PRIMITIVE WONDERFUL VARIETIES

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1. Introduction

This is a summary of the results of the papers [Lun01], [Bra13], [BP16]. We first recall Luna's result on classification of type A wonderful variwties and we review the result of Bravi and Pezzini on the classification of general wonderful varieties, they completed the proof of the Luna conjecture-isomorphism classes of wonderful varieties are classified by spherical systems. A relevant byproduct of the proof of the Luna conjecture is an explicit description of a generic stabilizer of a wonderful variety using only its spherical system.

The main theorem is

Theorem 1.1. *The map $X \mapsto \mathcal{S}_X$ induces a bijection between the G-isomorphism classes of wonderful G-varieties and spherical G-systems.*

2. Luna's reduction

In this section G will be an adjoint group of type A over an algebraically closed field k.

Let me give a brief summary of Luna's reduction for wonderful varieties of type A: Luna introduced 29 families of primitive spherical systems for adjoint groups of type A, here we note Luna doesn't have a general definition of the primitive spherical systems. Based on the following proposition, Luna reduced the proof for geometric realization of general spherical systems to primitive spherical systems.

We introduce the definition of distinguished subset of colors, let $(S^p, \Sigma, \mathcal{A})$ be a spherical system and $\rho : \Delta \to \mathscr{V} \subset \chi^* \otimes \mathbb{Q}$ vector space of colors

Definition 2.1. We say a subset $\Delta' \subseteq \Delta_X$ is distinguished if $-\mathscr{V}$ intersects the relative interior E_0 of the cone E generated by $\rho(D)$ for $D \in \Delta'$.

Proposition 2.2. *A subset of colors $\Delta' \subset \Delta$ is distinguished if and only if for all $D \in \Delta'$ there exists a positive integer a_D such that $\sum_{D\in\Delta'} a_D\langle\rho(D), \sigma\rangle \geq 0$ for all $\sigma \in \Sigma$.*

This is because Σ is the set of generators of $\mathscr{V}^\perp = \{\gamma \in \chi \mid \langle\gamma, \sigma\rangle \leq 0 \text{ for } \sigma \in \mathscr{V}\}$.

Definition 2.3. Let N be a vector subspace of χ^* and $\Delta' \subseteq \Delta_X$, then the couple (N, Δ') is a colored subspace of χ^* if N is finitely generated as a convex cone by finitely many elements of $\mathscr{V}$ and elements $\rho(D)$ with $D \in \Delta'$.

There is a bijection between spherical subgroups K with $H \subseteq K$ such that K/H is connected and the set of colored subspaces of χ^*.

We have a characterization of Δ' being distinguished

Proposition 2.4. *Let X be a wonderful variety and $\Delta' \subseteq \Delta_X$, then Δ' is distinguished if and only if there exists a unique vector subspace $\mathscr{C}$ of χ^* such that $(\mathscr{C}, \Delta')$ is a colored subspace and the image of $\mathscr{V}$ in $N/\mathscr{C}$ is strictly convex.*

Luna defined the following notion of quotient of spherical systems by distinguished subset

Definition 2.5. If $(S^p, \Sigma, \mathcal{A})$ is a spherical system and Δ' is a distinguished subset of Δ, we define χ/Δ' to be the element of χ which is annihilated by $N(\Delta')$. We define the quotient of the spherical system as: $(S^p, \Sigma, \mathcal{A})/\Delta'$

Date: May 2024.

- $S^p/\Delta' = \{\alpha \in \Sigma_G, \Delta(\alpha) \subset \Delta'\}$.
- Consider the linear combinations $\sum_{\gamma\in\Sigma} n(\gamma)\gamma \in \chi/\Delta'$, Σ/Δ' are indecomposable elements of this semigroup.
- we define $\mathcal{A}/\Delta'$ as the union of all $\mathcal{A}(\alpha)$ such that $\mathcal{A}(\alpha) \cap \Delta' = \emptyset$, and we define $\rho/\Delta' : \mathcal{A}/\Delta' \to (\chi/\Delta')^*$ as a restriction of ρ to $\mathcal{A}/\Delta'$ under the natural map $\chi^* \to (\chi/\Delta')^*$.

The following proposition is a result of Bravi, the second part of the proposition corresponds to the property $(*)$ introduced in Luna's paper section 3.3 on the discussion of quotient of spherical systems.

