SPHERICAL ROOT SYSTEMS

RUI CHEN

1. Introduction

This is a note for the paper [KK16], which is a unification of the Borel-Tits theory for reductive group over general fields and Luna's theory for spherical varieties over algebraically closed field of characteristic zero.

The main application in the [KK16] is to study the embedding theory of spherical varieties, while my main motivation is to study the spectrum of spherical varieties.

2. Notation

We will fix k a characteristic 0 field, and $\bar{k}$ its algebraic closure, $\Gamma = \mathrm{Gal}(\bar{k}/k)$ its Galois grouup, G a connected reductive group over k, X a G-spherical variety over k.

3. Root systems for reductive groups

In this section, I will introduce the absolute and restricted root systems for a reductive group G and Galois action on the absolute root system.

Let k be a characteristic 0 field, and let G be a connected quasisplit reductive group over k. We fix $B = MAU$ a Borel subgroup of G over k where $U = R_uB$ is the unipotent radical of B and A is a maximal split subtorus and M is an anisotropic torus, $T = MA$ is a maximal torus of G, the restriction of characters from T to A defines a surjective map

$$\mathrm{res}_A : X^*(T) \longrightarrow X^*(A)$$

for any $\Omega \subset X^*(T)$, we will use the notation $\mathrm{res}'_A\Omega := (\mathrm{res}_A\Omega)\backslash\{0\}$. Let $\Phi_{\bar{k}} = \Phi(G,T)$ be the associated root system and we have $\Phi_k := \mathrm{res}'_A\Phi$ the restricted root system with respect to A, we will denote the Weyl group of $\Phi_{\bar{k}}$ by $W_{\bar{k}}(G)$, and the Weyl group of Φ_k by $W_k(G)$.

The choice of the Borel subgroup B gives us a set $\Sigma_{\bar{k}} \subseteq \Phi$ of simple roots, and a set of simple restricted roots $\Sigma_k := \mathrm{res}'_A\Sigma_{\bar{k}}$.

Remark 3.1. For general reductive group G, we have the set of k-compact simple roots $\Sigma^0 := \{\alpha \in \Sigma_{\bar{k}} |\ \mathrm{res}_A\alpha = 0\}$, which is the set of simple roots for M, since our group G is quasisplit, we have $\Sigma^0 = \emptyset$.

The Galois group Γ acts on $X^*(T)$ and leaves the root system $\Phi_{\bar{k}}$ invariant, we have the following property on the restriction map

Proposition 3.2. *For α_1, α_2 two simple roots in $\Sigma_{\bar{k}}$, we have $res_A\alpha_1 = res_A\alpha_2$ if and only α_1 and α_2 belong to the same Γ-orbit.*

Remark 3.3. For general connected reductive group, there is a Γ^*-action, which is defined as: for any $\gamma \in \Gamma$, there is a unique element $\omega_\gamma \in W$ such that $\omega_\gamma\gamma\Sigma_k = \Sigma_k$, and $\gamma * \chi = \omega_\gamma\gamma(\chi)$. Since our group G is quasisplit, the Borel subgroup is defined over k hence the set of simple roots is Galois stable, we can choose $\omega_\gamma = 1$, the Γ^*-action and Γ are the same.

Example 3.4. (Root system and restricted root system for SU_n)

For the special unitary group $G = SU_3$ over k defined via $\{g \in SL_{3,\ell} |\ {}^t\overline{g}Jg = J\ \}$, with $J = \mathrm{adiag}(1, -1, 1)$. Then SU_3 splits over ℓ.

The maximal torus T is isomorphic to $\mathrm{Res}_{\ell/k}\mathbb{G}_m \times U(1)$, it contains a maximal split torus $A = \mathrm{diag}(a, 1, a^{-1})$. Over $\bar{k}$, $G_{\bar{k}} \cong SL_3$, we will choose the simple roots α_1, α_2 for $G_{\bar{k}}$, we have the root system $\Phi(G,T) \subset X^*(T)$,

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we also have the restricted root system $\Phi(G,A) \subset X^*(A)$, where $\Phi(G,A)$ is the weight space of the torus S action on the Lie algebra of A. The set of simple restricted roots are $\{a, 2a\} \subset X^*(A)$.

