ROOT SYSTEM

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1. Introduction

This is my personal study note for root systems from the chapter 2 of Knapp's book [1].

The structure is as follows: Given a complex semisimple algebra $\mathfrak{g}$, the choice of a Cartan algebra gives us an abstract root system Φ, the choice of a set of positive roots of an abstract root system gives us an abstract Cartan matrix A, using the Weyl group $W = W(\Phi)$ of an abstract root system, we can show that the Cartan matrix determines the root system 3.3, we can associate Dynkin diagram to Cartan matrix and we can obtain a classification of connected abstract Dynkin diagram from the properties of abstract Cartan matrix 4.1, finally, based on the Serre relation 4.4, we can establish the isomorphism theorem 4.5 and existence theorem 4.6. Above all, passing from the complex Lie algebra to its root system and the Cartan matrix, then to the Dynkin diagram, although we have to make a choice at each step, we don't loose information, and a classification of simple complex Lie algebras can be obtained from the classification of connected Dynkin diagrams.

Let k be a characteristic zero field, let's note that we can also associate root systems to symmetric varieties over k [2], and to spherical varieties over k [3].

2. Abstract root systems

The abstract root system is a way to axiomize the property of the complex semisimple Lie algebras.

Definition 2.1. An *abstract root system* Φ is a finite-dimensional real inner product space V with inner product $\langle \cdot, \cdot \rangle$ with norm squared $|\cdot|^2$, Δ is a finite set of nonzero elements of V such that

- Δ spans V.
- the orthogonal transformations $s_\alpha(\varphi) = \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|^2}\alpha$ sends Δ to itself.
- $\frac{2\langle \beta, \alpha \rangle}{|\alpha|^2}$ is an integer for $\alpha, \beta \in \Delta$.

an abstract root system is said to be *reduced* if $\alpha \in \Delta$ implies $2\alpha \notin \Delta$.

An abstract root system Φ is said to be *reducible* if Δ admits a nontrivial disjoint decomposition $\Delta = \Delta' \cup \Delta''$ with every member of Δ' orthogonal to every member of Δ''.

Example 2.2. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra, B the Killing form of $\mathfrak{g}$, Δ the set of roots of $\mathfrak{g}$, so we have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, the root space decomposition. The restriction of B on $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate and to each α we can find $H_\alpha \in \mathfrak{h}$ such that $\alpha(H) = B(H, H_\alpha)$.

We let V be the $\mathbb{R}$ span of Δ in $\mathfrak{h}^*$, then the restriction of $\langle \cdot, \cdot \rangle$ makes V into a real inner product space. For any root α, the simple reflection s_α sends Δ into itself and we can show that $\frac{2\langle \beta, \alpha \rangle}{|\alpha|^2} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$.

Hence we see that the root system for complex semisimple Lie algebra is indeed an abstract root system.

We have the following examples of reduced irreducible root systems

- type A_n: vector space $V = \{\Sigma_{i=1}^{n+1} a_i e_i\}$ with $\Sigma a_i = 0$, $\Delta = \{e_i - e_j | \ i \neq j\}$, the corresponding complex Lie algebra is $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$.
- type B_n: vector space $V = \{\Sigma_i^n a_i e_i\}$, $\Delta = \{\pm e_i \pm e_j | \ i \neq j\} \cup \{\pm e_i\}$, the correspnding complex Lie algebra is $\mathfrak{so}(2n+1, \mathbb{C})$.
- type C_n: vector space $V = \{\Sigma_{i=1}^n a_i e_i\}$, $\Delta = \{\pm e_i \pm e_j | \ i \neq j\} \cup \{\pm 2e_i\}$, the corresponding complex Lie algebra is $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$.
- type D_n: vector space $V = \{\Sigma_{i=1}^n a_i e_i\}$, $\Delta = \{\pm e_i \pm e_j | \ i \neq j\}$, the corresponding complex Lie algerba is $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$.

Definition 2.3. We will call a square matrix A a *Cartan matrix*, if it satisfies the following properties:

- A_{ij} is in $\mathbb{Z}$ for all i and j.
- $A_{ii} = 2$ for all i.
- $A_{ij} \leq 0$ for $i \neq j$.
- $A_{ij} = 0$ if and only if $A_{ji} = 0$.
- there exists a diagonal matrix D with positive diagonal entries such that DAD^{-1} is symmetric positive definite.

Example 2.4. We fix an abstract root system Φ and we assume Φ is reduced. Fix a positivity ordering, and let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}, \ell = \text{dim}V$ be the set of simple roots, then we can associate a Cartan matrix $A = \{A_{ij}\}$ to Φ with $A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}$.

Proposition 2.5. *The reduced root system Φ is reducible if and only if for some enumeration of indices, the cartan matrix is block diagonal with more than one block.*

To a reduced abstract root system Φ and Π a set of simple roots, we can associated a graph-Dynkin diagram. We associate to each simple root α_i a vertex of the graph, and we attach to that vertex a weight propotional to $|\alpha_i|^2$, if two simple vertices are given corresponding to distinct roots α_i and α_j, we connect these vertices by $A_{ij}A_{ji}$ edges. The resulting graph is the *Dynkin diagram* of Π.

