EXPLICIT FORMULAS FOR SPHERICAL FUNCTIONS ON SHALIKA MODELS

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1. Introduction

We summarize the various explicit formulas for spherical functions on Shalika models.

2. Shalika models for GL_{2n}

This is the result of Sakellaridis. We will denote F a non-archimedean local field, by $\mathfrak{o}$ its ring of integers, by q the order of its residue field and ϖ a uniformizing element. We denote P the Borel subgroup and $I(\chi)$ unramified principal series of G obtained by induction from χ.

The Shalika subgroup S consists matrices of the form

$$s = \begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} I & X \\ & I \end{pmatrix}$$

we use $\Psi(s) = \psi(trX)$.

For any irreducible representation π of G, a G-equivariant morphism

$$\pi \longrightarrow \mathrm{Ind}_S^G(\Psi)$$

is called a Shalika model on π.

We assume that χ is of the form

$$\chi = (\chi_1, \cdots, \chi_n, \chi_n^{-1}, \cdots, \chi_1^{-1})$$

Let Λ denote a Shalika function on $I(\chi)$ and let

$$\Omega(g) = \Lambda(R_g \phi_K)$$

denote the image of ϕ_K under the Shalika embedding defined by Λ.

We notice first that it suffices to compute it for a set of double coset representatives in $S \backslash G/K$, by an easy argument we see that such a set of representatives is given by matrices

$$g_\lambda = \begin{pmatrix} \varpi^\lambda & \\ & I \end{pmatrix}$$

where ϖ is a uniformizer of F and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and ϖ^λ denotes $\mathrm{diag}(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \cdots, \varpi^{\lambda_n})$.

Proposition 2.0.1. *Under certain normalization, for $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, we have*

$$\Omega(g_\lambda) = \sum_{\omega \ in \ \Gamma} (-1)^{l(\omega)} \prod_{\alpha \in \Phi_{Sp}^+, \ \omega\alpha < 0} \chi(a_\alpha) \prod_{\alpha \in \Phi_{Sp}^{S+}, \ \omega\alpha < 0} \frac{1 - q^{-1}\chi(a_{-\alpha})}{1 - q^{-1}\chi(a_\alpha)} \ {}^\omega\chi\delta^{\frac{1}{2}}(g_\lambda)$$

here Γ denote the Weyl group of Sp_{2n}.

Proposition 2.0.2. *Let $\omega = \omega_\alpha \omega_{\tilde{\alpha}}$, where $\omega_\alpha = (i, i+1)$, $1 \leq i < n$. Then*

$$\Lambda_{H,\chi}(T_{\omega^{-1}} \phi_{B, {}^\omega\chi}) = (-1)\chi(a_\alpha) \frac{1 - q^{-1}\chi(a_{-\alpha})}{1 - q^{-1}\chi(a_\alpha)} c_\alpha(\chi) c_{\tilde{\alpha}}(\chi)$$

Proposition 2.0.3. *For $\omega = (n, n+1)$, we have*

$$T^*_{\omega^{-1}} \Lambda_{H,\chi} = (-1)\chi(a_\alpha) c_\alpha(\chi) \Lambda_{H, {}^\omega\chi}$$

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3. Shalika models for U_4

Let π be a generic unramified irreducible representation of $PGU_{2,2}(F)$, we fix a spherical vector ϕ_0 of π which we normalize by $\phi_0(1) = 1$. We are interested in computing the values of $S_\pi(g) := \Lambda_S(\pi(g)\phi_0)$ as g varies, where Λ_S is the unique non-trivial Shalika functional for π. Note that $S_\pi(g)$ is the image of ϕ_0 under the embedding to $\mathrm{Ind}_{S(F)}^{GU_{2,2}(F)}(\chi_S)$.

It is sufficient to calculate S_π at a set of double coset representatives of $S(F)\backslash GU_{2,2}(F)/GU_{2,2}(\mathcal{O})$. Using the Iwasawa and Cartan decomposition, we reduce it to calculate the value of S_π at matrices

$$g_n := \begin{pmatrix} \varpi^n I & \\ & I \end{pmatrix}$$

Proposition 3.0.1. *We have the following equality*

$$T^*_{s_2^{-1}} \Delta_{\tilde{S}, {}^{s_2}\xi}(1_{I_\omega}) = -\chi^{-1}(a_{\alpha_1-\alpha_2}) c_{2\alpha_2-\alpha_0}(\xi)$$

Proposition 3.0.2. *We have the following equality*

$$T^*_{s_1^{-1}} \Delta_{\tilde{S}, {}^{s_1}\xi}(1_{I_w}) = -\chi^{-1}(a_{2\alpha_2-\alpha_0}) c_{\alpha_1-\alpha_2}(\xi) \frac{1+q^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+q^{-1}\chi^{-1}(a_{2\alpha_2-a_0})}$$

Theorem 3.0.3. *For all $n \geq 0$, a suitable normalized spherical Shalika functional satisfies*

$$S_\pi(\begin{pmatrix} \varpi^n I_2 & \\ & I_2 \end{pmatrix}) = \frac{q^{-2n}}{1+q^{-1}} \mathcal{A}((-1)^n e^{\check{\rho}+n(\check{\alpha}_1+\check{\alpha}_2)} \prod_{\alpha \in \Phi^{+,s}_{Sp_4}} (1+q^{-1}e^{-\check{\alpha}}))(g_\chi)(\mathcal{A}(e^{\check{\rho}})(g_\chi))^{-1}$$

here we have denoted $\mathcal{A}(\cdot) = \sum_{\omega \in W_{Sp_4}} (-1)^{\ell(\omega)} \omega(\cdot)$.

