PRINCIPAL SERIES REPRESENTATIONS OF SPECIAL UNITARY GROUPS

RUI CHEN

1. Introduction

In this note we study the reducibility for the principal series for $SU(3)$ following the paper [Key84].

F will be a non-Archimedean local field and E a separable quadratic extension of F with Galois automorphism $x \to \overline{x}$. Define $G = SU(3)$ to be the group

$$\{g \in SL_3(E) \mid {}^t g J \overline{g} = J\} \text{ where } J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

let A be the diagonal split torus, and $M = Z_G(A) \cong E^*$ and N the group of upper triangular unipotent matrices of G. $P = MN$ is a minimal parabolic subgroup of G.

Given a character λ of M we can define the principal series $\mathrm{Ind}_P^G \lambda$ via parabolic induction. Let $\mathfrak{a} = \mathrm{Hom}(X(A), \mathbb{Z}) \otimes \mathbb{R}$ be the real Lie algebra of A, with dual $\mathfrak{a}^* = X(A) \otimes \mathbb{R}$. For $\lambda \in \hat{M}$, we set $\lambda_s(x) = \lambda(x)|x|_E^s$.

2. A multiplicity one result

Let $V = {}^t N$ be the lower unipotent subgroup of G, then K_m has Iwahori factorization $V_m M_m N_m,\ m \geq 1$. Now let $\psi_0 \equiv 1$ on K_0, and define ψ_m on $K_m = V_m M_m N_m$ by $\psi_m(vln) = \psi_m(v)$ for $v \in V_m,\ l \in M_m,\ n \in N_m$.

Let $K_0 = K = G(\mathcal{O}_E)$ be the standard maximal compact subgroup of G and set

$$K_m = \{\ g \in G \mid g \equiv I \bmod p_E^m\}$$

for $m \geq 1$.

We want to show

Theorem 2.1. *Suppose π is an irreducible representation of G, then the multiplicity of $\overline{\psi}_m$ in $\pi|_{K_m}$ is 0 or 1.*

Define an algebra of (K_m, ψ_m)-spherical functions, $m \geq 0$ by $\mathscr{S}_m = \{f \in C_c^\infty(G) \mid f(k_1 g k_2) = \psi_m(k_1 k_2) f(g)$ for $k_1,\ k_2 \in K_m,\ g \in G\}$. Theorem 2.1 will follow from

Theorem 2.2. *S_m is a commutative algebra under convolution.*

the case $m = 0$ is well known as this is the spherical Hecke algebra.

3. Intertwining operators and Plancherel measure

Fix a coset representative $\overline{\omega}$ and define the usual intertwining operators between $\mathrm{Ind}_P^G \lambda$ and $\mathrm{Ind}_P^G \omega\lambda$ by

$$A(\overline{\omega}, \lambda) f(g) = \int_{N \cap \omega N \omega^{-1}} f(gn\overline{\omega})\ dn$$

these operators satisfy the cocycle relation

$$A(\overline{\omega}_1 \overline{\omega}_2, \lambda) = A(\overline{\omega}_1, \omega_2 \lambda) A(\overline{\omega}_2, \lambda)$$

provided $l(\omega_1 \omega_2) = l(\omega_1) + l(\omega_2)$.

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We can define a function $f_{m,\lambda,s}$ on VP which is open dense in G and transforms as $\overline{\psi}_m$ under K_m. Note $A(\omega,\lambda_s)f_{m,\lambda,s}$ converges, and is in $\mathrm{Ind}_P^G\omega\lambda_s$, and transform as $\overline{\psi}_m$ under K_m, by the multiplicity one result for $SU(3)$, we have

$$A(\omega,\lambda_s)f_{m,\lambda,s}(g) = \gamma_m(\lambda,s)f_{m,\omega\lambda,\omega s}(g)$$

for all $g \in G$.

In the later section, we will explicitly compute $\gamma_m(\lambda,s)$ for $G = SU(3)$ by setting $g = 1$ in and evaluating the integral

$$\gamma_m(\lambda,s) = \int_N f_{m,\lambda,s}(n\overline{\omega})\, dn$$

for Re $s > 0$ and $m \geq 1$.

For the rank one group $G = SU_3$, we have

$$\mu_\omega(\lambda,s) = \gamma_\omega^2(G|P)[\gamma_m(\omega\lambda,-s)\gamma_m(\lambda,s)]^{-1}$$

4. Reducible principal series of SU(3)

Reducibility of the unitary principal series $\mathrm{Ind}_P^G\lambda$ is determined by the theory of R-group and knowledge of the Plancherel measure. $\mathrm{Ind}_P^G\lambda$ is reducible if and only if $\omega\lambda = \lambda$ and $\gamma_\omega(\omega\lambda,-s)\gamma_m(\lambda,s)$ is holomorphic at $s = 0$. In this case, $\mathrm{Ind}_P^G\lambda$ splits as a sum of two inequivalent irreducible subrepresentations, each with multiplicity one.

The non-unitary principal series $\mathrm{Ind}_P^G\lambda$ is reducible if and only if $\gamma_m(\omega\lambda,-s)\gamma_m(\lambda,s)$ is zero at $s = 0$, we may assume Re $\lambda > 0$, then the kernel of $A(\omega,\lambda)$ is an irreducible invariant subspace of $\mathrm{Ind}_P^G\lambda$ which transforms as a special representation of G.

