divisor D^γ such that γ is the T-weight of T_zX/T_zD^γ. Consider the intersection of all G-invariant prime divisors of X different from D^γ, this intersection is a wonderful variety of rank 1, and having γ as its spherical root.

If H is wonderful then H has finite index in $N_G(H)$, and if $H = N_G(H)$ then it is wonderful.

We will denote the set of spherical roots of all wonderful G-varieties of rank 1 by $\Sigma(G)$, for G of adjoint type the elements of $\Sigma(G)$ are always linear combinations of simple roots with nonnegative integer coefficients.

Now let's recall some lemmas on colors: Let X be a wonderful G-variety, S the set of simple roots associated to B, for $\alpha \in S$, we let P_α be the standard parabolic subgroup associated to α. Let $\Delta_X(\alpha)$ denote the set of non P_α-stable colors, we will say that α moves the colors in $\Delta_X(\alpha)$, and a color is always moved by some simple roots.

Lemma 4.4. ([Lun96]) *For all $\alpha \in S$, $\Delta_X(\alpha)$ has at most two elements and only the following four cases can appear:*

(1) $\Delta_X(\alpha) = \emptyset$, this happens when the open Borel orbit $\mathring{X}$ is stable under P_α, and the set of all such α will be denote by S_X^p.

(2) $\Delta_X(\alpha)$ has two elements, this happens exactly when $\alpha \in \Sigma_X$, the two colors in $\Delta_X(\alpha)$ will be denoted by D_α^+, D_α^- and we have

$$\langle \rho(D_\alpha^+), \gamma\rangle + \langle \rho(D_\alpha^-), \gamma\rangle = \langle \alpha^\vee, \gamma\rangle$$

for every $\gamma \in \Sigma_X$. We will denote by $\mathcal{A}_X$ the union of all $\Delta_X(\alpha)$ for every $\alpha \in S \cap \Sigma_X$.
(3) $\Delta_X(\alpha)$ has one element and $2\alpha \in \Sigma_X$, the color in $\Delta_X(\alpha)$ is denoted by D'_α and we have:

$$\langle \rho(D'_\alpha), \gamma\rangle = \frac{1}{2}\langle \alpha^\vee, \gamma\rangle$$

(4) The remaining case, i.e. $\Delta_X(\alpha)$ has one element but $2\alpha \notin \Sigma_X$. In this case, the color in $\Delta_X(\alpha)$ is denoted by D_α and

$$\langle \rho(D_\alpha), \gamma\rangle = \langle \alpha^\vee, \gamma\rangle$$

for every $\gamma \in \Sigma_X$.

Lemma 4.5. *([Lun97]) For all $\alpha, \beta \in S$, the condition $\Delta_X(\alpha) \cap \Delta_X(\beta) \neq \emptyset$ occurs only in the following two cases:*

(1) if $\alpha, \beta \in S \cap \Sigma_X$ then it can happen that the cardinality of $\Delta_X(\alpha) \cup \Delta_X(\beta)$ is equal to 3.
(2) if α and β are orthogonal and $\alpha + \beta$ or $\frac{1}{2}(\alpha+\beta)$ belongs to Σ_X, then $D_\alpha = D_\beta$.

The relations in these two lemmas come from the study of some analysis of the cases in rank 1 and rank 2, they will appear in the next section as the axioms for *spherical systems*. The spherical systems for a wonderful variety X consists of S^p_X the simple roots moving no colors, Σ_X the set of spherical roots, and $\mathcal{A}_X$ a subset of colors.

5. Spherical systems

The following definition comes from the classification of wonderful varieties of rank less or equal to 2 and some geometric properties of colors studied by Luna 4.4, 4.5.

Definition 5.1. Given a root datum $\mathcal{R} = (\chi^*, \Phi, \chi_*, \Phi^\vee)$ of a connected reductive algebraic group G and a set of positive roots S, a triple $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ such that $S^p \subseteq S$, $\Sigma \subset \Sigma(G)$, $\mathcal{A}$ is a finite set endowed with a map $\rho : \mathcal{A} \longrightarrow \chi^\vee$, where $\chi = \langle \Sigma \rangle$, $\mathcal{S}$ will be called a *spherical systems* if the following axioms are satisfied:

(A1) $\forall D \in \mathcal{A}$, $\rho(D)(\alpha) \leq 1$ for all $\alpha \in \Sigma$, equality holds if and only if $\alpha \in S \cap \Sigma$.
(A2) $\forall \alpha \in S \cap \Sigma$, $\mathcal{A}(\alpha) := \{D \in \mathcal{A} | \ \rho(D)(\alpha) = 1\} = \{D_\alpha^+, \ D_\alpha^-\}$, and $\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee$.
(A3) $\mathcal{A} = \cup_{\alpha \in S \cap \Sigma} \mathcal{A}(\alpha)$.
(Σ1) If $2\alpha \in \Sigma \cap 2S$, then $\frac{1}{2}\langle \alpha^\vee, \beta\rangle$ is a non-positive integer, $\forall \beta \in \Sigma \backslash \{2\alpha\}$, furthermore $\alpha \notin \chi$ and $\frac{1}{2}\langle \alpha^\vee, \beta\rangle$ is an integer for all $\beta \in \chi$.
(Σ2) If $\alpha, \beta \in S$ are orthogonal and $\alpha + \beta$ belongs to Σ or 2Σ, then $\langle \alpha^\vee, \gamma\rangle = \langle \beta^\vee, \gamma\rangle$, $\forall \gamma \in \chi$.
(S1) For all $\alpha \in \Sigma$, there is a wonderful G-variety X of rank 1 with $S^p_X = S^p$, and $\Sigma_X = \{\alpha\}$.
(S2) For all $\gamma \in \Sigma$

$$\{\alpha \in \Sigma_G, \ \langle \alpha^\vee, \gamma\rangle = 0\} \cap \mathrm{supp}(\gamma) \subset S^p \subset \{\alpha \in \Sigma_G, \ \langle \alpha^\vee, \gamma\rangle = 0\}$$

The cardinality of Σ will be called the *rank* of the spherical system.

Let's note that for the spherical systems of a wonderful variety X, the spherical root system (Φ_X, Σ_X) is not part of the axiom.

The definition of the spherical system is such that the following lemmas holds:

Lemma 5.2. *For every wonderful G-variety X the triple $(S^p_X, \Sigma_X, \mathcal{A}_X)$ is a spherical system.*

Let's sketch the proof for this lemma: axioms $(A2), (A3)$ correspond to lemma 4.4 (2), axiom (Σ1) correspond to lemma 4.4 (3), axiom (Σ2) corresponds to 4.5 (2) and axiom (S) follows from the definition of Σ_X and $\Sigma(G)$.

