EMBEDDING THEORY OF SPHERICAL VARIETIES

RCHEN69

Contents

1. Introduction 1
2. Notation 1
3. Toric varieties 1
4. Spherical varieties 2
References 4

1. Introduction

We present the embedding theory of spherical varieties following Guido Pezzini's note.

2. Notation

We fix G be a connected reductive algebraic group over $\mathbb{C}$, $T \subseteq B \subseteq G$, maximal torus T and a fixed Borel. We denote Δ the set of simple roots for G.

For H an algebraic group, we will denote $\chi(H)$ the character lattice of H, $\Lambda(H) = Hom_{\mathbb{Z}}(\chi(H), \mathbb{Q})$ the weight lattice of H.

3. Toric varieties

Definition 3.1. A toric variety for $T = (\mathbb{C}^{\times})^n$ is a normal variety X where T acts with an open orbit.

Affine toric varieties X is completely determined by $\mathbb{C}[X]$, the ring of regular functions on X, we have an inclusion $\mathbb{C}[X] \hookrightarrow \mathbb{C}[T]$, as $\mathbb{C}[X]$ is multiplicity free and $\mathbb{C}[T]$ contains all irreducible T-modules once. As an $\mathbb{C}[T]$-module, $\mathbb{C}[X]$ is given by the characters of T extend to a regular functions on X. This gives us a correspondence:

$$\mathbb{C}[X] \longleftrightarrow \text{convex polyhedral cone } \sigma \subset \chi(T)$$

Example 3.2. For $\mathbb{G}_m \hookrightarrow \mathbb{A}^1 = X$, if we identified $\chi(\mathbb{G}_m)$ with $\mathbb{Z}$ via $n \mapsto (z \to z^n)$, then we have $\chi(X) = \mathbb{Z}_{\geq 0}$.

Note we also have a bijection between polyhedral cones in $\chi(T)$ and $\Lambda(T)$: σ corresponds to

$$\sigma^{\vee} := \{\lambda \in \Lambda(T) \mid \langle \lambda, \chi \rangle \geq 0 \text{ for all } \chi \in \sigma\}$$

Now we talk about how to go from the polyhedral cone $\sigma^{\vee}$ to affine toric varieties If we denote $\mathbb{C}[z, z^{-1}] = \mathbb{C}[z_1, \cdots z_n, z_1^{-1}, \cdots z_n^{-1}]$, then we have a map

$$\theta : \mathbb{Z}^n \to \mathbb{C}[z, z^{-1}]$$
$$a = (\alpha_1, \cdots, \alpha_n) \mapsto z^a = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

Definition 3.3. We define $R_\sigma = \{f \in \mathbb{C}[z, z^{-1}] : \text{supp(f)} \subset \sigma^{\vee}\}$. Here for $f = \sum \lambda_a z^a$, supp(f) is set $a \neq 0 \in \mathbb{Z}^n$.

We get $X_\sigma = \text{Spec}(R_\sigma)$ an affine toric variety.

Example 3.4. For the cone $\mathbb{Z}_{\geq 0}$ inside $\mathbb{Z}$, we get the polynomial algebra $\mathbb{C}[z]$, which is the coordinate ring of $\mathbb{A}^1$.

Date: April 2023.

Example 3.5. For the toric variety $X = \mathbb{P}^1$, we have two affine charts $U_\sigma = \{[x:1] \mid x \neq 0\}$, $U_\tau = \{[1:y] \mid y \neq 0\}$, we can identify the char lattice of U_σ and U_τ as the cones $\mathbb{Z}_{\geq 0}(1,-1)$ and $\mathbb{Z}_{\leq 0}(1,-1)$ of $\mathbb{Z}^2 \cong \chi(T)$, we see that the cones for $\mathbb{P}^1$ is $e_1^* \cup -e_1^*$.

For $X = \mathbb{P}^2$, we have three affine charts $U_{\sigma_1}, U_{\sigma_2}, U_{\sigma_3}$, if we choose coordinate $[x_0 : x_1 : x_2]$ for $\mathbb{P}^2$, then we can identify $U_{\sigma_1} = \{(z_1, z_2), z_1 = \frac{x_1}{x_0}, z_2 = \frac{x_2}{x_0}\}$, $U_{\sigma_2} = \{(z_1^{-1}, z_1^{-1}z_2)\}$, and $U_{\sigma_3} = \{(z_2^{-1}, z_1 z_2^{-1})\}$. The polyhedral cones for $\mathbb{P}^2$ is $\mathbb{Z}_{\geq 0}e_1^* + \mathbb{Z}_{\geq 0}e_2^*$, $\mathbb{Z}_{\geq 0}(-e_1^*) + \mathbb{Z}_{\geq 0}(-e_1^* + e_2^*)$, and $\mathbb{Z}_{\geq 0}(-e_2^*) + \mathbb{Z}_{\geq 0}(e_1^* - e_2^*)$.

