PLANCHEREL FORMULA FOR $SL_2(\mathbb{R})$

RUI CHEN

Contents

1. Introduction

This is a study note for Plancherel formula for $SL_2(\mathbb{R})$ following Varadarajan's book.

2. Plancherel formula for $SL_2(\mathbb{C})$

For $G = SL_2(\mathbb{C})$, let's denote H the maximal diagonal torus, we have a map $\hat{H} \to \mathrm{Ch}(G)$ and the fibers are precisely the orbits for Weyl group, then the Plancherel formula becomes

$$f(1) = \int_{\hat{H}} \hat{f}(\chi)\mu(\chi)\ d\chi$$

where μ is a Weyl group invariant nonnegative continuous function. An explicit formula for μ together with the explicit formula for Θ_χ will be a complete and far-reaching generalization of the compact theory.

We will introduce the orbital integrals on G and its Lie algebra. Let H be the diagonal matrices and $\mathfrak{h} \subset \mathfrak{g}$ its Lie algebra, let $\mathfrak{h}' \subset \mathfrak{h}$ be the elements with distinct diagonal entries.

For V an Euclidean space, we will denote $\mathcal{S}_0(V)$ the space of linear functions that

$$\sup_{v \in V}(1 + ||v||^2)^q |f(v)| < \infty$$

$\mathcal{S}(V)$ the space of C^∞ functions such that all derivatives are in $\mathcal{S}_0(V)$.

Definition 2.1. For $f \in \mathcal{S}_0(\mathfrak{g})$, the orbital integral of f on $\mathfrak{g}$ is defined as

$$\psi_f(X) = \pi(X) \int_{G/H} f(xXx^{-1}) d(G/H)$$

for $X \in \mathfrak{h}'$, here $\pi(X) = x_1 - x_2$.

We have the conjugacy map

$$\varphi : G/H \times \mathfrak{h}' \longrightarrow \mathfrak{g}', \quad xH, X \mapsto xXx^{-1}$$

φ is proper and all its fibers have cardinality 2, we have the following formula

$$\varphi^* \omega_{\mathfrak{g}} = \pm |\pi(X)|^4 d(G/H) d\mathfrak{h}$$

We have the following Lie algebra version of the Weyl integration formula

Lemma 2.2. *For all $f \in \mathcal{S}_0(\mathfrak{g})$*

$$\int_{\mathfrak{g}} f d\mathfrak{g} = \frac{1}{2} \int_{\mathfrak{h}'} |\pi(X)|^2 \psi_f(X)\ d\mathfrak{h}$$

Date: February 2024.

For any $g \in \mathcal{S}(\mathfrak{g})$, we can define the Fourier transform

$$\hat{\mathfrak{g}}(Y) = \int_{\mathfrak{g}} g(Z) e^{i\langle Y,Z\rangle} d\mathfrak{g}(Z)$$

we can also define Fourier transform on $\mathfrak{h}$

$$\hat{h}(Y) = \int_{\mathfrak{h}} e^{i\langle Y,Z\rangle} h(Z)\ d\mathfrak{h}(Z)$$

Proposition 2.3. *We have the following*

$$\psi_{\hat{f}}(X) = (2\pi)\hat{\psi}_f(X)$$

We can introduce a differential operator Δ on $\mathcal{S}(\mathfrak{g})$ and it satisfies

$$(-1)(\widehat{\Delta\overline{\Delta}u})(Y) = |\pi(Y)|^2 \hat{u}(Y)$$

Theorem 2.4. *(limit formula for $\mathfrak{g}$) Let*

$$\psi_f(X) = |\pi(X)|^2 \int_{G/H} f(xXx^{-1}) d(G/H)$$

$X \in \mathfrak{h}'$, then for all $f \in \mathcal{S}(G)$

$$f(0) = \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2(2-1)} \cdot 2!} (\Delta\overline{\Delta}\psi_f)(0)$$

for $X \in \mathfrak{h}'$.

