SATAKE ISOMORPHISM

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1. Introduction

This is a note summarize the Satake isomorphism following Shahidi's book [Sha10].

2. Satake isomorphism

Throughout this section k will be a p-adic local field and G is a connected reductive group over k. Let T be a maximal torus of G defined over k. We will assume that G is quasisplit. We will take T a maximal torus of G and A a maximal split torus of T which is also a maximal split torus of G, we will let $W_0 = W(G, A)$.

We will denote the character modules of T and A by $X^*(T)$ and $X^*(A)$. Let $X^*(T)_k = X^*(T)^{\Gamma_k}$ be the subgroup of k-rational characters of T, those defined over k. Observe that $X^*(A) = X^*(A)^{\Gamma_k}$ as A is split.

Next let $X_*(T) = \mathrm{Hom}(X^*(T), \mathbb{Z})$ and $X_*(T)_k = \mathrm{Hom}(X^*(T)_k, \mathbb{Z})$. We have the standard homomorphism

$$H_T : T(k) \longrightarrow X_*(T)_k$$
$$q^{\langle H_T(t),\ \chi\rangle} = |\chi(t)|_k$$

There is a pairing $\langle , \rangle$ between $X^*(T)$ and $X_*(T)$ which allows us to view $X_*(T)$ as the group of cocharacters of T. The embedding $\theta :\ A \subset T$ leads to an injection

$$0 \longrightarrow X_*(A) \longrightarrow X_*(T)$$

hence we have a map

$$0 \longrightarrow X_*(A) \longrightarrow X_*(T) \longrightarrow X_*(T)_k$$

We claim the following

- $X_*(A) = X_*(T)^{\Gamma_k}$
- the injection θ_* from $X_*(A)$ into $X_*(T)$ injects $X_*(A)$ into $X_*(T)_k = \mathrm{Hom}(X^*(T)_k, \mathbb{Z})$.

Example 2.1. Let's consider the torus $T = \mathrm{Res}_{K/k} GL_1$ for ℓ/k a quadratic extension. Then T maybe identified with $T = K^* \times K^*$ on which σ acts as

$$(t_1, t_2)^\sigma = (t_2^\sigma, t_1^\sigma)$$

its k-points can be defined as the Γ-fixed points, which is isomorphic to K^*. An arbitrary element χ of $X(T)$ is of the form

$$\chi(t_1, t_2) = t_1^{n_1} t_2^{n_2}$$

we define

$$X_0 = \{\ \chi \in X \mid \sum_{\sigma \in \Gamma} \chi^\sigma \}$$

then we can show that $A = X_0^\perp$ is a maximal split torus of T. In our example, we can claculate that the maximal split torus A of T equals

$$A = X_0^\perp = \{\ (t, t) \mid t \in K^*\ \}$$

and its k-points $A(k) \cong k^*$.

We can also define a maximal anisotropic torus T_0 of T as

$$T_0 = (X^\Gamma)^\perp$$

it can be checked that in our example, every $\chi \in X^\Gamma$ is of the form $\chi(t_1, t_2) = (t_1 t_2)^n$, hence

$$T_0 = \{\ (t, t^{-1}) \mid t \in K^*\}$$

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and $T_0(k) = K^1$.

Let's denote A^0 and T^0 the kernel of H_A and H_T, i.e. the largest compact subgroup of $A(k)$ and $T(k)$. We then have the following exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A^0 \arrow[r] \arrow[d] & A \arrow[r, "H_A"] \arrow[d, "\theta"] & X_*(A) \arrow[r, "id"] \arrow[d, "\overline{\theta}"] & X_*(A) \arrow[r] \arrow[d, "\theta_*"] & 0 \\
0 \arrow[r] & T^0 \arrow[r] & T \arrow[r, "H_T"] & H_T(T(k)) \arrow[r] & X_*(T)_k &
\end{tikzcd}$$

In general, we therefore have

$$X_*(A) \subset H_T(T(k)) \subset X_*(T)_k$$

Proposition 2.2. *For a unramified torus T, we have*

$$H_T(T(k)) = X_*(A)$$

Definition 2.3. A character $\chi : T(k) \longrightarrow \mathbb{C}^*$ is unramified if $\chi \mid T^0 = 1$, i.e. χ factors through Λ. We use $X_{un}(T(k)) = X_{un}$ to denote the set of unramified characters of $T(k)$.