Proposition 2.6. *(*[Bra13] *theorem 3.1) If $\Delta' \subset \Delta$ is distinguished then:*

- *the monoid $\{\sigma \in \mathbb{N}\Sigma \mid \langle \rho(D), \sigma\rangle$ for all $D \in \Delta'\}$ is free.*
- *setting $S^p/\Delta' = \{\alpha \mid \Delta(\alpha) \subset \Delta'\}$, then Σ/Δ' equal to the basis of the above monoid and $\mathcal{A}/\Delta' = \cup_{\alpha\in S\cap\Sigma/\Delta'}\mathcal{A}(\alpha)$, the triple $(S^p/\Delta', \Sigma/\Delta', \mathcal{A}/\Delta')$ is a spherical G-system.*

Definition 2.7. A spherical system $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ is called cuspidal if $\mathrm{supp}\Sigma = \Sigma_G$.

Let $f : X \to Y$ be a surjective G-morphism with connected fibers between wonderful G-varieties, then the subset $\Delta_f := \{D \in \Delta_X \mid f(D) = Y\}$ is distinguished, and $\mathcal{S}_Y = \mathcal{S}_X/\Delta_f$. The following holds:

Proposition 2.8. *Let X be a wonderful G-variety, then the map $f \mapsto \Delta_f$ induces a bijection between distinguished subsets of Δ_X and G-isomorphism classes of surjective G-morphisms with connected fibers from X onto another wonderful G-variety.*

This proposition will be very helpful for us to study the Galois descent properties of morphisms between wonderful varieties.

Remark 2.9. Let X, Y_1, Y_2 be three wonderful varieties, we say two G-morphisms $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are G-isomorphic if there exists a G-morphism $\varphi : Y_1 \to Y_2$ such that $\varphi \circ f_1 = f_2$.

We have the following proposition concerning the parabolic induction.

Proposition 2.10. *Let X be a wonderful G-variety, there exists a unique G-morphism $\phi : X \to G/G_{-S'}$ induces a parabolic induction structure on X if and only if supp$(\Sigma) \cup S^p \subset S'$, and Δ_ϕ is set of colors $\Delta(S') = \cup_{\alpha\in S'}\Delta(\alpha)$.*

The existence of the morphism Δ_ϕ follows from the proposition 2.8.

Proposition 2.11. *Let X be a wonderful variety G variety and the spherical system $\mathcal{S}$ of X is indecomposable, then X is cuspidal-can't be obtained as parabolic induction from proper parabolic subgroups if and only if the spherical system of X is cuspidal. Here if $X = G/G$ is a point $S^p = \Sigma_G$, we set X as not cuspidal.*

Proof. Suppose $\mathcal{S}$ is cuspidal, then it is clear that X is cuspidal.

Conversely, assume X is cuspidal, then according to the proposition 2.10, we only need to discuss the case $S^p \cup \mathrm{supp}\Sigma = \Sigma_G$. In this case, $\mathrm{supp}\Sigma$ and $S^p \backslash \mathrm{supp}\Sigma$ are orthogonal hence $G \cong G_{\mathrm{supp}\Sigma} \times G_{S^p\backslash\mathrm{supp}\Sigma}$ and X decompose as $X_1 \times X_2$ with X_1 a wonderful variety for $G_{\mathrm{supp}\Sigma}$ and X_2 a wonderful variety for $G_{S^p\backslash\mathrm{supp}\Sigma}$. If $\mathrm{supp}\Sigma \neq \emptyset$ and $S^p\backslash\mathrm{supp}\Sigma \neq \emptyset$, then $\mathcal{S}$ is decomposable, a contradiction. So we have $\mathrm{supp}\Sigma = \emptyset$ or $S^p\backslash\mathrm{supp}\Sigma = \emptyset$. In the first case, we have $S^p = \Sigma_G$ and X is a point in this case, and X is not cuspidal, a contradiction. In the second case, we have $\mathrm{supp}\Sigma = \Sigma_G$ and $\mathcal{S}$ is cuspidal. ☐

Definition 2.12. Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system, let $D \in \mathcal{A}$ we say D is a projective color of $\mathcal{A}$ if $\langle \rho(D), \gamma\rangle \in \{0, 1\}$ for all $\gamma \in \Sigma$.

for D a projective element, there is a G-morphism $\phi_D : X \to X_D$ corresponds to D with fiber projective space.