Proof. We have for $f \in \mathcal{S}(\mathfrak{g})$

$$\begin{aligned} f(0) &= \frac{1}{(2\pi)^{2(2^2-1)}} \int \hat{f}\ d\mathfrak{g} \\ &= \frac{1}{(2\pi)^{2(2^2-1)} 2!} \int |\pi|^2 \psi_{\hat{f}}\ d\mathfrak{h} \\ &= \frac{1}{(2\pi)^{2(2-1)+2^2-2} \cdot 2!} \int |\pi|^2 \hat{\psi}_f\ d\mathfrak{h} \\ &= \frac{1}{(2\pi)^{2(2-1)+2^2-2} \cdot 2!} \int (\widehat{\Delta\overline{\Delta}\psi_f})\ d\mathfrak{h} \\ &= \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2^2-2} \cdot 2!} (\Delta\overline{\Delta}\psi_f)(0) \end{aligned}$$

where we used the Weyl integration formula 2.2, the relation between the Fourier transform on $\mathfrak{g}$ and $\mathfrak{h}$ 2.3. ☐

Proposition 2.5. *We define $\mathfrak{g}(\epsilon) = \{u \in \mathfrak{g} |\ |\lambda| < \epsilon$ for all eigenvalues λ of $u\}$, $G(\epsilon) = \{z \in G |\ |\lambda| < \epsilon$ for all eigenvalues λ of $z - 1\}$, then $\mathfrak{g}(\epsilon)$ (resp. $G(\epsilon)$) form a basis for the family of invariant open neighborhoods of 0 in $\mathfrak{g}$ (resp. 1) in G, and if $\epsilon > 0$ is sufficiently small, exp is a diffeomorphism of $\mathfrak{g}(\epsilon)$ onto an invariant open neighborhood of 1 in G.*

We can recover $f(1)$ from $F_f(1)$

Theorem 2.6. *Define F_f by*

$$F_f(h) = |\Delta(h)|^2 \int_{G/H} f(xhx^{-1})\ d(G/H)$$

where $dG = d\mathfrak{g}$, $dH = d\mathfrak{h}$ then

$$f(1) = \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2^2-2} \cdot 2!} (\Delta\overline{\Delta}F_f)(1)$$

Proof. We can find $j \in C^\infty(G)$, for sufficiently small ϵ, we have $jf \in C_c^\infty(G)$, supp$jf \subset \exp(\mathfrak{g}(\epsilon))$, $jf = f$ in an invariant neighborhood of 1. We set $g(Z) = f(\exp Z)$, then $g \in C_c^\infty(\mathfrak{g})$, for $X \in \mathfrak{g}(\epsilon) \cap \mathfrak{h}'$, $h = \exp X \in H' \cap \exp(\mathfrak{g}(\epsilon))$, for such X

$$F_f(\exp\, X) = |\Delta(\exp\, X)|^2 \int_{G/H} f(x \exp X x^{-1}) d(G/H) = |\omega(X)|^2 \psi_{\mathfrak{g}}(X)$$

for $\omega(X) = \frac{\Delta(X)}{\pi(X)},\ X \in \mathfrak{h}'$.

We can find an invariant entire function ω_1 on $\mathfrak{g}$ which restricts to $\omega(X)^2$. For sufficiently small ϵ, $|\omega_1|$ is a real-analytic function, $|\omega_1|\mathfrak{g} \in C_c^\infty(\mathfrak{g}(\epsilon))$ for all $\mathfrak{g} \in C_c^\infty(\mathfrak{g}(\epsilon))$, and we have

$$F_f(\exp X) = \psi_{|\omega_1|g} X$$

for $X \in \mathfrak{h}' \cap \mathfrak{g}(\epsilon)$. Applying $\Delta\overline{\Delta}$ to both sides and let $X \to 0$, we get 2.6 from 2.4 as $|\omega_1|(0) = 1$ and

$$g(0) = \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2(2-1)} 2!} (\Delta\overline{\Delta}\psi_g)(0)$$