3. Weyl group

Definition 3.1. Let Φ be an abstract root system in a finite dimensional real vector space V, we let $W(\Phi)$ be the subgroup of the orthogonal group of V generated by the reflections s_α for $\alpha \in \Delta$, this is the *Weyl group* of Φ.

Example 3.2. We have the following examples of Weyl group

(1) For A_n, W consists of all permutations on e_i, $|W| = (n+1)!$. For B_n and C_n W is generated by all permutations of e_i and the sign changes of the coefficients of e_i, $|W| = n!2^n$. For D_n, W consists of all permutations of e_i and even sign changes, $|W| = n!2^{n-1}$.
(2) The Weyl group for the nonreduced root system of type $(BC)_2$ has order 8.
(3) The Weyl group of G_2 is of size 12 consists of 6 rotations through multiples of angles of $\pi/3$ and 6 reflections defined by sending a root to its negative leaving the orthogonal complement fixed.

Given a reduced root system, we can define a Cartan matrix, and we can use the Weyl group to show that a reduced root system is determined by its Cartan matrix up to isomorphism.

Proposition 3.3. *The Cartan matrix determines the reduced root system up to isomorphism.*

Proof. Let's first see that the Cartan matrix determines the set of simple roots up to a linear transformation on V, we may assume Φ is irreducible, and let $\alpha_1, \cdots \alpha_\ell$ be simple roots, the Cartan matrix determines $|\alpha_i|$ up to a common proportionality constant, let $\beta_1, \cdots \beta_\ell$ be another simple system for the same Cartan matrix, normalizing, we may assume $|\alpha_j| = |\beta_j|$ for all j. From the Cartan matrix, we get $\frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2} = \frac{2\langle \beta_i, \beta_j \rangle}{|\beta_i|^2}$ for all i, j, and hence $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle$, in other words, the linear transformation $L\alpha_i = \beta_i$ preserves the inner product on a basis and hence orthogonal.

We want to see that the set $\{\alpha_1 \cdots \alpha_\ell\}$ of simple roots determines the set of roots, let Δ' be another set of roots, then since for any $\alpha \in \Delta'$, we can find $s \in W(\Phi)$ such that $s\alpha_j = \alpha$, for some simple root α_j, we see $\Delta = \Delta'$. □

4. Classification result

We can classify the abstract Cartan matrices from the properties they satisfy and then we can show that every Cartan matrices arise from a reduced abstract root system. Since the Cartan matrix A determines the abstract Dynkin diagram up to a proportionality constant, we have the following classification result stated in terms of Dynkin diagram

Theorem 4.1. *(Classification) Up to isomorphism, the connected abstract Dynkin diagrams are A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4 and G_2.*

Abstract root systems that are not necessarily reduced arise in the structure theory of real semisimple Lie algebras and also the root system for symmetric varieties.

Example 4.2. Forming the union of the root systems B_n and C_n, we get a root system $(BC)_n$ with $V = \{\Sigma_{i=1}^n a_i e_i\}$, $\Delta = \{\pm e_i \pm e_j | \ i \neq j\} \cup \{\pm e_i\} \cup \{\pm 2e_i\}$.

Proposition 4.3. *Up to isomorphism the only irreducible abstract root systems Δ that are not reduced are of the form $(BC)_n$ for $n \geq 1$.*

Theorem 4.4. *(Serre) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $X = \{h_i, e_i, f_i\}_{i=1}^{\ell}$ be a set of canonical generators. Let $\mathfrak{F}$ be the free Lie algebra on 3ℓ generators h_i, e_i, f_i, let $\mathfrak{R}$ be the ideal generated by $\mathfrak{F}$ by Serre relations, then the canonical homomorphism of $\mathfrak{F}/\mathfrak{R} \to \mathfrak{g}$ is an isomorphism.*

We can lift the isomorphisms between root systems to isomorphisms between complex Lie algebras

Theorem 4.5. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex semisimple Lie algebras with corresponding Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ and root systems Φ and Φ', suppose there is a vector space isomorphism $\varphi : \mathfrak{h} \to \mathfrak{h}'$ which induces φ^t and $\varphi^t(\Delta') = \Delta$, fix Π a simple system for Δ, for each $\alpha \in \Pi$ select nonzero vectors $E_\alpha \in \mathfrak{g}$ and $E_{\alpha'} \in \mathfrak{g}'$ for α'. Then exists one and only one Lie algebra isomorphsim $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{g}'$ such that $\tilde{\varphi}|_{\mathfrak{h}} = \varphi$ and $\tilde{\varphi}(E_\alpha) = E_{\alpha'}$ for all $\alpha \in \Pi$.*

The proof of this theorem uses the theorem of Serre 4.4 on the generator relations of complex Lie algebras.

Next, we can show that any reduced root system is the root system of a complex semisimple Lie algebra.

Theorem 4.6. *If $A = (A_{ij})$ is an abstract Cartan matrix, then there exists a complex Lie algebra $\mathfrak{g}$ whose root system has A as Cartan matrix.*

References

[1] Anthony W Knapp and Anthony William Knapp. *Lie groups beyond an introduction*, volume 140. Springer, 1996.

[2] Aloysius G Helminck and Shu Ping Wang. On rationality properties of involutions of reductive groups. *Adv. Math*, 99(1):26–96, 1993.

[3] Friedrich Knop and Bernhard Krötz. Reductive group actions. *arXiv preprint arXiv:1604.01005*, 2016.