ARTHUR PACKETS AND RAMANUJAN CONJECTURE

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1. Arthur packets and the Ramanujan conjecture

1.1. **Introduction.** This is a study note for Shahidi's paper [Sha11].

Among the cuspidal representations of a quasisplit reductive group G over a number field k, those with nonzero Whittaker Fourier coefficients are quite important, they are called *global generic.* Globally generic automorphic representations are automatically locally generic.

It has been conjectured that when k is local, every tempered L-packet contains a unique generic representation with respect to a fixed generic character of $U(k)$, where U is the unipotent radical of a Borel subgroup of G over k, they parametrize the local tempered L-packets and can be used as base points. The main result of Shahidi's paper is that, under a part of the Arthur's A-packet conjecture 1.9, locally generic cuspidal automorphic representations of $G(\mathbb{A}_k)$ are tempered. In conclusion, generic representations also parametrize global tempered L-packets.

The trace formula is not sensitive to detecting globally generic representations and in practice we have to use a Poincare series, we hope to rule out the existence of locally generic representations which are not generic with respect to local components by any generic character of $U(k)\backslash U(\mathbb{A}_k)$ by means of multiplicity formula.

1.2. **Generic representations.** Let k be a number field, and $\mathbb{A}_k$ its ring of adeles, for each place v of k, we denote k_v the completion of k at v. Let $\mathcal{O}_v$ and $\mathcal{P}_v$ be the ring of integers and its maximal ideal. Let ϖ_v be a generator of $\mathcal{P}_v$ and normalized so that $|\varpi_v| = q_v^{-1}$, where q_v is the cardinality for $\mathcal{O}_v/\mathcal{P}_v$.

Let G be a quasisplit connected reductive group over k. We fix B a Borel subgroup over k and we write $B = TU$ for T a maximal torus of G isomorphic to the quotient B/U, where U is the unipotent radical of B.

The choice of Borel subgroup defines a set of positive roots of G, that is roots of T on Lie(U). We denote $\Delta = \Delta(G, T)$ the set of simple roots among them. Let $\{X_\alpha \mid \alpha \in \Delta\}$ be a choice of root vectors in Lie(U). This means that there exists a natural map

$$\phi : U \longrightarrow \prod \mathbb{G}_a$$

where the product runs over all the roots in Δ, sending $\exp(x_\alpha X_\alpha)$ to x_α, $x_\alpha \in \overline{k}$, whose kernel contains the derived subgroup of U. Composing ϕ with the map

$$\Sigma : \prod_{\alpha\in\Delta} \mathbb{G}_a \longrightarrow \mathbb{G}_a$$

defined by $\Sigma((x_\alpha)_\alpha) \mapsto \sum_{\alpha\in\Delta} x_\alpha$, we get a map from U to $\mathbb{G}_a$, since G is quasisplit, we may assume that the splitting is defined over k and then $\Sigma \circ \phi$ is defined over k.

According to k is local or global, we fix non-trivial character ψ_v or ψ of k_v or $k\backslash\mathbb{A}_k$, we can then define a generic character χ_v or χ of $U(k_v)$ or $U(k)\backslash U(\mathbb{A}_k)$ by

$$\chi = \psi \circ \Sigma \circ \phi$$

when k is global $\psi = \otimes_v \psi_v$ and hence $\chi = \otimes_v \chi_v$.

Definition 1.1. A representation σ of $G(k_v)$ on a complex vector space V_σ is called χ_v-generic if there exists a functional λ_v on the continuous dual V'_σ of V_σ such that

$$\lambda_v(\sigma(u)\omega) = \chi_v(u)\lambda_v(\omega)$$

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for every $\omega \in V_\sigma$.

Remark 1.2. When k is archimedean, one requires the continuity to be with respect to the seminorm topology on the space of differentiable vectors V_σ^∞.

Definition 1.3. A cuspidal representation $\pi = \otimes_v \pi_v$ is called locally generic if each π_v is generic with respect to a generic character χ_v of $U(k_v)$.

we are not requiring χ_v to be a local component of a global character χ of $U_k \backslash U(\mathbb{A}_k)$.

Now we introduce a notion of a globally generic cusp form. Let χ be a generic character of $U(k)\backslash U(\mathbb{A}_k)$, assume that π is a cuspidal representation of $G(\mathbb{A}_k)$ let φ be a cusp form in the space of π, let

$$W_\varphi(g) = \int_{U(k)\backslash U(\mathbb{A}_k)} \varphi(ug)\overline{\chi(u)}\, du$$

we define the following

Definition 1.4. A cuspidal representation $\pi = \otimes_v \pi_v$ is called globally generic with respect to $\chi = \otimes_v \chi_v$ if $W_\varphi(e) \neq 0$ for some $\varphi \in V_\pi$.

Conjecture 1.5. *Assume that $\pi = \otimes \pi_v$ is locally generic with respect to local components of a generic character $\chi = \otimes \chi_v$ of $U(k)\backslash U(\mathbb{A}_k)$, then π is globally generic.*

This conjecture is a well-known theorem for $G = GL_n$, since cusp forms on $GL_n(\mathbb{A}_k)$ are globally generic. On the other hand, there are many examples of nongeneric cuspidal representations for other groups, among them are the so-called CAP representations- *cuspidal representations associated to parabolics.* From the local results, one expect that generic representations completely parametrize global tempered L-packets. Using a part of a conjecture of Arthur and a rigidity conjecture, we can prove the following:

Assuming the conjectures 1.9 and 1.11, then locally generic cuspidal automorphic representations are tempered.

In other words there are no global obstructions for the equivalence of locally and globally generic cuspidal representations.

1.3. **Arthur conjecture.** We assume that k is local, let L_k be either W_k' the Weil-Deligne group of k if k is non-archimedean, or the Weil group, otherwise.