□

Lemma 2.7. *Let μ be the function on $\hat{H}$ given by*

$$\mu_{\chi_{m_1,m_2;i\rho_1,i\rho_2}} = [(m_1 - m_2)^2 + (\rho_1 - \rho_2)^2]$$

where $\chi_{m_1,m_2;i\rho_1,i\rho_2}$ is the character

$$diag(z_1, z_2) \mapsto \prod_{j=1}^{2} (z_j/|z_j|^{m_j})|z_j|^{i\rho_j}$$

the m_i are integers and ρ_j are real numbers with $\sum m_j = 0$ and $\sum \rho_j = 0$, then for any $f \in C_c^\infty(H)$, we have

$$(-1)^{2(2-1)/2} (\Delta\overline{\Delta}f\hat{)}(x) = \mu(\chi)\hat{f}(\chi)$$

Theorem 2.8. *(Plancherel formula for $SL_2(\mathbb{C})$) For any character χ of H and let T_χ be the distribution character of the principal series L_χ, computed with respect to the Haar measure $dG = d\mathfrak{g}$, let $d\chi$ be the measure on $\hat{H}$, then for any $f \in \hat{H}$, we have*

$$f(1) = \frac{1}{(2\pi)^{2^2-2} 2!} \int_{\hat{H}} T_\chi(f)\mu(\chi)\ d\chi$$

where

$$\mu_{\chi_{m_1,m_2;i\rho_1,i\rho_2}} = [(m_1 - m_2)^2 + (\rho_1 - \rho_2)^2]$$

Proof. We have by theorem 2.6,

$$\begin{aligned}
f(1) &= \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2^2-2} \cdot 2!} (\Delta\overline{\Delta}F_f)(1) \\
&= \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2^2-2} \cdot 2!} \int_{\hat{H}} (\Delta\overline{\Delta}F_f\hat{)}(\chi)\ d\chi \\
&= \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2^2-2} \cdot 2!} \int_{\hat{H}} \hat{F}_f(\chi)\mu(\chi)\ d\chi \\
&= \frac{(-1)^{2(2-1)/2}}{(2\pi)^{2^2-2} \cdot 2!} \int_{\hat{H}} T_\chi(f)\mu(\chi)\ d\chi
\end{aligned}$$

□

The key ingredient of this method is that *there is a single conjugacy class of Cartan subgroups.* In the general case, where there are several conjugacy classes of Cartan subgroups, the proof of the Plancherel formula becomes very difficult.

3. Plancherel formula for $SL_2(\mathbb{R})$

3.1. **Unitary representations of $SL_2(\mathbb{R})$ and their characters.** Let $G = SL_2$, $K = SO_2 = \{u_\theta = \begin{pmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{pmatrix}\}$ the maximal compact, $B = MAN$ the Levy-Langlands decomposition, $M = \{1, \gamma\}$, $\gamma = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \{a_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}\}$, $H = MA$.

Definition 3.1. A Cartan subgroup is the group of real points of a maximal torus of $G_{\mathbb{C}}$ defined over $\mathbb{R}$. They are precisely the centralizers of the regular elements.

The principal series of G are $\pi_{\epsilon,\lambda} = \mathrm{Ind}_B^G \chi$ with $\chi = (\epsilon, \lambda),\ \epsilon = 0, 1 \in \hat{M},\ \lambda \in \mathbb{C} = \hat{A}$

$$\chi_{\epsilon,\rho} :\ \mathrm{diag}(a, a^{-1}) \mapsto \mathrm{sign}(a)^\epsilon |a|^\rho$$

We denote $\hat{\bullet}$ the abelian Fourier transform on $H = MA$, then

Theorem 3.2. *Let χ be any character of H, $\chi = (\epsilon, \eta)$, for any $f \in C_c^\infty(G)$, we have*

$$T_\chi(f) = \hat{F}_{f,H}(\chi)$$

here

$$F_{f,H}(h) := |\Delta(h)|_{\mathbb{R}} \int_{G/H} f(xhx^{-1})\ d\dot{x}$$

here $\Delta(diag(a, a^{-1})) = |a - a^{-1}|$.