Lemma 5.3. *The map $X \mapsto (S^p_X, \Sigma_X, \mathcal{A}_X)$ is a bijection between rank one (resp. rank two) wonderful varieties (up to G-isomorphisms) and rank one (resp. rank two) spherical systems.*

This lemma is a reformulation of the result of Wasserman [Was96].

As we will see in the next section, in [Lun01] it is proven that spherical systems classify wonderful G-varieties for G adjoint of type A and he conjectured that wonderful varieties are classified by spherical systems, this program is completed in [BP16].

Theorem 5.4. *(*[BP16]*) There is a bijection $X \leftrightarrow (S^p_X, \Sigma_X, \mathcal{A}_X)$ between wonderful G-varieties and spherical systems.*

It is obvious that two G-isomorphic wonderful varieties have the same spherical systems, however, if two wonderful varieties are isomorphic, namely G-isomorphic up to outer automorphism of G, their spherical systems are equal up to *a permutation of of the set S of simple roots.*

Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system, we consider the couple (χ', ρ') with

- χ' is a subgroup of $\chi(T)$ contains Σ.
- The application $\rho' : \ \mathcal{A} \to (\chi')^*$.

For X a homogeneous spherical variety, we can still define the character lattice $\chi(X)$ and the Cartan pairing ρ_X, and the axioms for the spherical systems should also hold for $(\chi(X), \rho_X)$, this motivates the following definition of homogeneous spherical data

Definition 5.5. We say that (χ', ρ') is an augmentation of $(S^p, \Sigma, \mathcal{A})$ if it satisfies

- Under the natural map $(\chi')^* \to \chi^*$, ρ' is equal to ρ.
- For all $\alpha \in \Sigma \cap \Sigma_G$, $\mathcal{A}(\alpha) = \{D^+_\alpha, \ D^-_\alpha\}$, $\langle \rho'(D^+_\alpha), \gamma \rangle + \langle \rho'(D^-_\alpha), \gamma \rangle = \langle \alpha^\vee, \gamma \rangle$ for all $\gamma \in \chi'$.
- If $2\alpha \in \Sigma \cap 2\Sigma_G$, then $\alpha \notin \chi'$ and $\langle \alpha^\vee, \gamma \rangle$ is an even integer for all $\gamma \in \chi'$.
- If $\alpha + \beta \in \Sigma$ or $\frac{1}{2}(\alpha + \beta) \in \Sigma$, then α, β are orthogonal and $\langle \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle$ for $\gamma \in \chi'$.
- For all $\alpha \in S^p$, α annihilates χ'.

We will say that $(S^p, \Sigma, \mathcal{A}, \chi, \rho)$ is an augmented spherical system. An augmented spherical system such that every elements of Σ is primitive in χ' will be called a homogeneous spherical data.

5.1. **Uniqueness result.** We have the following uniqueness result of Losev: Given X_1, X_2 two spherical varieties, $\Delta_{X_1}, \ \Delta_{X_2}$ set of colors of X_1 and X_2, we will write $\Delta_{X_1} = \Delta_{X_2}$, if there is a bijection $\psi : \Delta_{X_1} = \Delta_{X_2}$ such that $G_D = G_{\psi(D)}$, $\rho_{X_1}(D) = \rho_{X_2}(\psi(D))$, here $G_D = \{g \in D \mid gD = D\}$. Here we note that $\{G_D\}$ although is not part of the spherical system, but it can be calculated from $\Delta_X(\alpha)$, hence can be read from the Luna diagram.

Theorem 5.6. *Let $H_1, \ H_2$ be two spherical groups, $X_1 = G/H_1, \ X_2 = G/H_2$, if $(\mathcal{S}^p_{X_1}, \Sigma_{X_1}, \mathcal{A}_{X_1}) = (\mathcal{S}^p_{X_2}, \Sigma_{X_2}, \mathcal{A}_{X_2}), \ \Delta_{X_1} = \Delta_{X_2}$, then H_1 and H_2 are G-conjugate.*

6. Luna's classification

From the classification of wonderful varieties of rank less or equal than two, Luna showed that any spherical systems can be obtained from 29 primitive systems via parabolic induction, fiber product, projective fibration, since these operations are compatible with the operations on wonderful varieties side, Luna reduced the existence of wonderful varieties to a given spherical system to the existence of wonderful varieties for primitive spherical systems.

The two main theorems in Luna's paper are

Theorem 6.1. *Suppose G is semisimple adjoint of type A. The map that sends a wonderful variety X to $(S^p_X, \Sigma_X, \mathcal{A}_X)$ is a bijection between the isomorphism classes of wonderful G-varieties and the set of spherical systems for G.*

The classification of spherical subgroups can be reduced to the classification of wonderful subgroups, actually, to the spherical closure.

Theorem 6.2. *Suppose that theorem 6.1 is true for adjoint group G, then the map that sends a homogeneous spherical G-variety X to the quintuple $(S^p_X, \Sigma_X, \mathcal{A}_X, \chi(X), \rho_X)$ is a bijection between the isomorphism classes of homogeneous G-spherical varieties and the homogeneous spherical data for G.*

Remark 6.3. The classification of general spherical varieties can be reduced to the homogeneous spherical varieties via the embedding theory of spherical varieties.

6.1. **Operations on spherical systems.** We introduce various operations on spherical systems and the corresponding geometric operation.

Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system and $\rho : \Delta \to \mathscr{V} \subset \chi^* \otimes \mathbb{Q}$ vector space of colors.

Definition 6.4. We say a subset Δ' is distinguished if $C(\rho(\Delta'))^\circ$ is a cone in $-\mathscr{V}$.

Lemma 6.5. *For a subset $\Delta' \subset \Delta$ is distinguished if and only if there exists a subspace N' of $\chi^* \otimes \mathbb{Q}$ satisfies*

- *the couple N', Δ' is a colored sub vector space.*
- *the intersection $N' \cap \mathscr{V}$ is a face of the cone $\mathscr{V}$.*

Definition 6.6. If $(S^p, \Sigma, \mathcal{A})$ is a spherical system and Δ' is a distinguished subset of Δ, we define χ/Δ' to be the element of χ which is annihilated by $N(\Delta')$. We define the quotient of the spherical system as: $(S^p, \Sigma, \mathcal{A})/\Delta'$

- $S^p/\Delta' = \{\alpha \in \Sigma_G, \Delta(\alpha) \subset \Delta'\}$.
- Consider the linear combinations $\sum_{\gamma \in \Sigma} n(\gamma)\gamma \in \chi/\Delta'$, Σ/Δ' are indecomposable elements of this semigroup.
- we define $\mathcal{A}/\Delta'$ as the union of all $\mathcal{A}(\alpha)$ such that $\mathcal{A}(\alpha) \cap \Delta' = \emptyset$, and we define $\rho/\Delta' : \mathcal{A}/\Delta' \to (\chi/\Delta')^*$ as a restriction of ρ to $\mathcal{A}/\Delta'$ under the natural map $\chi^* \to (\chi/\Delta')^*$.