Luna also introduced the fiber product of two spherical systems based on the property $*$, we will recall the equivalent definition given by Bravi-Pezzini later.

Luna also introduced the notion of Δ-connected

Definition 2.13. Let $(S^p, \Sigma, \mathcal{A})$ be a spherical system and Δ denote its colors, for $\gamma \in \Sigma$, denote $\Delta(\gamma)$ the union of $\Delta(\alpha)$, $\alpha \in \mathrm{supp}(\gamma)$. We will say that two elements $\gamma_1, \gamma_2 \in \Sigma$ are strongly Δ-connected if for all $D \in \Delta(\gamma_1)$, we have $\langle \rho(D), \gamma_2\rangle \neq 0$ and for all $D \in \Delta(\gamma_2)$ we have $\langle \rho(D), \gamma_1\rangle \neq 0$.

We will say that $\gamma_1, \gamma_2 \in \Sigma$ are Δ-neighbors if

- Either they are strongly connected.
- Or there exists $\gamma_3 \in \Sigma$ such that the system obtained by localization in supp$(\{\gamma_1, \ \gamma_2, \ \gamma_3\})$ is isomorphic to $ax(1+q+1)$ for $q \geq 1$.

A subset $\Sigma' \subset \Sigma$ is Δ-connected (resp. strongly Δ-connected) if the two of any elements in Σ' can be joined by a sequence of elements of Σ, and any two successive elements in the sequence are Δ-neighbors (resp. strongly Δ-neighbors). A Δ-connected component of Σ is a maximal Δ-connected subset.

Proposition 2.14. *Let $\mathcal{S}$ be a spherical system and suppose $\mathcal{S}$ is Δ-connected, cuspidal and $\mathcal{S}$ doesn't have projective colors then $\mathcal{S}$ is primitive.*

We will see a general definition of the primitive spherical systems for all adjoint groups in next section 3.11.

3. Primitive spherical systems

In this section, G will be a connected adjoint group.

Bravi and Pezzini introduced the following definition of localization of spherical systems

Definition 3.1. ([BP16] definition 2.1.1) Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical G-system. For all subsets of simple roots $S' \subset \Sigma_G$, consider a semi-simple group $G_{S'}$ with set of simple roots S'. We define the localization $\mathcal{S}_{S'}$ of $\mathcal{S}$ in S' as the spherical $G_{S'}$-system $((S')^p, \Sigma', \mathcal{A})$ as

- $(S')^p = S^p \cap S'$.
- $\Sigma' = \{\sigma \in \Sigma \mid \text{supp}\sigma \subseteq S'\}$.
- $\mathcal{A}' = \cup_{\alpha \in \Sigma_G \cap \Sigma'} \mathcal{A}(\alpha)$.

There is a geometric counter part of the localization of spherical system

Proposition 3.2. *(*[BP16] *proposition 2.1.3) For all subsets of simple roots $S' \subset \Sigma_G$, we define the localization $X_{S'}$ of X in S' to be the subvariety X^{P^r} of points fixed by the radical P^r of P, here P is the parabolic subgroup containing B_- and corresponding to S'.*

Under the action of $G_{S'} = P/P^r$ the variety $X_{S'}$ is wonderful and

$$\mathcal{S}_{X_{S'}} = (\mathcal{S}_X)_{S'}$$

The following proposition discusses the compatibility of geometric realization with localization of spherical systems.

Proposition 3.3. *Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical G-system and let S' be a subset of simple roots containing suppΣ, if there exists a wonderful $G_{S'}$ with spherical system $\mathcal{S}_{S'}$, then there exists a wonderful G-variety with spherical system $\mathcal{S}$.*

There is another localization at spherical roots by taking G-stable variety of a wonderful G-variety.

Definition 3.4. ([BP16] definition 2.1.7) Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A}_\Sigma)$ be a spherical G-system and Σ' a subset of Σ. The *localization* of $\mathcal{S}$ in Σ' is the spherical system $\mathcal{S}_{\Sigma'} = (S^p, \Sigma', \mathcal{A}_{\Sigma'})$ where

$$\mathcal{A}_{\Sigma'} = \cup_{\alpha \in \Sigma_G \cap \Sigma'} \Delta(\alpha)$$

and the Cartan pairing of $\mathcal{S}_{\Sigma'}$ is given by the restriction from the Cartan pairing of $\mathcal{S}$.