We denote G'_H as all $x \in G$ with distinct eigenvalues in $\mathbb{R}$. We have the following explicit formula for T_χ

Theorem 3.3. *Let χ be a quasicharacter of H, then there is a unique invariant function θ_χ on G which is zero outside G'_H and coincides with*

$$(\sum_{\omega \in W} \chi^\omega) / \ |\Delta|_{\mathbb{R}}$$

on H', locally integrable on G and is the character of the principal series π_χ

$$T_\chi(f) = \int_G \theta_\chi f\ dx$$

Proof. We omit the proof for the existence of θ_χ with desired property, we show that θ_χ is the character of $\pi_\chi = \mathrm{ind}_B^G \chi$.

Since θ_χ is outside G'_H, from Weyl integration formula

$$\int_{G'_H} \theta_\chi f\ dx = \frac{1}{2} \int \int_{G/H \times H'} \theta_\chi(xhx^{-1}) f(xhx^{-1}) |\Delta(h)|^2_{\mathbb{R}}\ dxdh$$

use $F_{\theta_\chi f,H} = |\Delta(h)|_{\mathbb{R}} \int_{G/H} \theta_\chi f(xhx^{-1})\ dx$, we have

$$\begin{aligned} \int_G \theta_\chi f\ dx &= \frac{1}{2} \int_H |\Delta|_{\mathbb{R}} F_{\theta_\chi f,H}\ dh \\ &= \frac{1}{2} \int_H (\sum_\omega \chi^\omega) F_{f,H}\ dh \\ &= \int F_{\chi,H} \chi\ dh \\ &= \hat{F}_{f,H}(\chi) \\ &= T_\chi(f) \end{aligned}$$

here we used that $\theta_\chi = \frac{\sum_\omega \chi^\omega}{|\Delta|_{\mathbb{R}}}$ on G'_H and $F_{\theta_\chi f,H}(h) = \theta_\chi(h) F_{f,H}(h)$. □

More explicitly, we have

$$\theta_\chi(\mathrm{diag}(a,a^{-1})) = \frac{\chi(a)+\chi(a^{-1})}{|a-a^{-1}|}$$

We also have the character formula for discrete series-the unitary representations with matrix coefficients square-integrable on G

Theorem 3.4. *Let's denote $\chi_n : u_\theta \mapsto e^{in\theta}$ the characters of K, there is a discrete series representation associated with χ_n for $n \neq 0$ and its character Θ_χ is a locally integrable function with the following formula*

$$\Theta_n(u_\theta) = -sgn(n)\frac{e^{in\theta}}{e^{i\theta}-e^{-i\theta}},\ (\theta \neq 0,\pi)$$

$$\Theta_n(diag(a,a^{-1})) = sgn(a)\frac{a^{-|n|}}{|a-a^{-1}|}\ (|a|>1)$$

These formula obtained by Harish-Chandra, strongly suggested that the discrete series representations associated with the compact torus K in the same way as the principal series associated with H the split torus.

3.2. **Orbital integrals.**

3.2.1. *Orbital integrals for hyperbolic elements.*

Definition 3.5. For L the Cartan subgroup, we define the orbital integral

$$F_{f,L}(ma_t) := \frac{1}{2}|e^t - e^{-t}|\int_{G/A} f(xma_tx^{-1})d(G/A)$$

then use $dG = \frac{1}{2}e^{2t}d\theta ds dt$, we can show

$$F_{f,L}(a_t) = \frac{1}{4}e^t\int \overline{f}(a_tn_s)ds$$

for $\overline{f}(x) = \int f(u_\theta x u_{-\theta})d\theta$, hence $f \mapsto F_{f,L}$ extends to a continuous map of $C_c^\infty(G) \to C_c^\infty(L)$.

Letting $t \to 0\pm$, and note $F_{f,L}$ is an even function, we obtain

Theorem 3.6. *We have*

- $F_{f,L}(1) = \frac{1}{4}\int_{-\infty}^{\infty}\overline{f}(n_s)ds$
- $F'_{f,L}(1) = 0.$

We can also define the orbital integral for Lie algebra.