It is not clear that whether the quotient $(S^p, \Sigma, \mathcal{A})/\Delta'$ is still a spherical system, so we have the following definition

Definition 6.7. We will say that Δ' has the property $(*)$ if the elements Σ/Δ' forms a $\mathbb{Z}$-basis of the module χ/Δ'.

We will see later that 6.27 as a corollary of the existence of wonderful varieties to spherical systems, for all adjoint groups of type A, any distinguished subset Δ' satisfies the property $(*)$, hence the quotient triple $(S^p, \Sigma, \mathcal{A})/\Delta'$ is still a spherical system.

Definition 6.8. Suppose X is a wonderful G-variety, and $(S^p, \Sigma, \mathcal{A})$ is a spherical system for X, suppose X' is another G-wonderful variety, and $\phi : X \to X'$ is a dominant G-morphism, we put $\Delta(\phi) = \{D \in \Delta_X,\ \phi(D) = X'\}$.

Proposition 6.9. *The map $\phi \mapsto \Delta(\phi)$ induces a bijection between the G-morphisms with connected fibers between wonderful varieties and distinguished subset of Δ with property $(*)$, moreover for $\phi : X \to X'$ associated with $\Delta(\phi)$, the spherical system of X' is $(S^p, \Sigma, \mathcal{A})/\Delta(\phi)$.*

We let $G/H = \mathring{X}_G$ the open G-orbit of X, then it follows from the embedding theory of homogeneous spherical varieties, we have a bijection between G-morphisms with connected fiber $G/H \to G/H'$ and colored subvector spaces N', Δ' of $\rho : \Delta \to \chi^* \otimes \mathbb{Q}$, and G/H' is wonderful if and only if $N' \cap \mathscr{V}$ is a face of the cone $\mathscr{V}$.

Definition 6.10. We say the distinguished subset $\Delta' \subset \Delta$ is parabolic if $N(\Delta') = N$.

Proposition 6.11. *Let X be a wonderful G-variety and $\Sigma' \subset \Sigma_G$, we have a bijection between $\phi : X \to G_{-\Sigma'}$ and the distinguished parabolic subsets $\Delta' \subset \Delta$ with $\Sigma' = \Sigma \backslash (S^p/\Delta')$.*

Here we note that a distinguished subset $\Delta' \subset \Delta$ is parabolic if and only if $\Sigma/\Delta' = \emptyset$.

Let Q be a parabolic subgroup of G contains B_-, then there exists a subset $\Sigma' \subset \Sigma_G$ such that $Q = G_{-S'}$, put $L = G_{S'} \cap G_{-S'}$.

Definition 6.12. We say X is a parabolic induction of X' from Q to G if $X \cong G \times_Q X'$ where $G \times_Q X'$ is the fiber product with

$$q \cdot (g, x) = (gq^{-1}, qx)$$

Let X be a wonderful G-variety, and $\Sigma' \subset \Sigma_G$, let $(S^p, \Sigma, \mathcal{A})$ be the spherical system for X, we denote $\Delta(S')$ the union of $\Delta(\alpha),\ \alpha \in \Sigma'$.

Proposition 6.13. *There exists a G-morphism $\phi : X \to G/G_{-S'}$ induces a parabolic induced structure on X if and only if $supp(\Sigma) \cup S^p \subset \Sigma'$, the morphism is unique and $\Delta(\phi) = \Delta(\Sigma')$.*

The existence of ϕ is given by the previous proposition 6.11.

Definition 6.14. We say a spherical system $(S^p, \Sigma, \mathcal{A})$ is cuspidal if $\mathrm{supp}(\Sigma) = \Sigma_G$.

Proposition 6.15. *Let X be a wonderful G-variety, suppose the spherical system of X is cuspidal, then X can't be obtained via parabolic induction.*

Definition 6.16. Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system and Δ the set of colors, Δ_1, Δ_2 two distinguished sets of Δ, we say that Δ_1, Δ_2 decomposes the spherical system $(S^p, \Sigma, \mathcal{A})$ if

(1) $\Delta_1 \neq \emptyset,\ \Delta_2 \neq \emptyset$ and $\Delta_1 \cap \Delta_2 = \emptyset$.
(2) $\Delta_1,\ \Delta_2,\ \Delta'$ satisfies property $(*)$.
(3) $\Sigma(\Delta_1) \cap \Sigma(\Delta_2) = \emptyset$.
(4) $S^p(\Delta_1)$ is orthogonal to $S^p(\Delta_2)$.
(5) Δ_1 and Δ_2 are smooth.

Proposition 6.17. *Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical system, suppose a couple Δ_1, Δ_2 decomposes $\mathcal{S}$, suppose there exists wonderful varieties $X_1,\ X_2,\ X'$ with spherical systems $\mathcal{S}/\Delta_1,\ \mathcal{S}/\Delta_2,\ \mathcal{S}/\Delta'$. Then we have dominant G-morphisms $\phi_1 : X_1 \to X'$, $\phi_2 : X_2 \to X'$, then we have*

- *The fiber product $X_1 \times_{X'} X_2$ is a wonderful variety with spherical system $\mathcal{S}$.*
- *If X is a wonderful variety with spherical system $\mathcal{S}$, then X is isomorphic to $X_1 \times_{X'} X_2$.*

The existence of $\phi_1,\ \phi_2$ exists as $\mathcal{S}/\Delta'$ is a quotient of $\mathcal{S}/\Delta_1$ and $\mathcal{S}/\Delta_2$.

Definition 6.18. Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system. Let $\delta \in \mathcal{A}$ satisfies $\delta(\Sigma) \subset \{0, 1\}$, then we say δ is a projective element of $\mathcal{A}$.

Let δ be a projective element of $\mathcal{A}$, we put $S_\delta = \delta^{-1}(1) \subset \Sigma_G \cap \Sigma$, we define the quotient $(S^p, \Sigma, \mathcal{A})/\{\delta\}$ as

- $S^p/\{\delta\} = S^p$.
- $\Sigma/\{\delta\} = \Sigma \backslash S_\delta$.
- $\mathcal{A}/\{\delta\}$ is the restriction of $\mathcal{A}(\Sigma \backslash S_\delta)$ to $\Sigma/\{\delta\}$.