We fix the isomorphisms $X^*(T) \cong \mathbb{Z}^3/\mathbb{Z}$, $X^*(A) \cong \mathbb{Z}$, and we have the restriction map

$$\begin{aligned} \mathrm{res}_A : X^*(T) &\longrightarrow X^*(A) \\ (n_1, n_2, n_3) &\longmapsto n_1 - n_3 \end{aligned}$$

In particular, we see that $\alpha_1, \alpha_2 \mapsto 1$, this corresponds to the fact that the simple roots on the same Galois orbits are mapped to the same image in $X^*(A)$. $\alpha_1 + \alpha_2 \mapsto 2$. The restricted Weyl group W_k is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and it is the Weyl group for the root system $\Phi(G,A)$.

Now we consider the special unitary group SU_{2n} defined via $\{g \in SL_{2n,\ell} |\ {}^t\bar{g}Jg = J\}$, where $J = \mathrm{adiag}(1,-1,1,-1,\cdots)$. Then SU_{2n} splits over ℓ.

The maximal torus T is isomorphic to $(\mathrm{Res}_{\ell/k}\mathbb{G}_m)^n$, and the maximal split torus $A = \mathrm{diag}(a_1, a_2, \cdots a_n, a_n^{-1}, a_{n-1}^{-1}, \cdots a_1^{-1})$. We have the root system $\Phi(G,T) \subset X^*(T)$ and the restricted root system $\Phi(G,A) \subset X^*(A)$.

We fix the isomorphisms $X^*(T) \cong \mathbb{Z}^{2n}/\mathbb{Z}$, and $X^*(A) \cong \mathbb{Z}^n$, then we have the restriction map

$$\begin{aligned} \mathrm{res}_A :\ X^*(T) &\longrightarrow X^*(A) \\ (x_1, x_2, \cdots x_{2n}) &\longmapsto (x_1 - x_{2n}, x_2 - x_{2n-1}, \cdots, x_n - x_{n+1}) \end{aligned}$$

In particular, we note that α_i and α_{2n-i} have the same image. The restricted Weyl group is isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$.

4. k-local structure theorem

In this section, we will assume X is a k-dense G-spherical variety. We will define the maximal k-split torus $A_k(X)$ and the k-valuation cone $\mathscr{V}_k(X)$ in this section. As we know, these objects play important roles in the study of root system over algebraic closure, and the study of the properties of these objects are based on the parallel approach to algebraic closure, and sometimes reduced to the algebraic closure.

I will state a simplified version of the local structure theorem for spherical varieties for quasisplit groups, from now on, I will assume G is a quasisplit group over k and B a Borel subgroup over k, T a maximal torus of B, A a maximal k-split torus of T.

Let's denote $\mathring{X}$ the unique open Borel orbit of X. As a consequence of the generic local structure theorem, from [KK] corollary 4.6, we get an isomorphism

$$\mathring{X} \cong T_X \times U_{P(X)}$$

here T_X is the T-orbit of a point $x_0 \in \mathring{X}(k)$.

Definition 4.1. We define the maximal split torus of X to be $A_k(X) := A\ x_0$, and we define the k-character lattice of X to be

$$\chi_k(X) = X^*(A_k(X))$$

the k-rank of X is defined to be

$$\mathrm{rk}_k X := \mathrm{rk}\chi_k(X) = \dim\ A_k(X)$$

The anisotropic kernel X_{an} of X can be defined to be $X_{an} := Mx_0$.

It is immediate to see that these defnitions generalize the notion of maximal split torus and anisotropic kernel in the group case if we view the group as a symmetric spherical variety.

We have the folowing connection between the $\bar{k}$-character lattice and k-character lattice

Lemma 4.2. *We have*

$$res_A \chi_{\bar{k}}(X) = \chi_k(X)$$

here $\chi_{\bar{k}}(X) = X^*((T_X)_{\bar{k}})$.

as a corollary of this lemma, we see that the torus $A_k(X)$ is the image of $A \subset T$ in T_X under the quotient map

$$T \longrightarrow T_X$$

We denote $\mathfrak{a}_k(X) = \chi_k(X)^* \otimes \mathbb{Q} = \mathrm{Hom}(\chi_k(X), \mathbb{Q})$.