Let $\Phi(G)$ be the set of Langlands parameters, that is the equivalence classes of homomorphisms

$$\phi : L_k \longrightarrow {}^L G = \hat{G} \rtimes L_k$$

satisfy certain conditions. We let $\Phi_{temp}(G)$ denotes those $\phi \in \Phi(G)$ with bounded image in $\hat{G}$.

We let $\Psi(G)$ be the set of $\hat{G}$-orbits of maps

$$\psi : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G = \hat{G} \rtimes L_k$$

such that the projection of $\psi(L_k)$ onto $\hat{G}$ is bounded. Moreover, we assume that $\phi = \psi | L_k \in \Phi_{temp}(G)$.

For each $\psi \in \Psi(G)$, we can define a Langlands parameter $\phi_\psi \in \Phi(G)$ by

$$\phi_\psi(\omega) = \psi(\omega, \begin{pmatrix} |\omega|^{1/2} & 0 \\ 0 & |\omega|^{-1/2} \end{pmatrix})$$

The map

$$\psi \mapsto \phi_\psi$$

is an injection from $\Psi(G)$ to $\Phi(G)$. Arthur conjectured the existence of a finite set $\Pi(\psi)$ of irreducible admissible representations of $G(k)$ satisfying a list of properties. In particular, he demanded that the L-packet $\Pi(\phi_\psi)$ defined by the Langlands parameter ϕ_ψ to be contained in $\Pi(\psi)$, the members in $\Pi(\psi)$ are rather mysterious and are there to supplement $\Pi(\phi_\psi)$ to produce stable distributions.

Let's assume k is a p-adic field, let $I_k' \subset L_k$ be $I_k' = I_k \times \mathrm{SL}_2(\mathbb{C})$, for I_k the inertia group of W_k. Assume that $\phi | I_k' = 1$, then $\phi_\psi | I_k' = 1$ and $\Pi(\phi_\psi)$ consists of unramified representations of $G(k)$, each for a hyperspecial maximal compact subgroup of G. If G is defined over $\mathcal{O}$, the ring of integers of k, then $\Pi(\phi_\psi)$ has a unique unramified representation with respect to $G(\mathcal{O})$.

The map from each L_{k_v} to L_k then allows us to define $\psi_v \in \Psi(G/k_v)$ and $\phi_{\psi_v} \in \Phi(G/k_v)$. Given $\psi \in \Psi(G/k)$, we may define the global Arthur packet

$$\Pi(\psi) = \{\ \pi = \otimes_v \pi_v \mid \pi_v \in \Pi(\psi_v)\}$$

where for almost all v, $\pi_v = \pi_v^0$, the unique $G(\mathcal{O}_v)$-spherical representation in $\Pi(\phi_\psi)$. Arthur's conjecture states that every automorphic representation must belong to $\Pi(\psi)$ for some $\psi \in \Psi(G/k)$.

1.4. **Local main result.** In this section we will assume k is a characteristic zero local field, and G is a quasisplit connected reductive group over k.

Theorem 1.6. *If a member of the L-packet defined by ϕ_ψ is unramified and generic, then it is tempered.*

The unramified condition in 1.6 can be removed whenever the local Langlands conjecture holds for the Levi subgroup defined by tempered parameter ϕ, as we will see in the next theorem

Theorem 1.7. *Assume the validity of the local Langlands conjecture for every proper Levi subgroup M of G to the extent that every irreducible generic tempered representation σ of $M(k)$ is parameterized by the homomorphism $\phi : L_k \to {}^L M$ with bounded image in $\hat{M}$ such that*

$$L(s, r \cdot \phi_v) = L(s, \sigma_v, \tilde{r})$$

where r and v are defined before. Let $\psi \in \Psi(G/\mathbb{R})$ and let $\Pi(\phi_\psi)$ be the packet attached to ϕ_ψ, suppose that $\Pi(\phi_\psi)$ has a generic member, then ϕ_ψ is tempered.

Corollary 1.8. *Assume $k = \mathbb{R}$ (or $\mathbb{C}$). Let $\psi \in \Psi(G/\mathbb{R})$, then every generic member of $\Pi(\phi_\psi)$ is tempered.*

This is because the full Langlands conjecture for real groups is now a theorem.

1.5. **Ramanujan conjecture.** We now assume k is a number field and G is quasisplit connected reductive group defined over k. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A}_k)$.

We will assume the following conjecture

Conjecture 1.9. *For almost all finite primes v, $\pi_v \in \Pi(\phi_v)$ where ψ_v is the Arthur parameter of π_v, suppose $\psi_v = (\phi_v, \rho_v)$ and $\phi_v | I_k' = 1$, then the unramified members of $\Pi(\psi_v)$ are precisely those in $\Pi(\phi_{\psi_v})$.*

We will now assume that π is locally generic, that is each π_v is generic with respect to a generic character χ_v of $U(k_v)$. We have the following from theorem 1.7

Theorem 1.10. *Assuming conjecture 1.9, and π is locally generic cuspidal automorphic representation of $G(\mathbb{A}_k)$, then π_v is tempered for almost all k.*

We make the following conjecture

Conjecture 1.11. *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_k)$, assume that π_v is tempered for almost all many places, then π is tempered.*

We have the following result toward the proof of conjecture 1.11

Theorem 1.12. *Assume the Ramanujan conjecture for $GL_N(\mathbb{A}_k)$, then globally generic cuspidal representation of $G(\mathbb{A}_k)$ are all tempered for G a classical group.*

References

[Sha11] Freydoon Shahidi. Arthur packets and the Ramanujan conjecture. *Kyoto Journal of Mathematics*, 51(1):1 – 23, 2011.