Let X be a wonderful G-variety with spherical system $(S^p, \Sigma, \mathcal{A})$, and δ a projective element pf $\mathcal{A}$, and $\phi_\delta : X \to X_\delta$ a G-morphism corresponds to $\{\delta\}$, ϕ_δ is a projective fibration: it is smooth, all fibers are isomorphic to $\mathbb{P}^n$ with $\mathrm{rk}\ X = n + \mathrm{rk}\ X_\delta$.

Proposition 6.19. *Suppose G is of type A. Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system and δ a projective element, let X_δ be a wonderful variety with spherical system $(S^p, \Sigma, \mathcal{A})/\{\delta\}$, then there exists a wonderful variety X unique up to isomorphism, satisfies:*

- *$(S^p, \Sigma, \mathcal{A})$ is a spherical system for X.*
- *X_δ is G-isomorphic to X'.*

6.2. **Primitive spherical systems and geometric realizations.** First we note there are 5 wonderful varieties of rank 1 for type A groups

- type a_1: this is the spherical root of $X = PGL_2/T$.
- type $a_n,\ n > 1$: this is the spherical root of $X = PGL_n/GL_n$.
- type a': this is the spherical root of $X = PGL_2/N(T)$.
- type d_3: this is the spherical root of SL_4/Sp_4.
- type $a_1 \times a_1$: this is the spherical root of $SL_2 \times SL_2/SL_2^{\mathrm{diag}}$.

From the axiom for spherical system, we know that any spherical roots of a spherical system $\mathcal{S}$ for adjoint groups of type A are necessarily of type $a_1,\ a_n\ n > 1,\ a',\ d_3,\ a_1 \times a_1$.

We have the following list of spherical systems, which we will call primitive, in the following $\Sigma_n = \{\alpha_1, \cdots, \alpha_n\}$ will be the simple roots for root system of type A_n.

(1) family ao(n), $n \geq 1$

- $S^p = \emptyset$.
- $\Sigma = \{2\alpha_1, \cdots, 2\alpha_n\}$.
- $\mathcal{A} = \emptyset$.

(2) family ac(n), n odd, $n \geq 3$

- $S^p = \{\alpha_i,\ i\text{odd},\ 1 \leq i \leq n\}$.
- $\Sigma = \{\alpha_1 + 2\alpha_2 + \alpha_3,\ \alpha_3 + 2\alpha_4 + \alpha_5, \cdots, \alpha_{n-2} + 2\alpha_{n-1} + \alpha_n\}$.
- $\mathcal{A} = \emptyset$.

Families aa

(3) family aa(p+q+p), $n = 2p+q,\ p \geq 1,\ q \geq 1$

- $S^p = \{\alpha_{p+2}, \cdots, \alpha_{p+q-1}\}$.
- $\Sigma = \{\alpha_1 + \alpha_n, \alpha_2 + \alpha_{n-1}, \cdots, \alpha_p + \alpha_{n-p+1}, \alpha_{p+1}\}$.
- $\mathcal{A} = \emptyset$ if $q \geq 2$.

(4) aa(p,p) localization of aa(p+q+p) at $S' = \{\alpha_1, \cdots, \alpha_p, \alpha_{p+q+1}, \cdots, \alpha_n\}$.

(5) aa(q) localization of aa(p+q+p) at $S' = \{\alpha_{p+1}, \cdots, \alpha_{p+q}\}$.

(6) family aa^*(p+1+p) $p \geq 1$

- $S^p = \emptyset$.
- $\Sigma = \{\alpha_1 + \alpha_n, \alpha_2 + \alpha_{n-1}, \cdots, \alpha_p + \alpha_{n-p+1}, 2\alpha_{p+1}\}$.
- $\mathcal{A} = \emptyset$.

(7) family ac^*(n), $n \geq 3$

- $S^p = \emptyset$.
- $\Sigma = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \cdots, \alpha_{n-1} + \alpha_n\}$.
- $\mathcal{A} = \emptyset$.

Families ax

(8) family ax(1+p+1+1+q+1) $n = p+q+3,\ p \geq 1,\ q \geq 1$.

- $S^p = \{\alpha_3, \cdots, \alpha_p, \alpha_{p+4}, \cdots, \alpha_{p+q+1}\}$.
- $\Sigma = \{\alpha_1, \alpha_2 + \cdots + \alpha_{p+1}, \alpha_{p+2}, \alpha_{p+3} + \cdots + \alpha_{p+q+2}, \alpha_n\}$.
- $\mathcal{A}$ is associated with $\Sigma \cap S$.

(9) family ax(1+p+1,1) is localization of ax(1+p+1+q+1) at $S' = \{\alpha_1, \cdots, \alpha_{p+2}, \alpha_n\}$.

(10) family ax(1+p+1) is localization of ax(1+p+1+q+1) at $S' = \{\alpha_1, \cdots, \alpha_{p+2}\}$.

(11) family ax(1,1,1) is localization of ax(1+p+1+q+1) at $S' = \{\alpha_1, \alpha_{p+2}, \alpha_n\}$.

(12) family ay(p+q+p)

- $S^p = \{\alpha_{p+2}, \cdots, \alpha_{p+q-1}\}$.
- $\Sigma = \{\alpha_1, \cdots, \alpha_p, \alpha_{p+1} + \cdots + \alpha_{p+q}, \alpha_{p+q+1}, \cdots, \alpha_n\}$.
- $\mathcal{A}$ is associated with $S \cap \Sigma$.

(13) family ay(p+q+p-1) is the localization of ay(p+q+p) at $S' = \{\alpha_1, \cdots, \alpha_{n-1}\}$.

(14) family ay(p,p) is the localization of ay(p,p-1) at $S' = \{\alpha_1, \cdots, \alpha_p, \alpha_{p+q+1}, \cdots, \alpha_n\}$.

(15) family ay(p,p-1) is the localization of ay(p+q+p) at $S' = \{\alpha_1, \cdots, \alpha_p, \alpha_{p+q+1}, \cdots, \alpha_{n-1}\}$.

(16) family $\widetilde{ay}(p+q+p),\ n = 2p+q,\ p \geq 2,\ q \geq 1$

- $S^p = \{\alpha_{p+2}, \cdots, \alpha_{p+q-1}\}$.
- $\Sigma = \{\alpha_1, \cdots, \alpha_p, \alpha_{p+1} + \cdots + \alpha_{p+q}, \alpha_{p+q+1}, \cdots, \alpha_n\}$.
- $\mathcal{A}$ is associated with $S \cap \Sigma$.

(17) family $ay^*(2+q+2),\ n = 4+q,\ q \geq 1$

- $S^p = \{\alpha_4, \cdots, \alpha_{q+1}\}$.
- $\Sigma = \{\alpha_1, \alpha_2, \alpha_3 + \cdots + \alpha_{q+2}, \alpha_{n-1}, \alpha_n\}$.
- $\mathcal{A}$ is associated with $\alpha_1, \alpha_2, \alpha_{n-1}, \alpha_n$.