Corollary 4.3. *The space* $\mathfrak{a}_k(X)$ *is the image of* $\mathfrak{a}_k \subseteq \mathfrak{a}_{\bar{k}}$ *in* $\mathfrak{a}_{\bar{k}}(X)$.

We will introduce the notion of k-invariant valuation.

Definition 4.4. An invariant valuation of a k-dense G-variety X is called k-central if it is trivial on the subfield $\bar{k}(X_{an}) = \bar{k}(X)^{AU}$. The set of k-central valuations is denoted by $\mathscr{V}_k(X)$.

We have the following short exact sequence

$$1 \longrightarrow \bar{k}(X_{an})^* \longrightarrow \bar{k}(X)^{(AU)} \longrightarrow \chi_k(X) \longrightarrow 1$$

The k-central valuation induces a homomorphism

$$\lambda_v : \ \chi_k(X) \to \mathbb{Q} : \ \chi_f \mapsto v(f)$$

and we get a map $\iota_k : \mathscr{V}_k(x) \to \mathfrak{a}_k(X)$.

According to corollary 4.3, we may view $\mathfrak{a}_k(X)$ as a subspace of $\mathfrak{a}_{\bar{k}}(X)$

Proposition 4.5. *Let X be a k-dense G-spherical variety, then:*

- *The map ι is injective.*
- *Considering $\mathscr{V}_k(X), \mathscr{V}_{\bar{k}}(X)$ as subsets of $\mathfrak{a}_k(X)$ and $\mathfrak{a}_{\bar{k}}(X)$, then $\mathfrak{a}_k(X) = \mathscr{V}_{\bar{k}}(X) \cap \mathfrak{a}_k(X)$.*

Note over algebraic closure $\iota_{\bar{k}}$ is injective, together with the inclusion $\mathfrak{a}_k(X) \to \mathfrak{a}_{\bar{k}}(X)$ we know that ι is injective.

From 4.5, we see that $\mathscr{V}_k(X)$ is a finitely generated convex cone in $\mathfrak{a}_k(X)$.

We denote $\mathfrak{a}_k^-$ the antidominant Weyl for G with respect to the restricted root system.

Proposition 4.6. *Let $\pi : \mathfrak{a}_k \to \mathfrak{a}_k(X)$ be the canonical projection, then $\pi(\mathfrak{a}_k^-) \subseteq \mathscr{V}_k(X)$.*

This follows from proposition 4.5 and the corresponding result over $\bar{k}$.

5. The Weyl group

In this section, we will assume X is a k-dense G-spherical variety over k and G a quasisplit connected reductive group over k, B a fixed Borel subgroup over k, T a maximal torus of B, and we choose the associated parabolic subgroup $P(X)$ over k.

There is a Borel subgroup B and a point $x \in X(k)$ such that

$$\begin{array}{ccccc} A & \subseteq & T & \subseteq & L_k \\ \downarrow & & \downarrow & & \downarrow \\ A_k(X) & \subseteq & T_X & \xrightarrow{\cong} & Tx \end{array}$$

To define the k-Weyl group for X. The strategy is to study the valuation cone $\mathscr{V}_k(X)$, which over algebraic closure we know it is the Weyl chamber for the little Weyl group W_X.

First let's note that the associated parabolic subgroup $P(X)$ can be choosen to be defined over k. We get a set of k-roots $\Sigma_k^p(X) \subseteq \Sigma_{\bar{k}}$, since $P(X)$ is defined over k, we have $\Sigma_k^p(X)$ is Γ-stable.

We make the following definition of k-little Weyl group

Definition 5.1. We define

$$W_k(X) := N_{W_{\bar{k}}(X)}(\mathfrak{a}_k(X))/C_{W_{\bar{k}}(X)}(\mathfrak{a}_k(X))$$

We have a parallel result over algebraic closure

Theorem 5.2. *Let X be a k-dense G-spherical variety, then $\mathscr{V}_k(X)$ is a fundamental domain for the action of $W_k(X)$ on $\mathfrak{a}_k(X)$.*

6. The root system over k

We will fix X an k-dense G-spherical variety, G a connected reductive group over k.

