SCHWARTZ SPACE OF BASIC AFFINE SPACE

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1. Introduction

In this note, we want to categorify the result of Braverman and Kazhdan on spherical and Iwahori Hecke modules associated with basic affine spaces [BK18].

2. The basic affine space

Let's set up the notation. Let F be a local non-archimedean field with ring of integers $\mathcal{O}$, we fix a generator κ of $\mathcal{O}$. Let G be a split connected reductive group over F and B its Borel subgroup with U its unipotent radical, $T = B/U$ its Cartan torus.

Let Λ be the lattices of cocharacters of T and let $\Lambda^\vee$ be the lattice of characters of T. We fix a Haar measure dg on G and denote by $\mathcal{H}(G)$ the Hecke algebra of locally constant compactly supported functions on G. As well-known, the category $\mathcal{M}(G)$ of smooth G-modules is equivalent to the category of non-degenerate $\mathcal{H}(G)$-modules.

Let P be a parabolic subgroup of G with a Levi subgroup M and unipotent radical U_P, let $X_P = G/U_P$, this space has a natural $G \times M$-action, therefore the space $\mathcal{S}_c(X_P)$ of locally constant supported functions on X_P becomes $G \times M$ module. We are going to twist the M-action by the square root of the absolute value of the determinant of the M-action on the Lie algebra $\mathfrak{u}_P$ of U_P. It is easy to see that X_P possesses unique G-invariant measure, hence we can talk about $L^2(X_P)$ and it has a natural unitary action of $G \times M$.

We have defined certain algebra $\mathcal{J}(G)$ of functions on G which contains the Hecke algebra $\mathcal{H}(G)$ and can be thought as an algebraic version of the Harish-Chandra Schwartz space $\mathcal{C}(G)$, in particular $\mathcal{J}(G)$ is a smooth $G \times G$-module. There is a $\mathcal{J}(G)$-action on $L^2(X_P)$ for any P so we can set

$$\mathcal{S}(X_P) = \mathcal{J}(G) \cdot \mathcal{S}_c(X_P)$$

here $\mathcal{S}_c(X_P)$ stands for the space of locally constant functions with compact support on X_P. The space $\mathcal{S}(X_P)$ is a smooth $G \times M$ module.

Note we have the following theorem which generalizes the classical Fourier transform on the space $SL_2/U_{SL_2} \cong \mathbb{A}^2$

Theorem 2.1. *Let P and Q be two associate parabolics, i.e. two parabolics with the same Levi subgroup M, then there exists a $G \times M$-equivariant unitary isomorphism $\Phi_{P,Q} : L^2(X_P) \cong L^2(X_Q)$, these isomorphism satisfy the following properties*

- $\Phi_{P,P} = id$.
- *For three parabolic subgroups P, Q, R with the same Levi subgroup M we have $\Phi_{Q,R} \circ \Phi_{P,Q} = \Phi_{P,R}$.*

Note the operator $\Phi_{P,Q}$ is not canonical.

For a parabolic subgroup P of G with Chosen Levi subgroup M let $Ass(P)$ be the set of all parabolics Q containing M as a Levi subgroup, we now define another version of the space $\mathcal{S}'(X_P)$ of the Schwartz space of functions on X_P

$$\mathcal{S}'(X_P) = \sum_{Q \in Ass(P)} \Phi_{Q,P}(\mathcal{S}_c(X_Q))$$

The expected relationship between the two versions of the Schwartz space is described in the following conjecture

Conjecture 2.2. *We have*

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- $\mathcal{S}'(X_P) \subset \mathcal{S}(X_P)$.
- $\mathcal{S}'(X_P)_{cusp} = \mathcal{S}(X_P)_{cusp}$, *here by* $\mathcal{S}(X_P)_{cusp}$ *we denote the* M*-cuspidal part of* $\mathcal{S}(X)_{cusp}$*, similarly for* $\mathcal{S}'(X_P)$.
- *The operator* $\Phi_{P,Q}$ *defines an isomrophism between* $\mathcal{S}(X_P)$ *and* $\mathcal{S}(X_Q)$.
- *The right action of* $\mathcal{J}(M)$ *on* $L^2(X_P)$ *preserves* $\mathcal{S}(X_P)$*, thus making* $\mathcal{S}(X_P)$ *into a* $(\mathcal{J}(G), \mathcal{J}(M))$*-bimodule.*

Example 2.3. Let's look at two extreme cases. First we consider the case $P = G$, then in this case $Ass(P)$ consists of 1 element and therefore $\mathcal{S}'(X_P) = \mathcal{H}(G)$, similarly $\mathcal{S}(X_P) = \mathcal{J}(G)$, we have $\mathcal{S}'(X_P) \subset \mathcal{S}(X_P)$.

Second, let's consider the extreme case when P is a Borel subgroup B of G. Then M is a maximal split torus of G and we have $\mathcal{H}(T) = \mathcal{J}(T)$, since any representation of T is cuspidal, so we have

$$\mathcal{S}(X_B) = \mathcal{S}(X_B)_{cusp}, \ \mathcal{S}'(X_B) = \mathcal{S}'(X_B)_{cusp}$$

now the second assertion in conjecture 2.2 states that $\mathcal{S}'(X_B) = \mathcal{S}(X_B)$. Braverman and Kazhdan proved this conjecture for Iwahori part of both spaces.