(18) family $\widetilde{az}(3+q+3),\ n = 6+q,\ q \geq 1$

- $S^p = \{\alpha_5, \cdots, \alpha_{2+q}\}$.
- $\Sigma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 + \cdots + \alpha_{q+3}, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$.
- $\mathcal{A}$ is associated with $\alpha_1, \alpha_2, \alpha_3, \alpha_{n-2}, \alpha_{n-1}, \alpha_n$.

(19) family $\widetilde{az}(3+q+2)$ is the localization of $\widetilde{az}(3+q+3)$ at $S' = \{\alpha_1, \cdots, \alpha_{n-1}\}$.

(20) az(3,3) is the localization of $\widetilde{az}(3+q+3)$ at $S' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$.

(21) az(3,2) is the localization of $\widetilde{az}(3+q+3)$ at $S' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_{n-2}, \alpha_{n-1}\}$.
(22) az(3,1) is the localization of $\widetilde{az}(3+q+3)$ at $S' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_{n-1}\}$.
(23) $ae_6(6)$.

- $S^p = \emptyset$.
- $\Sigma = \{\alpha_1, \cdots, \alpha_6\}$.
- $\mathcal{A}$ is the set of colors associated with Σ.

(24) $ae_6(5)$ is the localization of $ae_6(6)$ at $S' = \{\alpha_1, \cdots, \alpha_5\}$.
(25) $ae_6(4)$ is the localization of $ae_6(6)$ at $S' = \{\alpha_2, \cdots, \alpha_5\}$.
(26) $ae_7(7)$

- $S^p = \emptyset$.
- $\Sigma = \{\alpha_1, \cdots, \alpha_7\}$.
- $\mathcal{A}$ is the set of colors associated with Σ.

(27) $ae_7(6)$ is the localization of $ae_7(7)$ at $S' = \{\alpha_1, \cdots, \alpha_6\}$.
(28) $ae_7(5)$ is the localization of $ae_7(7)$ at $S' = \{\alpha_2, \cdots, \alpha_6\}$.
(29) af(4)

- $S^p = \emptyset$.
- $\Sigma = \{\alpha_1, \cdots, \alpha_4\}$.
- $\mathcal{A}$ is the set of colors associated with Σ.

In conclusion there are 29 primitive spherical systems, there are 6 classical ones

- ao(n), $n \geq 1$.
- ac(n), n odd ≥ 3.
- aa(p+q+p), aa(p,p), aa(q), $aa^*(p+1+p)$ $p \geq 1,\ q \geq 1$.

and 16 spherical systems obtained as localizations from the previous ones

- $ac^*(n)$ $(n \geq 3)$.
- ax(1+p+1+q+1), ax(1+p+1,1), ax(1+p+1), ax(1,1,1) $(p \geq 1, q \geq 1)$.
- ay(p+q+p), ay(p+q+p-1), ay(p,p), ay(p,p-1).
- $\widetilde{ay}(p+q+p)$, $ay^*(2+q+2)$, $(p \geq 2,\ q \geq 1)$.
- $\widetilde{az}(3+q+3)$, $\widetilde{az}(3+q+2)$, az(3,3), az(3,2), az(3,1) $(q \geq 1)$.

And seven exceptional cases

- $ae_6(6), ae_6(5), ae_6(4)$.
- $ae_7(7), ae_7(6), ae_7(5)$.
- $af(4)$.

We have the following geometric realizations of the classical primitive spherical systems
(1) ao(n), $n \geq 1$

$$H = N_G(SO_{n+1}),\ G = SL_{n+1}$$

(2) ac(n) , n odd ≥ 3

$$H = N_G(Sp_{n+1}),\ G = SL_{n+1}$$

(3) aa(p+q+p) $n = 2p+q$, $p \geq 1,\ q \geq 1$

$$H = N_G(SL_{p+q} \times SL_{p+1})^0 \cdot C(G),\ G = SL_{n+1}$$

(4) aa(p,p), $p \geq 1$, $H = SL_{p+1}^{\text{diag}} C(G)$ inside $SL_{p+1} \times SL_{p+1}$.
(5) aa(q), $q \geq 1$,

$$H = GL_q,\ G = SL_{q+1}$$

(6) $aa^*(p+1+p),\ n = 2p+1,\ p \geq 1$

$$H = N_G(SL_{p+1} \times SL_{p+1}),\ G = SL_{n+1}$$

For the geometric realizations of other spherical systems see the table in [BP05].

6.3. **Properties of spherical systems.** Although we have seen many properties of the spherical systems that are directly related to geometry, we have the following two properties Δ-connected and erasable which are not directly related geometry but are very helpful for reduction type argument for spherical systems.

Definition 6.20. Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system and Δ denote its colors, for $\gamma \in \Sigma$, denote $\Delta(\gamma)$ the union of $\Delta(\alpha),\ \alpha \in \mathrm{supp}(\gamma)$. We will say that two elements $\gamma_1,\ \gamma_2 \in \Sigma$ are strongly Δ-connected if for all $D \in \Delta(\gamma_1)$, we have $\langle \rho(D), \gamma_2 \rangle \neq 0$ and for all $D \in \Delta(\gamma_2)$ we have $\langle \rho(D),\ \gamma_1 \rangle \neq 0$.

We will say that $\gamma_1, \gamma_2 \in \Sigma$ are Δ-neighbors if

- Either they are strongly connected.
- Or there exists $\gamma_3 \in \Sigma$ such that the system obtained by localization in $\mathrm{supp}(\{\gamma_1,\ \gamma_2,\ \gamma_3\})$ is isomorphic to $ax(1+q+1)$ for $q \geq 1$.

A subset $\Sigma' \subset \Sigma$ is Δ-connected (resp. strongly Δ-connected) if the two of any elements in Σ' can be joined by a sequence of elements of Σ, and any two successive elements in the sequence are Δ-neighbors (resp. strongly Δ-neighbors). A Δ-connected component of Σ is a maximal Δ-connected subset.

Proposition 6.21. *Let $\mathcal{S}$ be a spherical system, suppose $\mathcal{S}$ is Δ-connected, cuspidal, and $\mathcal{S}$ doesn't have projective colors, then $\mathcal{S}$ is primitive.*

Proof. Let $\mathcal{S}$ be a Δ-connected, cuspidal, spherical systems without projective colors. Looking at the list of spherical systems of rank ≤ 2, we see that if Σ contains a spherical root of type d_3, then $(S^p, \Sigma, \mathcal{A})$ is isomorphic to ac(n) for $n \geq 3$, similarly, we can see that if Σ contains a spherical root of type a', then $\mathcal{S}$ is isomorphic to ao(n), $n \geq 1$, or to aa^*(p+1+p), $p \geq 1$. If Σ contains a spherical root of type $a_1 \times a_1$, then $\mathcal{S}$ is isomorphic to aa(p+q+p), $p \geq 1$, $q \geq 1$ or to aa(p,p), $p \geq 1$, aa^*(p+1+p), $p \geq 1$.