In this section, I will recall the construction of root systems in the paper [KK16], just like the group case, we want to define the restricted root system for spherical varieties and the Galois action on spherical root system over $\bar{k}$. We will construct an integral root system for $W_k(X)$ 5.1 the k-little Weyl group, as in the case over algebraic closure, we face an issue of normalization.

There is an obvious normalization

Definition 6.1. A primitive k-spherical root of X is a primitive element $\sigma \in \chi_k$ such that $\mathscr{V}_k(X) \cap \{\sigma \geq 0\}$ is a facet of $\mathscr{V}_k(X)$.

We will denote the set of primitive k-spherical roots by $\Sigma_k^{pr} := \Sigma_k^{pr}(X)$, it is in one to one correspondence with the facets of $\mathscr{V}_k(X)$.

The *primitive k-root system of* X is defined to be

$$\Phi_k^{pr} = \Phi_k^{pr}(X) := W_k \cdot \Sigma_k^{pr}$$

Since the valuation cone $\mathscr{V}_k(X)$ is defined by a set of linearly independent linear inequalities, we have the following description of $\mathscr{V}_k(X)$ based on the primitive k-spherical roots

Proposition 6.2. $\mathscr{V}_k(X) = \{a \in \mathfrak{a}_k |\ \sigma(a) \leq 0,\ \textit{for all}\ \ \sigma \in \Sigma_k^{pr}\}$.

We have the following lemma

Lemma 6.3. *Every* $\sigma \in \Sigma_k^{pr}$ *is a linear combination* $\sigma = \sum_{\alpha \in \Sigma_k} c_\alpha \alpha$ *with* $c_\alpha \in \mathbb{Q}_{\geq 0}$.

Definition 6.4. The set of $\alpha \in \Sigma_G$ such that $c_\alpha > 0$ is called the *support* of σ.

A *weight lattice* for a root system Φ with Weyl group W is a lattice Ξ containing Φ with $(1 - s_\sigma)\Xi \subset \mathbb{Z}\sigma$ for all $\sigma \in \Phi$, here s_σ denotes the reflection about σ. If Φ is a reduced root system, this is equivalent to W acts trivially on $\Xi/\langle\Phi\rangle_{\mathbb{Z}}$.

Proposition 6.5. Φ_k^{pr} *is a reduced root system with Weyl group* $W_k(X)$*, the set* Σ_k^{pr} *is a set of simple roots for* Φ_k^{pr}*, the valuation cone* $\mathscr{V}_k(X)$ *is an antidominant Weyl chamber with respect to* Σ_k^{pr}*. The lattice* χ_k *is a weight lattice for* Φ_k^{pr}*.*

Proof. Since we know $\mathscr{V}_k(X)$ is the fundamental domain for W_k. We only need to show χ_k is W_k stable, over $\bar{k}$, this is the result of [Kno94], over k, this follows from $\chi_k = \mathrm{res}'_A \chi_{\bar{k}}$. □

For any $\sigma \in \chi_{\bar{k}}$, we let $\overline{\sigma} := \mathrm{res}_A \sigma$ be the restriction of σ to A and

$$\chi_{\bar{k}}^0 = \{\sigma \in \chi_{\bar{k}}(X) |\ \overline{\sigma} = 0\}$$

elements of $\Sigma_{\bar{k}}^0 := \Sigma_{\bar{k}} \cap \chi_{\bar{k}}^0$ will be called the compact spherical roots. The compact spherical spherical roots can be recovered from the compact simple roots

$$\Sigma_{\bar{k}}^0 = \{\sigma \in \Sigma_{\bar{k}} |\ \mathrm{supp}(\sigma) \subseteq \Sigma^0\}$$

we will see later $\Sigma_{\bar{k}}^0$ is the set of $\bar{k}$-spherical roots of X_{el}.

Example 6.6. If G is a quasisplit group over k, X a G-spherical k-variety, then $\Sigma_{\bar{k}}^0 = \emptyset$ and in this case $X_{el} = T_X$.