Theorem 4.1. *Let $G = SU(3)$, $\lambda \in \hat{(E^*)} = \hat{M}$, then*

- *The unitary principal series $\mathrm{Ind}_P^G\lambda$ is reducible if and only if $\lambda \neq 1$, $\omega\lambda = \lambda$ and $\lambda|_F x \equiv 1$.*
- *Suppose $\lambda \in \hat{(E^*)}$ and Re $s > 0$, the reducible non-unitary principal series $\mathrm{Ind}_P^G\lambda_s$ are the following: assume E/F is unramified*
  - *The unramified $\lambda_s(x) = |x|_E^s$ for $s = 1$ or $s = \frac{1}{2} + \pi i(2\ln q)^{-1}$.*
  - *λ ramified degree $h \geq 1$, $\lambda|_F x \equiv 1$ and $s = \frac{1}{2} + \pi i(2\ln q)^{-1}$.*

  now assume E/F is ramified
  - *unramified $\lambda_s(x) = |x|_E$ for $s = 1$.*
  - *λ ramified of degree h $\lambda|_{\mathcal{O}} x$ of order 2 and $s = \frac{1}{2}$.*

Proof. This follows from the previous computation on $\gamma_m(\lambda,s)$. □

5. Computation of some c-functions

Theorem 5.1. *Let $m \geq max\{1,h\}$, for $G = SU(3)$, char$F \neq 2$, then $\gamma_m(\lambda,s)$ are given as follows: first assume E/F is unramified*

- *If λ is unramified $\lambda_s(x) = |x|_E^s$ then*

$$\gamma_m(\lambda,s) = |2|_F^{1-2s}q_F^{-8ms}\frac{1-q^{2s-2}}{1-q^{-2s}}\frac{1+q^{2s-1}}{1+q^{-2s}}$$

- *If λ is ramified of degree $h \geq 1$ and $\lambda|_F\ x \equiv 1$ then*

$$\gamma_m(\lambda,s) = |2|_F^{1-2s}\lambda(\tau)q^{-8ms}q^{h(2s-1)}\frac{1+q^{2s-1}}{1+q^{-2s}}$$

- *If λ is ramified of degree $h \geq 1$, $\lambda|_F x$ ramified of degree $h' \geq 1$ then $\omega\lambda_s \neq \lambda_s$ for any s and $\gamma_m(\lambda,s) =$*

$$|2|_F^{1-2s}\lambda^{-1}(2)q^{-8ms}\lambda(\pi^{4m})\Gamma_E(\lambda(x\overline{x})|x|_E^{2s})\Gamma_E(\lambda_s)\Gamma_F(\lambda_F^{-1}x\cdot|\cdot|_F^{-2s+1})$$

  Now assume E/F is ramified
  - *If $\lambda(x) = |x|_E^s$ is unramified then*

$$\gamma_m(\lambda,s) = |2|_F^{1-2s}q^{-4ms}q^s\frac{1-q^{s-1}}{1-q^{-s}}$$

– *If λ is ramified of degree $h \geq 1$ and $\lambda|_F\ x \equiv 1$ then*

$$\gamma_m(\lambda, s) = |2|_F^{1-2s} q^{-4ms} \lambda(\tau) q^s q^{h/2(2s-1)}$$

– *If λ is ramified of degree $h \geq 1$ and $\lambda|_{\mathcal{O}_F}\ x$ has order 2 then*

$$\gamma_m(\lambda, s) = |2|_F^{1-2s} \lambda^{-1}(2) q^{-4ms} c_1 q^{h(s-\frac{1}{2})} c_2 \times q^{\frac{1}{2}-2s} \frac{1-q^{2s-1}}{1-q^{-2s}}$$

– *If λ is ramified of degree $h \geq 1$ and $x \mapsto \lambda(x\overline{x})$ is ramified on $E^\times$ of degree h' then $\gamma_m(\lambda, s) = |2|_F^{1-2s} \lambda^{-1}(2) q^{-4ms} \lambda(\pi\overline{\pi})^{2m}$.*

$$\Gamma_E(\lambda(x\overline{x})| \cdot |_E^{2s}) \Gamma_E(\lambda_s) \Gamma_F(\lambda^{-1}|_F x \cdot | \cdot |_F^{-2s+1})$$

This theorem is proved by computing the unipotent integral

$$\begin{aligned}
\gamma_m(\lambda, s) &= \int_N f_{m,\lambda,s}(n\overline{\omega})\ dn \\
&= \int f_{m,\lambda,s}(\begin{pmatrix} 1 & x & y \\ & 1 & -\overline{x} \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}) \\
&= \int f_{m,\lambda,s} \begin{pmatrix} 1 & & \\ -\frac{\overline{x}}{y} & 1 & \\ \frac{1}{y} & \frac{x}{\overline{y}} & 1 \end{pmatrix} \lambda_s^{-1} \delta^{-\frac{1}{2}}(y)
\end{aligned}$$

the difficulty is that the integral is over those $x, y \in E$ satisfying $y + \overline{y} + x\overline{x} = 0$.

6. R-groups for SU(3)

Recall that $R = \{\omega \in W_\lambda \mid \alpha \in \Delta' \text{ and } \alpha > 0 \text{ imply that } \omega\alpha > 0\}$

Theorem 6.1. *For $G = SU(3)$, $R \cong \mathbb{Z}_2$. Explicitly, any λ with non-trivial R-group is conjugate to a character λ defined by specifying distinct characters λ of $E^\times$ satisfying $\lambda(x\overline{x}) = 1$, $\lambda \neq 1$ and $\lambda|_{F^\times} = 1$.*

The commuting algebra of $\mathrm{Ind}_P^G \lambda$ is given as the group algebra $\mathbb{C}[R]$, thus $\mathrm{Ind}_P^G \lambda$ decomposes into $|R| \leq 2$ irreducible inequivalent components.

References

[Key84] David Keys. Principal series representations of special unitary groups over local fields. *Compositio mathematica*, 51(1):115–130, 1984.