It remains to examine when all the spherical roots are of type a_n, $n \geq 2$, looking at the spherical systems of rank ≤ 2, we see that $\mathcal{S}$ is isomorphic to aa(q), $q \geq 2$, or ac^*(n), $n \geq 3$. ☐

Lemma 6.22. *Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system which is strongly Δ-connected, cuspidal, without projective color. Suppose that all spherical roots are of type a_n, $n \geq 1$, and there exists a spherical root of type a_1 and that Σ contains at least two elements. Then all the spherical roots are of type a_1, and the system is isomorphic to one of the following 12 cases*

- $ax(1,1,1)$*;*
- $ae_6(6), ae_6(5), ae_6(4)$*;*
- $ae_7(7), ae_7(6),\ ae_7(5)$*;*
- $ay(p,p),\ ay(p,p-1)\ (p \geq 2)$*;*
- $az(3,3),\ az(3,2),\ az(3,1)$.

Definition 6.23. Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system, and Σ' a Δ-connected component of Σ, let's denote $\Delta(\Sigma')$ the set of $D \in \Delta(\mathrm{supp}(\Sigma'))$ such that $\rho(D)$ is zero on $\Sigma \backslash \Sigma'$. We say that the component Σ' is erasable if:

- $\Delta(\Sigma')$ is a distinguished smooth subset of Δ.
- $\Sigma(\Delta(\Sigma')) = \Sigma'$.

We say a Δ-connected component Σ' of Σ is isolated if $\mathrm{supp}(\Sigma) = \mathrm{supp}(\Sigma') \cup \mathrm{supp}(\Sigma \backslash \Sigma')$ is a factorization of spherical system that obtained as a localization of $\mathcal{S}$ at $\mathrm{supp}(\Sigma)$.

Proposition 6.24. *Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical system adjoint of type A without projective color, let Σ' be a Δ-connected component of Σ:*

(1) If the localization of $\mathcal{S}$ at suppΣ' is isomorphic to ao(n), ac(n), aa(p+q+p), aa(p,p), aa^*(p+1+p), ax(1+p+1+q+1), ax(1+p+1,1), ax(1,1,1), $\widetilde{ay}$(p+q+p), ae_6(4), and ae_7(5), then Σ' is isolated.
(2) If the localization of $\mathcal{S}$ at suppΣ' is not isomorphic to aa(p) or ac^*(n) (n even), then Σ' is erasable.

Proof. The verification of (1) can be done case by case using the table of rank two spherical systems.

For (2), according to proposition 6.21, it remains to consider the cases where the localization in suppΣ is isomorphic to ac^*(n) (n odd), ax(1+p+1), ay(p+q+p), ay(p+q+p-1), ay(p,p), ay(p,p-1), ay^*(2+q+2), az(3,3), az(3,2), az(3,1), ae_6(5), ae_7(7), ae_7(6) and af(4), this can be done case by case. ☐

6.4. **Reduction to the primitive spherical systems.** In this section, we prove the theorem 6.1.

Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical system adjoint of type A, we will show the existence (up to isomorphism) of a wonderful variety X with spherical system $\mathcal{S}$.

We will use induction on the rank of $\mathcal{S}$, using proposition 6.11, we may assume that $\mathcal{S}$ is cuspidal. Using proposition 6.19, we may assume that $\mathcal{S}$ doesn't have projective colors.

From the argument in the proof of proposition 6.21 , we know that any Δ-connected component which contain a spherical root of type a', $a_1 \times a_1$, d_3 is isolated and generates a classical spherical system.

So we are reduced to the case when $\mathcal{S}$ is a cuspidal spherical system, without projective colors and all spherical roots are of type a_n, $n \geq 1$, which we will assume now on.

First assume that $\mathcal{S}$ contains several Δ-connected components erasable, let's denote Σ_1, Σ_2 two of these components, and $\Delta_1 = \Delta(\Sigma_1)$, $\Delta_2 = \Delta(\Sigma_2)$. Let's show that Δ_1, Δ_2 decompose the spherical system. By definition 6.16, Δ_1, Δ_2 are distinguished and the pair has property (1), (2), (3), (5). Let $\alpha_i \in \operatorname{supp}(\Sigma_i)$, $i = 1, 2$, if α_1 is not orthogonal to α_2 which is the case for a_n spherical root, then $\Delta(\alpha_1)$ is not a subset of Δ_1 which implies (4).

Proposition 6.17 and the reduction hypothesis reduce to the case when $\mathcal{S}$ contains only one erasable Δ-connected component. The following lemma brings us back to the case in proposition 6.17, or $\mathcal{S}$ is a primitive spherical system.

Lemma 6.25. *Let $(S^p, \Sigma, \mathcal{A})$ be a cuspidal system, without projective colors, all the spherical roots are of type $a(n)$ with $n \geq 1$, we assume that a single Δ-connected component Σ_1 of Σ is erasable. Put $\Sigma_2 = \Sigma \backslash \Sigma_1$, $\Delta_i = \Delta(\Sigma_i)$, $i = 1, 2$, then*

- *Either $(S^p, \Sigma, \mathcal{A})$ is primitive.*
- *Or Δ_1, Δ_2 decomposes $(S^p, \Sigma, \mathcal{A})$.*

Proof. If $\Delta_2 \neq \emptyset$, then Σ_1 generate a subsystem isomorphic to $ay(p, p)$ or $az(3, 3)$, $az(3, 2)$, the system generated by Σ_2 is quite simple, as it contains no projective element, its Δ-connected components generate subsystems isomorphic to $aa(q)$, $ac^*(n)$ (n even), and the Dynkin diagram of $\operatorname{supp}(\Sigma_2)$ is connected. It is easy to see that Δ_2 is distinguished and $(S^p, \Sigma, \mathcal{A})/\Delta_2$ is isomorphic to $\widetilde{ay}(p+q+p)$, $\widetilde{az}(3+q+3)$, $\widetilde{az}(3+q+2)$. Finally, we can check that Δ_1, Δ_2 decomposes $\mathcal{S}$. $\square$