We have the following result

Proposition 6.7. *For* X *a* k*-dense* G*-spherical variety, then*

$$\chi_{\bar{k}}(X_{an}) = \chi_{\bar{k}}^0(X),\ \Sigma_{\bar{k}}(X_{an}) = \Sigma_{\bar{k}}^0(X)$$

We define $\Sigma_k := \Sigma_k(X) := \mathrm{res}'_A \Sigma_{\bar{k}} = \{\overline{\sigma},\ \overline{\sigma} \neq 0\}$. In general Σ_k and Σ_k^{pr} are different, see for example in the group case.

Now we consider the Galois action on spherical roots, the key is the commutative diagram that we used in the construction of k-little Weyl group: There is a Borel subgroup B and a point $x \in X(k)$ such that

$$\begin{array}{ccccc} A & \subseteq & T & \subseteq & L_k \\ \downarrow & & \downarrow & & \downarrow \\ A_k(X) & \subseteq & T_X & \xrightarrow{\cong} & Tx \end{array}$$

Proposition 6.8. *The set $\Sigma_k \subseteq \chi_k$ is linearly independent, moreover, $\overline{\sigma} = \overline{\tau} \neq 0$ for $\sigma, \tau \in \Sigma_{\bar{k}}$ if and only if σ and τ are in the same Γ-orbit.*

There is a map $\Sigma_k^{pr} \to \mathbb{Z}_{>0} : \ \sigma \mapsto n_\sigma$ such that $\Sigma_k = \{n_\sigma \sigma | \ \sigma \in \Sigma_k^{pr}\}$

This generalize the proposition 3.2 from the group case to general spherical varieties.

Proposition 6.9. *There is a map $\Sigma_k^{pr} \to \mathbb{Z}_{>0} : \ \sigma \mapsto n_\sigma$ such that $\Sigma_k = \{n_\sigma \sigma \mid \sigma \in \Sigma_k^{pr}\}$.*

Proof. $\mathscr{V}_k(X)$ is also defined by the inequalities $\overline{\sigma} \leq 0$ with $\sigma \in \Sigma_{\bar{k}}$, since they are linearly independent, they form a minimal set of inequalities, so we see that every $\overline{\sigma}$ is an integral multiple of an element of Σ_k^{pr}. ☐

Now we proceed to the construction of a root system for $W_k(X)$, we define $\Phi_k := W_k(X)\Sigma_k = W_k(X)\mathrm{res}'_A\Sigma_k$, $\Phi_k^{res} := \mathrm{res}'_A \Phi_{\bar{k}} = \mathrm{res}'_A W_{\bar{k}}(X)\Sigma_{\bar{k}}$. In general, $\Phi_{\bar{k}}^{res}$ is not a root system.

However, we have the following result

Theorem 6.10. *Let X be a k-dense G-variety, then*

(1) (Φ_k, χ_k) is an integral root system, its Weyl group is $W_k(X)$ and Σ_k is a system of simple roots.
(2) $n_\sigma \in \{1, 2\}$ for all $\sigma \in \Sigma_k^{pr}$.

Proof. For (1): Let $R_{\bar{k}} = \langle \Sigma_{\bar{k}} \rangle$ and $R_k := \mathrm{Res}_A R_{\bar{k}} = \langle \Sigma_k \rangle$ be the root lattices, since $R_{\bar{k}}$ is $W_{\bar{k}}$-stable and hence $N(\mathfrak{a}_k)$-stable, we see R_k is W_k-stable. Since the elements of Σ_k are primitive in R_k, we conclude Φ_k is a root system for W_k and Σ_k is a set of simple roots.

It remains to show χ_k is a set of weights for Φ_k, and since χ_k is W_k stable this means to show W_k acts trivially on χ_k/R_k. This clearly holds over $\bar{k}$, the assertion now follows from the fact that χ_k/R_k is a quotient of $\chi_{\bar{k}}/R_{\bar{k}}$.