Note that we have

$$X_{un}(T(k)) = \mathrm{Hom}(\Lambda, \mathbb{C}^*) = \hat{A}$$

Definition 2.4. A connected reductive group G over k is unramified if G is quasisplit and split over an unramified cyclic extension of k.

Definition 2.5. Let π be an irreducible admissible representation of $G(k)$ on a complex vector space $\mathcal{H}$, where G is unramified, then π is called unramified if $\mathcal{H}$ has a vector fixed by $G(\mathcal{O})$.

The Satake transform is the linear map

$$S : \mathcal{H}(G(k), K) \longrightarrow \mathcal{H}(T(k), T^0)$$
$$Sf(t) = \delta(t)^{1/2} \int_U f(tu)\, du$$

Theorem 2.6. *The Satake transform S is an algebra isomorphism of $\mathcal{H}(G(k), K)$ onto $\mathbb{C}[\Lambda]^{W_0}$.*

3. Connection with L-groups and local unramified L-functions

Let G be an unramified group over k splits over the unramified extension k'/k. Let $T^\vee$ and $G^\vee$ be the connected components of T and G. Let σ be the Frobenius element and thus $\Gamma_{k'/k} = \langle\ \sigma\ \rangle$, let $N^\vee$ be the normalizer of $T^\vee$ in $G^\vee$ and $W^\vee = N^\vee/T^\vee$, let $W = W(G,T)$, we may identify W with $W^\vee$, we also denote $W_0 = W(G, A)$, it is the subgroup of W consisting of elements which send A to itself, let $N^\vee$ be the corresponding subgroup of $W^\vee$.

Proposition 3.1. *Let $\nu^\vee : T^\vee \to A^\vee$ be the canonical surjection, and $\nu' : T^\vee \rtimes \sigma \to A^\vee$ be defined by $\nu' : \nu'(t \times \sigma) = \nu(t)$, then ν' induces a bijection*

$$\overline{\nu} :\ T^\vee \rtimes \sigma / Int\ N^\vee \cong A^\vee / W_0$$

let $(G^\vee \rtimes \sigma)_{ss}$ be the conjugacy classes of semisimple elements in $G^\vee \rtimes \sigma$, then the map

$$\overline{\mu} :\ T^\vee \rtimes \sigma / Int\ N^\vee \longrightarrow (G^\vee \rtimes \sigma)_{ss} / Int\ G^\vee$$

induces by the inclusion is a bijection, and hence

$$\alpha = \overline{\mu} \cdot \overline{\nu}^{-1} :\ A^\vee / W_0 \longrightarrow (G^\vee \rtimes \sigma)_{ss} / Int\ G^\vee$$

is a bijection.

Let π be an irreducible unramified representation of $G(k)$ on $\mathcal{H}(\pi)$, then dim $\mathcal{H}(\pi)^K = 1$ which is an module for $\mathcal{H}_K = \mathcal{H}(G(k), G(\mathcal{O}_k))$, for $K = G(\mathcal{O}_k)$, hence there exists an algebra homomorphism ω from $\mathcal{H}_K$ to $\mathbb{C}$ such that

$$\pi(f)v = \omega(f)v$$

for $v \in \mathcal{H}(\pi)^K$.

This homomorphism ω is of the form

$$\omega_\chi(f) = \int_{T(k)} Sf(t)\chi(t)\ dt$$

for χ an unramified character of $T(k)$, unique up to conjugation by elements of W_0, class π is uniquely determined by the orbit $\chi \in A^\vee/W_0$.

Given an irreducible unramified representation π, let $A(\pi) \in A^\vee/W_0$ be the W_0-orbit of $A^\vee$ determining the class of π, we will set

$$c(\pi) :=\ \overline{\mu} \cdot \overline{\nu}^{-1}(A(\pi))$$

Definition 3.2. The local Langlands L-function attached to π and r is

$$L(s, \pi, r) = \det(I - r(c(\pi))q^{-s})^{-1}$$

With definition, we can calculate some examples of local L-functions.