Theorem 3.6. *We have:*

- *Any toric variety is the union of affine T-stable open subsets, which are also T-toric varieties.*
- *Toric varieties are classified by families of strictly convex polyhedral cones in $\Lambda(T)$, called fans, satisfying: any face of a cone of the fan belongs to the fan and the intersection of two cones of the fan is always a face of each other.*

Example 3.7. In the $\mathbb{P}^2$ example, the cone for $\tau = \sigma_1 \cap \sigma_2$ is given by $S_\tau = S_{\sigma_1} + \mathbb{Z}_{\geq 0}(-e_1^*)$, so we have $U_\tau = \mathbb{C}_{z_1}^* \times \mathbb{C}_{z_2}$ in U_{σ_1}. We can also glue U_{σ_1} and U_{σ_2} to obtain $\mathbb{P}^2 \backslash \{[0:0:1]\}$.

4. Spherical varieties

Definition 4.1. A G-variety X is called spherical if it is normal and B has an open orbit on X.

We denote the open B-orbit by $\mathring{X}$, and for $x \in \mathring{X}$, we get an inclusion

$$Gx \cong G\backslash H \hookrightarrow X$$

the group H is called a *spherical subgroup* of G, and $G\backslash H \hookrightarrow X$ is called a *spherical embedding* of $G\backslash H$. So we see that the classification of spherical varieties is reduced to the study of spherical embeddings and classification of spherical subgroups, a general theory of embeddings is developed by Luna and Vust.

Example 4.2. For $G\backslash H = SL_2\backslash T$, for T the diagonal maximal split torus, we have H is isomorphic to the stabilizer of $(p,q) \in \mathbb{P}^1 \times \mathbb{P}^1$ in the open B-orbit. There is a unique closed G-orbit $Z = diag(\mathbb{P}^1)$.

Later we will see that $G\backslash H$ has two simple embeddings, the trivial one and the one embedds into $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 4.3. (1) we define the B semi-invariants $\mathbb{C}(X)^B$ as

$$\mathbb{C}(X)^{(B)} = \{f \in \mathbb{C}(X)^* \mid bf = \chi(b)f, \text{for all} b \in B\}$$

for $f \in \mathbb{C}(X)^B$, the associated character χ will be denoted by χ_f.
(2) We define $\chi(X) := \{\chi_f \mid f \in \mathbb{C}(X)^B\}$, the character lattice of X.

we note that if f_1 and f_2 has the same weight χ, then f_1/f_2 will be B-invariant, hence must be a constant function on X.

Definition 4.4. A spherical embedding X is called *simple*, if it has a unique closed G-orbit.

Example 4.5. For $G = SL_2$ acts on $\mathbb{P}^1 \times \mathbb{P}^1$, we consider the closed G-orbit Z, the orbit Z doesn't exist any G-stable neighborhood except the whole X, let's consider the action of the Borel group, there are two B-stable divisors other than Z:

$$D^+ = \mathbb{P}^1 \times \{[1:0]\},\ D^- = \{[1:0]\} \times \mathbb{P}^1$$

there exists an B-stable affine open set intersects with Z, we call it $Z_{Z,B}$, t is

$$X_{Z,B} = X\backslash(D^+ \cup D^-) = \{([x:1],[y:1])\} \cong \mathbb{A}^2$$

the action of B on $X_{Z,B}$ is easy to describe:

$$(Z \cap X_{Z,B}) \times M \cong X_{Z,B}$$

here M is the line $\{x + y = 0\} \subset \mathbb{A}^2$.

This example can be generalized to neighborhood for closed G-orbit for general spherical varieties.

Definition 4.6. For any G-spherical variety X and any closed G-orbit $Y \subset X$, we can define

$$X_{Y,B} = X \backslash \cap D$$

for all B-stable prime divisors D that do not contain Y.

Theorem 4.7. *For X a spherical G-variety and $Y \subset X$ a closed G-orbit, then we have*

- *$X_{Y,B}$ is affine B-stable and is equal to $\{x \in X | \ Y \subseteq \overline{B \cdot x}\}$.*
- *define parabolic P the stabilizer of $X_{Y,B}$, and choose a Levi subgroup L of P, then there exists an affine L-stable L-spherical closed variety M of $X_{Y,B}$ such that*

$$P_u \times M \to X_{Y,B}$$

given by $p(v,m) = (ulbl^{-1}, lm), p = ul, u \in P_u, l \in L$ is a P-equivariant isomorphism, and $\chi(X) = \chi(M)$.

Definition 4.8. A spherical G-variety X is called *simple* if it contains a unique closed G-orbit.

Definition 4.9. We have a map

$$\rho_X : \{\text{discrete valuations on } X\} \to \Lambda(X)$$

this is given by $\rho_X(v) = v(f_\chi)$, for $\chi \in \chi(X)$.

Proposition 4.10. *The map ρ_X restricts to an injective map on the set of G-invariant valuations, $\rho_X : \mathscr{V}(X) \to \Lambda(X)$.*

Definition 4.11. A prime divisor D on X is called a *color* if it is B-stable but not G-stable. We denote by $\Delta(X)$ the set of all colors of X.