6.5. **Some corollaries.**

Corollary 6.26. *Let G be a group adjoint of type A, and H a wonderful subgroup of G, we assume $\mathcal{S}_{G/H} = (S^p_{G/H}, \Sigma_{G/H}, \mathcal{A}_{G/H})$ is cuspidal and irreducible. Then H is connected, unless $\mathcal{S}_{G/H}$ is isomorphic to*

- *$ao(n)$, for n odd ≥ 3, in these cases H° is not wonderful.*
- *$aa^*(p+1+p)$, $p \geq 1$ or to $ao(1)$, in these cases H° is wonderful.*

Corollary 6.27. *Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system adjoint of type A, and let Δ be its set of colors, then for any distinguished subset Δ' of Δ, it has property $(*)$.*

Given $\phi : X \to X'$ a dominant G-morphism between wonderful G-varieties, set $\Delta(\phi) = \{D \in \Delta_X,\ \phi(D) = X'\}$ and $S(\phi) = S^a_X \cap S^{a'}_{X'}$, the following is a corollary of proposition 6.9

Corollary 6.28. *Let G be a group adjoint of type A, and X a wonderful G-variety, the association ϕ to the couple $(\Delta(\phi), S(\phi))$ is a bijection between dominant G-morphisms ϕ of X to another wonderful G-variety and the distinguished couple (Δ', S') of $(S^p_X, \Sigma_X, \mathcal{A}_X)$.*

We have the following characterization of reductive wonderful subgroups of type A groups

Corollary 6.29. *Let G be an adjoint group of type A and let H be a wonderful subgroup of G. For H to be reductive, it is necessary and sufficient that the spherical system of G/H is a product of classicial systems and systems $ac^*(n)$ (n even), $ax(1{+}p{+}1{,}1)$ ($p \geq 1$), $ax(1{,}1{,}1)$ and $ay(p{,}p{-}1)$ ($p \geq 2$).*

6.6. **Classification of general homogeneous spherical varieties.** The group of automorphisms of G/H can be identified with $N_G(H)/H$, and it acts naturally on $\Delta_{G/H}$, $\overline{H}$ are the elements of G/H which acts trivially on $\Delta_{G/H}$. $\overline{H}$ is a spherical subgroup of G and we have natural morphism $G/H \to G/\overline{H}$ induces a bijection $\Delta_{G/H} = \Delta_{G/\overline{H}}$.

Definition 6.30. We say the group $\overline{H}$ is the spherical closure of H, and we say H is *spherically closed* if $\overline{H} = H$.

The interests of these groups comes from the fact that any spherically closed subgroup is wonderful from a result of Knop [Kno96], We have a combinatorial characterization of spherical subgroups with a given spherical closure $\overline{H}$.

Let H be a spherically closed subgroup of G, we will denote $\mathcal{S} = (\chi, \Sigma_{G/H}, \mathcal{A}_{G/H})$ the spherical system of G/H, $\Delta_{G/H}$ the set of colors, $\rho_{G/H}$ the Cartan pairing, and H° the connected component of H, $\overline{H^\circ}$ its spherical closure. We put $(\chi^\circ, \Sigma^\circ, \mathcal{A}^\circ)$ the spherical system of $G/\overline{H^\circ}$, Δ° its set of colors and $\Sigma_G^\circ = \Sigma_G \cap \Sigma^\circ \cap \frac{1}{2}\Sigma$, $m = \mathrm{card}\Sigma_G^\circ$.

Lemma 6.31. *The group $H/\overline{H^\circ}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$, and χ° is the subgroup of $\chi(T)$ generated by χ and Σ°.*

The natural morphism G/H° sends colors of $\Delta^\circ(\alpha)$ to $\Delta(\alpha)$ for $\alpha \in \Sigma_G$. The natural action of H on Δ° exchanges the two elements of $\Delta^\circ(\alpha)$ and fixes other colors, hence we get an injection $H/\overline{H^\circ} \to (\mathbb{Z}/2\mathbb{Z})^m$.

After replacing G be a finite covering, we may assume that $G = C \times G_1$ for C a torus and G_1 a semisimple simply-connected group, $B = C \times B_1$ for B_1 a Borel subgroup of G_1, and $H = C \times H_1$, H_1 a spherically closed subgroup of G_1.

We denote the functions $f \in k(G)$, proper under the left action of B and the right action of H by ${}^{(B)}k(G)^{(H)}$. Let $\pi : G \to G/H$ be the natural projection, for all $D \in \Delta$, let f_D be the equation of $\pi^{-1}(D)$ in $k[G]$ which is constant equal to 1 on C. For all $\chi \in \chi(C)$, let f_χ be the character of G corresponding to χ under the identification $\chi(G) = \chi(C)$.

Lemma 6.32. *We have*

- *Any $f \in {}^{(B)}k(G)^{(H)}$ is of the form*
$$f = cf_\chi \prod_{D\in\Delta} (f_D)^{n(D)}$$
with $c \in k^*$, $\chi \in \chi(C)$ and $n(D)_{D\in\Delta} \in \mathbb{Z}^\Delta$, and this factorization is unique.
- *The sets $\pi^{-1}(D)$ are irreducible, unless $D = D'_\alpha$ for $\alpha \in \Sigma_G^\circ$, in which case $\pi^{-1}(D)$ has two connected components which are exchanged by H.*

We have a map $\tau :\ \chi(C) \times \mathbb{Z}^\Delta \to \chi(H)$

$$\tau((\chi, n(D)_{D\in\Delta})) \in \chi(H)$$

We denote H^* the subgroup of H such that H/H^* is the largest quotient group of H which is multiplicative. For H' a subgroup of H contains H^*, we put $\Phi' = \tau^{-1}(\chi(H/H'))$ and $\Phi = \tau^{-1}(\{e\}) = \overline{\rho}$, for $\overline{\rho}$

$$\overline{\rho} :\ \chi \longrightarrow \chi(C) \times \mathbb{Z}^\Delta,\ \ \overline{\rho}(\gamma) = (\gamma|_{\chi(C)}, \langle \rho(D), \gamma\rangle_{D\in\Delta}),\ \gamma \in \chi$$

Lemma 6.33. *The assignment from H' to Φ' is a bijection between subgroup H' of H contains H^* and subgroups Φ' of $\chi(C) \times \mathbb{Z}^\Delta$ contains Φ.*

Definition 6.34. We say that a couple (χ', ρ') is adapted to ρ if the following conditions are satisfied

- For all $\alpha \in \Sigma^\circ$, $\alpha \notin \chi'$.
- The restriction of $\overline{\rho'}$ to χ is equal to $\overline{\rho}$.
- For $\sigma : \chi(C) \times \mathbb{Z}^\Delta \to \chi(B)$, $\sigma((\chi, n(D)_{D\in\Delta})) = f_\chi \prod_{D\in\Delta}(f_D)^{n(D)}$, we have $\sigma \circ \overline{\rho'}$ is identity on χ'.