Example 3.3. Let $G = SL_2$, for r the adjoint representation of $G^\vee(\mathbb{C}) = PGL_2(\mathbb{C})$, we have

$$L(0, \pi, r) = \frac{1}{(1 - q^{-2s})(1 - q^{2s})} = (1 - \mu(a_\alpha))^{-1}(1 - \mu(a_\alpha)^{-1})^{-1}$$

here $\mu(a_\alpha) = |\omega|^{-s} = q^{-2s}$ with $\mu(\mathrm{diag}(a, a^{-1})) = |a|^{2s}$, $a_\alpha = \mathrm{diag}(\varpi, \varpi^{-1})$.

Example 3.4. For $G = SU_3$, assume it is defined by E/k, with $[E : k] = 2$, and let $\Gamma_{E/k} = \mathrm{Gal}(E/k)$, the maximal torus T has its k-points as

$$T(k) = \{\mathrm{diag}(a, \overline{a}/a, \overline{a}^{-1}) \mid a \in E^*\}$$

for $\mu(a) = |a|_k^{2s}$, the elements A_μ can be represented by

$$A_\mu = \mathrm{diag}(q_k^s, 1, q_k^{-s})$$

dnote ${}^L\mathfrak{n}$ the Lie algebra of ${}^LN$, it has basis $X_{\alpha^\vee}$ and $X_{\beta^\vee},\ X_{\alpha^\vee+\beta^\vee}$ with Galois action. Then we have

$$\sigma(X_{\alpha^\vee}) = X_{\beta^\vee},\ \sigma(X_{\beta^\vee}) = X_{\alpha^\vee},\ \sigma(X_{\alpha^\vee+\beta^\vee}) = -X_{\alpha^\vee+\beta^\vee}$$

we can calculate that for the adjoint representation $\tilde{r}$ on ${}^L\mathfrak{n}$, we get

$$L(0, \pi, \tilde{r}) = \det(I - \tilde{r}(A_\mu \times \sigma))^{-1} = \det(\begin{pmatrix} 1 & -q^{-s} & 0 \\ -q_k^{-s} & 1 & 0 \\ 0 & 0 & 1 + q_k^{-2s} \end{pmatrix})^{-1} = (1 - \mu(a_\alpha)^2)^{-1}$$

here $a_\alpha = \mathrm{diag}(\varpi, 1, \varpi^{-1})$.

4. Connection with intertwining operators

Proposition 4.1. *Assume $G = SL_2$, choose $f_0 \in I(\mu)$, where μ is an unramified character of k^* such that $f_0(gk) = f_0(g)$ for $k \in SL_2(\mathcal{O}_k)$, $f_0(e) = 1$ then*

$$\int_{N^-(k)} f_0(n)\ dn = (1 - q^{-1}\mu(a_\alpha))/(1 - \mu(a_\alpha)) = L(0, \mu, \tilde{r})/L(1, \mu, \tilde{r})$$

here $L(0, \pi, \tilde{r})$ is calculated in 3.3.

Proposition 4.2. *Assume $G = SU_3$, choose $f_0 \in I(\mu)$, where μ is an unramified character of $T(k)$ such that $f_0(gk) = f_0(g)$ for $k \in SU_3(\mathcal{O}_k)$, $f_0(e) = 1$ then*

$$\int_{N^-(k)} f_0(n)\ dn = L(0, \mu, \tilde{r})/L(1, \mu, \tilde{r})$$

here $L(0,\pi,\tilde{r})$ is calculated in 3.4.

Let's also include the archimedean case

Proposition 4.3. *Let $G = SL_2$ and take $K = SO_2(\mathbb{R})$, let $\mu(diag(a, a^{-1})) = |a|^s$, denote f the K-invariant function of $V(\mu)$, normalized by $f(\mu) = 1$, then*

$$\int_{\mathbb{R}} f(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}) = L(s)/L(s+1)$$

for $L(s) = \pi^{-s/2}\Gamma(s/2)$.

References

[Sha10] Freydoon Shahidi. *Eisenstein series and automorphic L-functions*, volume 58. American Mathematical Soc., 2010.