Example 4.12. For $G \backslash H = SL_2 \backslash T \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, we have the two colors D^+ and D^-, denote v_Z the G-invariant valuation corresponds to Z, $f(x,y) = (x - y)$ is a local equation of Z in $X_{Z,B} = \{([x:1],[y:1]) \in \mathbb{P}^1 \times \mathbb{P}^1\}$ $\mathscr{V}(X) = \mathbb{Q}_{\geq 0} v_Z$. It follows that $\langle \rho(Z), \alpha_1 \rangle = -1$, and we have $\mathscr{V}(X) = \mathbb{Q}_{\geq 0} v_Z$, and since $f(x,y)$ has poles of order along both divisors, we get $\langle \rho(D^+, \alpha_1) \rangle = \langle \rho(D^-, \alpha_1) \rangle = 1$.

Definition 4.13. We say that (X,x) is an embedding of G/H if X is spherical and $G \cdot x$ is open in X, H is the stabilizer of x.

Definition 4.14. For X a simple G-spherical variety, and with Y the closed G orbit. We can define

$$\mathscr{D}(X) = \{D \in \Delta(X) | \ Y \subset D\}$$

Proposition 4.15. *Let (X,x) be a simple embedding of $G \backslash H$, then*

- *For $f \in \mathbb{C}(X)$, $f \in \mathbb{C}[X_{Y,B}]$ if and only if f is regular on the open B-orbit and $v_D(f) \geq 0$, for all D G-invariant prime divisors of X and all elements of $\mathscr{D}(X)$.*
- *Among simple embeddings, (X,x) is determined by $\mathscr{D}(X)$ and the valuations of G-stable prime divisors of X.*

For the first part, we note that $X_{Y,B}$ is normal, so any function is regular if it is regular in codim 1. For the second part, suppose (X,x) and (X',x') have the same $\mathscr{D}(X)$ and the valuations of G-stable prime divisors, we note that from the first part, we get $\mathbb{C}[X_{Y,B}] = \mathbb{C}[X'_{Y,B}]$, now since $GX_{Y,B} = X$ and $GX'_{Y,B} = X'$, we get $(X,x) \cong (X',x')$.

Similar to the cone associated with the toric varieties, we have the similar definition of colored cones for spherical varieties, here the extra information comes from the set of colors.

Definition 4.16. Let (X,x) be a simple embedding of $G \backslash H$, we define $\mathscr{C}(X) \subset \Lambda(X)$ to be the convex cone generated by $\rho_X(\mathscr{D}(X))$ and all G-invariant valuations associated to G-stable prime divisors of X. The couple $(\mathscr{C}(X), \mathscr{D}(X))$ is called the *colored cones* of X.

The following theorem is an analog of the fact that affine toric varieties are determined by polyhedral cone $\sigma^\vee$

Theorem 4.17. *The map $(X,x) \mapsto (\mathscr{C}(X), \mathscr{D}(X))$ induces a bijection between simple embeddinds of $G \backslash H$ and colored cones in $\Lambda(G \backslash H)$.*

The injectivity follows from the proposition 4.

Example 4.18. Let $G \backslash H = SL_2 \backslash T$, then we have two simple embeddings

- $(\mathscr{C}_1, \mathscr{D}_1) = (\{0\}, \emptyset)$, this corresponds to the simple embedding.

- $(\mathscr{C}_2, \mathscr{D}_2) = (\mathscr{V}(G\backslash H), \emptyset)$, this corresponds to the embedding $\mathbb{P}^1 \times \mathbb{P}^1$.

The next example shows that it is necessary to consider $\mathscr{D}(X)$ to distinguishes different embeddings.

Example 4.19. For $G\backslash H = SL_2\backslash U$, there is a unique color D, with $\langle \rho(D), \alpha_1 \rangle = 1$

- $(\mathscr{C}_1, \mathscr{D}_1) = (\mathbb{Q}_{\geq 0}\rho(D), \{D\})$, this corresponds to the embedding $SL_2\backslash U \hookrightarrow \mathbb{C}^2$.
- $(\mathscr{C}_2, \mathscr{D}_2) = (\mathbb{Q}_{\geq 0}\rho(D), \emptyset)$, this corresponds to the embedding given by blow up at $(0,0)$ in $\mathbb{C}^2$.

Similar to the fan in the toric varieties case, we can define a *colored fan* in $\Lambda(G\backslash H)$.

Definition 4.20. A *colored fan* in $\Lambda(G\backslash H)$ is a collection $\mathcal{F}$ of colored cones such that

- any face of a colored cone of $\mathcal{F}$ is in $\mathcal{F}$.
- the relative interiors of the colored cones do not intersect.

Given a embedding (X, x) we define the colored fans $\mathcal{F}(X)$ associated with X to be the colored cones associated with $X_{Y,G}$ for any G-orbit Y, here $X_{Y,G} = \{x \in G | Y \subseteq \overline{G \cdot x}\} = GX_{Y,B}$, is a simple embedding of Y.

Theorem 4.21. *The map $(X, x) \mapsto \mathcal{F}(X)$ induces a bijection between embeddings of $G\backslash H$ and colored fans in $\Lambda(G\backslash H)$.*

This is theorem 3.3 of [1]

References

[1] Friedrich Knop. The luna-vust theory of spherical embeddings. In *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*, volume 225, page 249. Citeseer, 1991.