For (2): Let $\sigma \in \Sigma_k^{pr}$ and $\tilde{\sigma} = n_\sigma \sigma \in \Sigma_k$ as $\sigma \in \chi_k$, it follows from (1) that

$$\frac{2}{n_\sigma} = \langle \frac{1}{n_\sigma}\tilde{\sigma}, \tilde{\sigma}^\vee \rangle = \langle \sigma, \tilde{\sigma}^\vee \rangle \in \mathbb{Z}$$

☐

Moreover, one can show that Φ_k consists precisely of the indivisible elements of Φ_k^{res}.

6.1. **Remaining question.**

Definition 6.11. We will call $(\chi_{\bar{k}}, \Sigma_{\bar{k}}, \Sigma_{\bar{k}}^0)$ together with the Γ-action the spherical index of X.

It behaves very much like the classical Borel-Tits index but there are some counterexamples. It will be interesting to investigate this phenomenon further, say from the dual side.

7. Examples of spherical root systems

7.1. **Spherical root systems for some spherical varieties of type A.** In this section, we examine some examples of quasisplit forms for symmetric varieties of type A. For the notations related to the special unitary groups, we keep the same as the previous example 3.4.

In all the examples, G will be a quasisplit connected group over k, B a Borel subgroup of G defined over k, T a maximal k-torus of G, A a maximal k-split torus of T.

Example 7.1. Let ℓ/k be a quadratic extension of characteristic zero fields, $G = \mathrm{Res}_{\ell/k}GL_2$, $X = \mathrm{Res}_{\ell/k}GL_2/GL_2$, we have $X_{\bar{k}} \cong GL_2 \times GL_2/GL_2$, we have $\Sigma_{\bar{k}} = \{\alpha_1 + \alpha_2\}$, here α_1, α_2 are the two simple roots of the first and second copy of GL_2 with B the product of upper triangular matrices. We have $P(X)_{\bar{k}} = B_{\bar{k}}$.

X is a Galois symmetric variety with $\theta(g) = \overline{g}$, for the conjugate θ' of θ such that $\theta'(g) = \omega_\ell \overline{g} \omega_\ell$, we have $\theta'(B) = \overline{B}$, hence we have $A_k(X) \cong A/A^{\theta'} \cong \mathbb{G}_m^2/\mathbb{G}_m$ with the diagonal embedding of $\mathbb{G}_m$. $\Sigma_k = \mathrm{Res}'_A \Sigma_{\bar{k}} = \{2\overline{\alpha_1}\}$ as α_1 and α_2 are in the same Γ-orbit. The k-little Weyl group of X is $N_{W_{\bar{k}}(X)}(\mathfrak{a}_k)/C_{W_{\bar{k}}(X)}(\mathfrak{a}_k) = \mathbb{Z}/2\mathbb{Z}$.

We conclude the k-spherical root system of X is of type A_1.

Example 7.2. Let ℓ/k be a quadratic extension of characteristic zero fields, $G = \mathrm{Res}_{\ell/k}GL_2$, $X = \mathrm{Res}_{\ell/k}GL_2/U_2$, here U_2 is the quasisplit two dimensional unitary group over k, we have $P(X) = B$ as $X_{\bar{k}} \cong GL_2 \times GL_2/GL_2^{\iota}$ with $\iota : g \mapsto (g, g^{-t})$. We have $\Sigma_{\bar{k}}(X) = \{\alpha_1 + \alpha_2\}$.

X is a Galois symmetric variety with $\theta(g) = J^{-1}g^{-t}J$, for the conjugate θ' of θ with $\theta'(g) = \overline{g}^{-t}$, we have $\theta'(B) = \overline{B}$, hence $A_k(X) \cong A/A^{\theta'} \cong \mathbb{G}_m^2/\{(\pm 1)^2\}$. $\Sigma_k = \{2\overline{\alpha_1}\}$. The k-little Weyl group of X is $\mathbb{Z}/2\mathbb{Z}$.

We conclude the k-spherical root system of X is of type A_1.