4. Shalika models for orthogonal groups

In this section we study Suzuki's result on Shalika models for special orthogonal groups.

Let $G = SO_{4n}(F)$, the F-split $4n$-dimensional special orthogonal group where F is a nonarchimedean local field of characteristic 0.

By P we denote the Siegel parabolic subgroup of G and by N, the nilpotent radical of P. We identify G with a subgroup of the quadratic form defined by

$$\xi = \begin{pmatrix} & 1_{2n} \\ 1_{2n} & \end{pmatrix}$$

N is identified with the subgroup consisting of the matrices of the form

$$\begin{pmatrix} 1_{2n} & X \\ & 1_{2n} \end{pmatrix}$$

with $X + {}^tX = 0_{2n}$.

Let M be the Levi component of P consisting of the matrices of the form

$$\begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}$$

There is a notion of a generalized Shalika model for the representations of G as follows, take any nontrivial additive character ψ of F with conductor 0. Take any nontrivial additive character ψ of F with conductor 0. The expression $\psi(\frac{1}{2}\mathrm{tr}(JX))$ defines a character Ψ on N, where $J = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$. The stabilizer of this character in M is naturally isomorphic to $Sp_{2n}(F)$, the symplectic group with respect to J.

Define the subgroup-the generalized Shalika subgroup H of P by $H = \mathrm{stab}_M(\Psi)N \cong Sp_{2n}(F) \ltimes N$ and extend Ψ to a character of H, which we still denote by Ψ.

Definition 4.0.1. Take a nontrivial element $\Lambda \in \mathrm{Hom}(I(\chi), \Psi)$, we define a generalized Shalika function

$$\Omega(g) = \Lambda(R_g \phi_K)$$

We will briefly explain the main result of Suzuki. Since the function Ω satisfies

$$\Omega(hgk) = \Psi(h)\Omega(g)$$

for every $h \in H,\ g \in G$ and $k \in K$, it suffices to compute it for a set of double coset representatives in $H\backslash G/K$. By the Iwasawa decomposition

$$H\backslash G/K = H\backslash PK/K \cong Sp_{2n}(F)\backslash GL_{2n}(F)/GL_{2n}(\mathfrak{o})$$

Proposition 4.0.2. *We have the following double coset decomposition*

$$GL_{2n}(F) = \bigsqcup_{\lambda} Sp_{2n}(F) g_\lambda GL_{2n}(\mathfrak{o})$$

where $g_\lambda = diag(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \cdots, \varpi^{\lambda_n}, 1, \cdots, 1)$ *with* $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ *and* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Let's fix the notation for the root system. Let $E_i = e_{2i-1} + e_{2i},\ \beta_i = E_i - E_{i+1},\ 1 \leq i < n$ and $\beta_n = E_n$, then $\Phi := \{E_i - E_j,\ \pm E_k \mid 1 \leq i, j, k \leq n,\ i \neq j\}$ is a root system of type B_n and $\{\beta_i \mid 1 \leq i \leq n\}$ is a basis of Φ.

Definition 4.0.3. For each long root $\beta = E_i - E_j \in \Phi$, let $a_\beta = a_{e_{2i-1} - e_{2j-1}}$. For a short root $\beta \in \Phi$, a_β is already defined since $\beta \in \Sigma$.

We define $d_\beta(\chi)$ for each $\beta \in \Phi$ as follows: if β is a short root

$$d_\beta(\chi) = \chi(a_\beta)$$

and if β is a long root

$$d_\beta(\chi) = \chi(a_\beta)\frac{1 - q^{-2}\chi(a_{-\beta})}{1 - q^{-2}\chi(a_\beta)}$$

We will consider the generalized Shalika model for unramified principal series of G, the Borel subgroup of G consisting of matrices of the form $\begin{pmatrix} x & \\ & {}^t x^{-1} \end{pmatrix}\begin{pmatrix} 1_{2n} & X \\ & 1_{2n} \end{pmatrix}$ with upper triangular $x \in GL_{2n}(F)$. Let $\chi = (|\cdot|^{z_1}, |\cdot|^{z_2}, \cdots, |\cdot|^{z_{2n}})$ be an unramified character of P_ϕ.

Our main theorem is as follows

Theorem 4.0.4. *Let* $\chi = (|\cdot|^{z_1}, |\cdot|^{z_2}, \cdots, |\cdot|^{z_{2n}})$ *be an unramified character on* P_ϕ *and assume that this character satisfies* $z_{2i-1} = 1 + z_{2i}$ *for all* $a \leq i \leq n$.

- *If* χ *is not of the form as above then* $I(\chi)$ *does not have a generalized Shalika model.*
- *For every* $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ *with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

$$\Omega(g_\lambda) = Q^{-1} \prod_{\alpha > 0} c_\alpha(\chi) \sum_{\omega \in \Gamma} (-1)^{l_\Gamma(\omega)} (\omega\chi)^{-1} \delta^{1/2}(h_\lambda) \prod_{\beta > 0,\ \omega\beta < 0} d_\beta(\chi)$$

here $h_\lambda = diag(\varpi^{\lambda_1}, 1, \varpi^{\lambda_2}, 1, \cdots, \varpi^{\lambda_n}, 1) \in M$ *and* l_Γ *is the length function of* Γ.

5. Further questions

It will be interesting to check the relative Langlands duality for these Shalika models and it will be nice to have a uniform proof for these explicit formulas.