Proposition 3.5. *(*[BP16] *proposition 2.1.8) Let X be a wonderful G-variety and $\Sigma' \subseteq \Sigma_X$, then X has a unqiue irreducible G-stable closed subvariety Y with spherical roots Σ' and $\mathcal{S}_Y = \mathcal{S}_{\Sigma'}$.*

Definition 3.6. ([BP16] definition 2.2.2) Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical G-system with set of colors Δ, two distinguished sets of colors Δ_1, Δ_2 are said to decompose $\mathcal{S}$ if

- $S^p/\Delta_1 \cap S^p/\Delta_2$ is equal to S^p.
- every connected component of $S^p/(\Delta_1 \cup \Delta_2)$ is contained in either S^p/Δ_1 or S^p/Δ_2.
- $\Sigma \subset \Sigma/\Delta_1 \cup \Sigma/\Delta_2$.

If $\mathcal{S}$ admits two non-empty subsets of colors then $\mathcal{S}$ is called *decomposable.*

Proposition 3.7. *(*[BP16] *proposition 2.3.4) Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical G-system with set of colors Δ, and let Δ_1, Δ_2 be distinguished subsets of Δ that decompose $\mathcal{S}$. If X_1 and X_2 are wonderful G-varieties with spherical systems $\mathcal{S}/\Delta_1$ and $\mathcal{S}/\Delta_2$ with surjective morphisms into wonderful G-variety X_3 with spherical G-system $\mathcal{S}/(\Delta_1 \cup \Delta_2)$, then the G-variety $X_1 \times_{X_3} X_2$ is a wonderful variety with spherical system $\mathcal{S}$.*

Bravi and Pezzini introduced the notion of positive comb which is a refinement of Luna's notion of projective color 2.12

Definition 3.8. Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical G-system, a *positive comb* of $\mathcal{S}$ is an element $D \in \mathcal{A}$ such that $\langle \rho(D), \sigma \rangle \geq 0$ for all $\sigma \in \Sigma$, and it is called an n-comb if $n = \text{card}\{\alpha \in \Sigma_G \cap \Sigma \mid \langle \rho(D), \alpha \rangle = 1\}$.

For D a positive comb, we set $S_D = \{\alpha \in \Sigma_G \cap \Sigma \mid \langle \rho(D), \alpha \rangle = 1\}$, then for $\alpha \in S_D$, we can define $\mathcal{S}_\alpha = (S^p, \Sigma_\alpha, \mathcal{A}_\alpha)$ where $\Sigma_\alpha = \Sigma \backslash (S_D \backslash \{\alpha\})$ and $\mathcal{A}_\alpha = \cup_{\beta \in \Sigma_G \cap \Sigma_\alpha} \mathcal{A}(\beta)$, the spherical system $\mathcal{S}_\alpha$ has a positive 1-comb $\mathcal{A}_\alpha(\alpha)$.

Proposition 3.9. *Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical G-system with a positive n-comb D, if for all $\alpha \in S_D$, there exists a wonderful G-variety with spherical system $\mathcal{S}_\alpha$, then there exists a wonderful G-variety with spherical system $\mathcal{S}$.*

Bravi and Pezinni also introduced a notion of tail [BP16] definition 2.4.1 as a subset $\tilde{\Sigma} \subset \Sigma$.

Proposition 3.10. *Let $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ be a spherical G-system with a tail $\tilde{\Sigma}$, set $S' = supp(\Sigma \backslash \tilde{\Sigma})$. If there exists a wonderful $G_{S'}$-variety with spherical system $\mathcal{S}_{S'}$ then there exists a wonderful G-variety with spherical system $\mathcal{S}$.*

They introduced the notion of the primitive spherical system

Definition 3.11. A spherical G-system is *primitive* if it is cuspidal, indecomposable without positive combs and without tails. Correspondingly, a wonderful variety is *primitive* if the spherical system is.

A positive 1-comb of a spherical G-variety $\mathcal{S}$ is called *primitive* if $\mathcal{S}$ is cuspidal, indecomposable and without tails.

Based on the proposition 3.3 on localization, the proposition 3.7 on fiber product, the proposition on the n-comb 3.9, the proposition on tail 3.10, the proof of the theorem 1.1 is reduced to the case of primitive spherical systems and all spherical systems with a positive 1-comb.

References

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