Example 7.3. Let ℓ/k be a quadratic extension of characteristic zero fields, $G = \mathrm{Res}_{\ell/k}GL_3$, $X = \mathrm{Res}_{\ell/k}GL_3/GL_3$. Over algebraic closure, let's choose B the upper triangular Borel subgroup, and α_1, α_2 the simple roots of the first copy of GL_3, α'_1, α'_2 simple roots of second copy GL_3, we have $\Sigma_{\bar{k}}(X) = \{\alpha_1 + \alpha'_2, \alpha_2 + \alpha'_1\}$ and $\Sigma_{\bar{k}} \subset X^*((T_X)_{\bar{k}})$.

X is a Galois symmetric variety with $\theta(g) = \overline{g}$, for the conjugate $\theta'(g) = \omega_\ell \overline{g} \omega_\ell$, we have $\theta'(B) = B$, hence $A_k(X) \cong A/A^{\theta'} \cong \mathbb{G}_m^3/\mathbb{G}_m \times \mathbb{G}_m$, $\Sigma_k = \{\overline{\alpha_1} + \overline{\alpha_2}'\}$. The k-little Weyl group of X is $\mathbb{Z}/2\mathbb{Z}$.

We conclude the k-spherical root system of X is of type A_1.

Example 7.4. Let ℓ/k be a quadratic extension of characteristic zero fields, $G = \mathrm{Res}_{\ell/k}GL_3$, $X = \mathrm{Res}_{\ell/k}GL_3/U_3$, here U_3 is the quasisplit unitary group associated with ℓ/k. Over algebraic closure, let's choose B the upper triangular Borel subgroup, and α_1, α_2 the simple roots of the first copy of GL_3, α'_1, α'_2 the simple roots of second copy of GL_3, we have $\Sigma_{\bar{k}}(X) = \{\alpha_1 + \alpha'_1,\ \alpha_2 + \alpha'_2\}$ and $\Sigma_{\bar{k}} \subset X^*((T_X)_{\bar{k}})$.

X is a Galois symmetric variety with $\theta(g) = J^{-1}\overline{g}^{-t}J$. For the conjugate θ' of θ with $\theta'(g) = \overline{g}^{-t}$, we have $\theta'(B) = \overline{B}$. Hence $A_k(X) = A/A^{\theta'} = \mathbb{G}_m^3/(\pm 1)^3$, we have $\Sigma_k(X) = \{2\overline{\alpha}_1,\ 2\overline{\alpha}_2\}$. The k-little Weyl group of X is $N_{W_{\bar{k}}(X)}(\mathfrak{a}_k)/C_{W_{\bar{k}}(X)}(\mathfrak{a}_k) = S_3$.

We conclude the k-spherical root system of X is of type A_2.

Example 7.5. The spherical variety $X = SL_{2(p+1)}/SL_{p+1} \times SL_{p+1}$, the Γ-action induced from $SU_{2(p+1)}$ preserves Ω_X, hence it admits a k-form $SU_{2(p+1)}/SU_{p+1} \times SU_{p+1}$.

The set of spherical roots is $\Sigma_{\bar{k}} = \{\alpha_1 + \alpha_{2p+1}, \cdots 2\alpha_{p+1}\}$, the set of restricted spherical roots is $\Sigma_k = \{2\overline{\alpha}_1, 2\overline{\alpha}_2, \cdots 2\overline{\alpha}_{p+1}\}$.

Example 7.6. The spherical variety $X = SL_{2p+q+1}/SL_{p+q} \times SL_{p+1}$, the Γ-action preserves Ω_X, hence it exists a k-form $SU_{2p+q+1}/SU_{p+q} \times SU_{p+1}$.

The set of simple spherical roots is $\Sigma_{\bar{k}} = \{\alpha_1 + \alpha_{2p+q}, \cdots \alpha_p + \alpha_{p+q+1}, \alpha_{p+1} + \cdots + \alpha_{p+q}\}$, the set of simple restricted roots Σ_k is $\{2\overline{\alpha}_1, \cdots 2\overline{\alpha}_p, \overline{\alpha}_{p+1} + \cdots \overline{\alpha}_{p+q}\}$.

As a particular example, the symmetric variety SL_{n+1}/SL_n has a spherical root $\alpha_1 + \cdots + \alpha_n$ and it is Galois stable.

References

[KK16] Friedrich Knop and Bernhard Krötz. Reductive group actions. *arXiv preprint arXiv:1604.01005*, 2016.