3. The spherical part

Let's define the spherical part of $\mathcal{S}(X_P)$ by setting

$$\mathcal{S}_{sph}(X_P) = \mathcal{S}(X_P)^{G(\mathcal{O}) \times M(\mathcal{O})}$$

we would like to describe this space more explicitly, for this note set-theoretically we have

$$G(\mathcal{O}) \backslash X_P / M(\mathcal{O}) = M(\mathcal{O}) \backslash M / M(\mathcal{O})$$

hence elements of $\mathcal{S}_{sph}(X_P)$ can be thoguht as $M(\mathcal{O}) \times M(\mathcal{O})$ invariant functions on M. Recall the Satake isomorphism for M says that the spherical Hecke algebra $\mathcal{H}_{sph}(M)$ consisting of compactly supported $M(\mathcal{O}) \times M(\mathcal{O})$-invariant functions on M is isomorphic to the complexified Grothendieck ring of the category of finite dimensional representations of the Langlands dual group $M^\vee$. We shall denote the corresponding map by $K_0(\mathrm{Rep}(M^\vee)) \to \mathcal{H}_{sph}(M)$ by $Sat(M)$.

Let $G^\vee$ be the Langlands dual group of G and let $P^\vee$ be the corresponding parabolic subgroup of $G^\vee$ with unipotent radical $U_{P^\vee}$. Let $\mathfrak{u}_{\mathfrak{p}^\vee}$ denote the Lie algebra of $U_{P^\vee}$, it has a natural action of $M^\vee$.

Now let

$$f_P = \sum_{i=0}^{\infty} \mathrm{Sat}_M([\mathrm{Sym}^i(\mathfrak{u}_{\mathfrak{p}^\vee})])$$

using the Satake isomorphism we can regard f_P as a $G(\mathcal{O}) \times M(\mathcal{O})$ invariant function on X_P.

Conjecture 3.1. $\mathcal{S}_{sph}(X)$ *is a free right* $\mathcal{H}_{sph}(M)$*-module generated by* f_P.

Let's introduce the local unramified conjecture from relative Langlands duality, we are in the setting $M = T^*X$ polarized and the dual $\check{M}$ is endowed with the neutral $\mathbb{G}_{gr}$-action, we ignore the shearing in the formulation.

Conjecture 3.2. *There is an equivalence of categories*

- *(small version)*

$$\mathbb{L}_X : \ Shv(X_F/G_\mathcal{O}) \longrightarrow perfect\ \check{G} - equivariant\ modules\ for\ \mathcal{O}_{\check{M}}$$

- *(large version)*

$$\mathbb{L}_X : \ SHV(X_F/G_\mathcal{O}) \longrightarrow QC(\check{M}/\check{G})$$

the equivalence is required to be compatible with pointings, Hecke actions, Galois actions and Poisson structures.

Example 3.3. For $M = T^*(G/U) = T^*X$ as a $G \times T$ space with $B \subset G$ the Borel subgroup and $B = TU$ the Levi decomposition, we have $\check{M} = T^*(\check{G}/\check{U}) = T^*\check{X}$ as a $\check{G} \times \check{T}$-space. Using the sheaf-function dictionary, we can see that the local unramified BZSV conjecture 3.2 is a categorification of 3.1.

4. Iwahori part and K-theory

Let $\mathcal{H}_{aff}(G)$ denote the affine Hecke algebra of G, this is an algebra over $\mathbb{C}[v, v^{-1}]$, its specilization at $v = q^{1/2}$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}(G, I)$ of G.

Let $\mathcal{N}_{G^\vee}$ be the nilpotent cone in the Lie algebra of $G^\vee$, let $\mathcal{B}_{G^\vee}$, $\mathcal{B}_{M^\vee}$ denote the corresponding flag varieties. The cotangent bundle $T^*\mathcal{B}_{G^\vee}$ maps naturally to $\mathcal{N}_{G^\vee}$ thus we can define

$$St_{G^\vee, M^\vee} = T^*\mathcal{B}_{G^\vee} \times_{\mathcal{N}_{G^\vee}} T^*\mathcal{B}_{M^\vee}$$

This variety is acted on by the group $G^\vee \times \mathbb{C}^\times$, thus we can consider the complexified equivariant K-theory $K_{M^\vee \times \mathbb{C}^\times}(St_{G^\vee, M^\vee})$, this is a vector space over $\mathbb{C}[v, v^{-1}] = K_{\mathbb{C}^\times}(pt)$, it is easy to see that it has a module structure of $\mathcal{H}_{aff}(G) \otimes \mathcal{H}_{aff}(M)$

Conjecture 4.1. *The specialization of $K_{M^\vee \times \mathbb{C}^\times}(St_{G^\vee, M^\vee})$ at $v = q^{1/2}$ is isomorphic to $\mathcal{S}'(X_P)^I$.*

Question: For $P = B$, can we formulate a categorical version of this conjecture?

References

[BK18] Alexander Braverman and David Kazhdan. Schwartz space of parabolic basic affine space and asymptotic hecke algebras. *arXiv preprint arXiv:1804.00336*, 2018.