Proposition 6.35. *Let H be a spherically closed subgroup of G, the map which associates a spherical subgroup H' to a couple $(\chi', \rho') = (\chi_{G/H'}, \rho_{G/H'})$ is a bijection between the spherical subgroups H' of G having H as spherical closure and the set of couples (χ', ρ') apapted to H.*

Recall that we have defined the notion of augmentation of (χ', ρ') to the spherical system $(S^p, \Sigma, \mathcal{A})$.

The notion of augmentation and adeptness are closely related

Lemma 6.36. *We have*

- *If (χ', ρ') is a couple adapt to H, then $(\chi', \rho'|_{\mathcal{A}})$ is an augmentation of $(S^p, \Sigma, \mathcal{A})$.*

- *If (χ', ρ') is an augmentation of $(S^p, \Sigma, \mathcal{A})$, and Δ set of colors, $\rho' : \Delta \to (\chi')^*$, then the couple (χ', ρ') is adapted to H.*

Proposition 6.37. *The map which associates a spherical subgroup H' to a couple (χ', ρ') is a bijection between spherical subgroups of G with spherical closure H and augmentation of spherical system $(S^p, \Sigma, \mathcal{A})$.*

Proof. This follows from lemma 6.36 and the proposition 6.35. □

We can define the spherical closure in terms of spherical systems

Definition 6.38. For $(S^p, \Sigma, \mathcal{A})$ a spherical system, we denote $2\Sigma(S^p)$ to be those $\gamma \notin \Sigma \backslash \Sigma_G$ with $2\gamma \in \Sigma_G$ and $(S^p, 2\gamma)$ is a couple for a wonderful variety of rank one. For $\gamma \in \Sigma$, put

$$\overline{\gamma} = 2\gamma \text{ for } \gamma \in 2\Sigma(S^p), \ \overline{\gamma} = \gamma \text{ otherwise}$$

we let $\overline{\Sigma} = \{\overline{\gamma}\}$ and $\overline{\chi} = \langle \overline{\Sigma} \rangle$, we say $(S^p, \overline{\Sigma}, \mathcal{A})$ is the spherical closure of the spherical system $(S^p, \Sigma, \mathcal{A})$.

Let H' be a spherical subgroup of G and H its spherical closure

Lemma 6.39. *Assume theorem 6.1 for adjoint groups G, then the spherical system of G/H is the spherical closure of the spherical system of G/H'.*

Remark 6.40. This lemma can also be seen from Losev's study of automorphism groups.

Well-definedness: Let's show that the map in the theorem is well-defined. We need to show that $\mathscr{L}' = (S^p_{G/H'}, \Sigma_{G/H'}, \mathcal{A}_{G/H'}, \chi_{G/H'}, \rho_{G/H'})$ is a homogeneous spherical data. We can first show that $\mathcal{S}' = (S^p_{G/H'}, \Sigma_{G/H'}, \mathcal{A}_{G/H'})$ is a spherical system. Let H be the spherical closure of H' and $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ its spherical system. Note $\Delta_{G/H'}(\alpha) \to \Delta(\alpha)$ is bijective for $\alpha \in \Sigma_G$, hence we deduce $S^p_{G/H'} = S^p$ and $\Sigma_G \cap \Sigma_{G/H'} = \Sigma_G \cap \Sigma$, and $\mathcal{A}_{G/H'}$ can be identified with $\mathcal{A}$. We can deduce that $(S^p, \Sigma, \mathcal{A}, \chi(G/H'), \rho_{G/H'})$ is an augmented spherical system, we get $\mathcal{A}_{G/H'}$ is adapted to $\Sigma_{G/H'}$ and $\mathcal{S}'$ satisfies (Σ_1) and (Σ_2). By definition, every $\gamma \in \Sigma_{G/H'}$ can be realized as a spherical root of a wonderful subvariety X of G/H', we can choose X so that $S^p_X = S^p_{G/H'}$, and we can use this to show that $\mathcal{S}'$ satisfies condition (S).

Since $(\chi_{G/H'}, \rho_{G/H'})$ is an augmentation of $(S^p, \Sigma, \mathcal{A})$, we see that $(\chi_{G/H'}, \rho_{G/H'})$ is an augmentation of $\mathcal{S}'$. By definition, elements of $\Sigma_{G/H'}$ are primitive in $\chi(G/H')$.

Injectivity: Let's show that the map in the theorem 6.2 is injective, let's denote H the spherical closure of H', the spherical system of G/H' determines that of G/H by lemma 6.39, and from theorem 6.1, since H is a wonderful subgroup, this determines H up to conjugation. According to proposition 6.4, $(\chi(G/H'), \rho_{G/H'})$ is an augmentation of $(S^p, \Sigma, \mathcal{A})$ and this determines H'. In summary, H' is determined up to conjugation by the homogeneous spherical data of G/H.

Surjectivity: Let's show that the map in theorem 6.2 is surjective. Let $(S^p, \Sigma, \mathcal{A}, \chi', \rho')$ be a homogeneous spherical data and let $(S^p, \overline{\Sigma}, \mathcal{A})$ be the spherical system of spherical closure of $(S^p, \Sigma, \mathcal{A})$, we can assume that G/H is a spherically closed variety with spherical system $(S^p, \overline{\Sigma}, \mathcal{A})$. The couple (χ', ρ') is an augmentation of $(S^p, \overline{\Sigma}, \mathcal{A})$, and H' is the subgroup of H corresponds to it according to proposition 6.37. Then by construction G/H' has homogeneous spherical data $(S^p, \Sigma, \mathcal{A}, \chi', \rho')$, and the proof of theorem 6.2 is finished.

7. Some questions

Let $\mathscr{G}$ be a connected reductive group over $\overline{k}$, and G a k-form of $\mathscr{G}$. Let $\mathscr{X}$ be a spherical variety for $\mathscr{G}$. The existence criterion of spherical variety X over k, a G-equivariant model of $\mathscr{X}$ has been proven by [BG22], so given Luna's classification, we only need to understand $H^1(k, \mathrm{Aut}^G(X))$, $\mathrm{Aut}^G(X)$ can be calculated by result of Losev [Los09]. One can think $\mathrm{Aut}^G(X)$ as the "center" of X, and the structure of $\mathrm{Aut}^G(X)$ over k needed to be studied further.

The first thing we need is an analog of Losev's uniqueness result over k. It will also be interesting to have a classification of spherical varieties over k as parallel to Luna's classification over $\overline{k}$ instead of just applying the existence criterion to the known classification over $\overline{k}$, this approach will be more useful to harmonic analysis for spherical varieties over k.

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