Definition 3.7. We define

$$\psi_{f,\mathfrak{a}}(tH) = 2|t|\int_{G/L} f(x(tH)x^{-1})d(G/L)$$

for $t \neq 0$.

We have the Fourier transform on $\mathfrak{g}$ and $\mathfrak{h}$

Definition 3.8. For $g \in \mathcal{S}(\mathfrak{g})$ and $h \in \mathcal{S}(\mathfrak{a})$, we define

$$\hat{g}(u) = \int_{\mathfrak{g}} g(v)e^{i\langle u,v\rangle}d\mathfrak{g}$$

$$\hat{h}(u) = \int_{\mathfrak{a}} h(v)e^{i\langle u,v\rangle}d\mathfrak{a}$$

where $\langle u,v\rangle = \frac{1}{2}\mathrm{tr}(uv)$.

Theorem 3.9. *The map $f \to \psi_{f,\mathfrak{a}}$ from $\mathcal{S}(\mathfrak{g})$ to $\mathcal{S}(\mathfrak{a})$ is continuous and*

- $\psi_{f,\mathfrak{a}}(0) = \frac{1}{4}\int_{-\infty}^{\infty}\overline{f}(sX)ds$ *and* $\psi'_{f,\mathfrak{a}}(0) = 0.$
- $\psi_{\hat{f},\mathfrak{a}}(tH) = 2\pi\hat{\psi}_{f,\mathfrak{a}}(tH).$

Finally, similar to the complex group case, $F_{f,L}$ and $f_{f,\mathfrak{a}}$ are connected by exponential map, and for sufficiently small t, we have

$$F_{f,L}(a_t) = |\frac{e^t - e^{-t}}{2t}|\psi_{g,\mathfrak{a}}(tH)$$

for $g = f \circ \exp$.

3.2.2. *Orbital integrals for elliptic elements.* We introduce $\Delta(u_\theta) = e^{i\theta} - e^{-i\theta}$.

Definition 3.10. We define the orbital integral for elliptic elements to be

$$F_{f,K}(u_\theta) = \Delta(u_\theta)\int_{G/K} f(xu_\theta x^{-1})d(G/K)$$

we note that $F_{f,K}$ is not continuous at 1.

Definition 3.11. We define the orbital integral for Lie algebra as

$$\psi_{\mathfrak{g},\mathfrak{k}}(\theta(X-Y)) = 2i\theta\int_{G/K} g(\theta x(X-Y)x^{-1})d(G/K)$$

we have the following formula for $\psi_{g,\mathfrak{k}}$

$$\begin{aligned}\psi_{g,\mathfrak{k}}(\theta(X-Y)) &= \frac{1}{2\pi}i\theta\int\int\int_{0\le\theta<2\pi,\ t>0} g(\theta(X-Y)^{u_{\theta_1}a_t})\text{sinh}2t\ d\theta_1\ d\theta_2\ dt\\ &= \frac{i\theta}{2}\int_0^\infty \overline{g}(\theta(e^{2t}X - e^{-2t}Y))(e^{2t}-e^{-2t})\ dt\end{aligned}$$

Proposition 3.12. *If $E \subset K'$ is a compact set, then the map*

$$f \mapsto F_{f,K}|_E$$

is a continuous map from $C_c^\infty(G)$ to $C_c^\infty(E)$.

For all small $\theta \neq 0$, we have

$$F_{f,K}(u_\theta) = \frac{(e^{i\theta} - e^{-i\theta})}{2i\theta}\psi_{g,\mathfrak{k}}(\theta(X-Y))$$

where $g = f \circ exp$.

The problem now is to investigate the bahaviour of $F_{f,K}$ and its derivatives near 1 and relate this to $F_{f,L}$, this comes down to the relation at the Lie algebra level.

3.2.3. *Some integration formulas.*

Proposition 3.13. *If we denote $\hat{\bullet}$ the Fourier transform on L with respect to $dL = dMdt$, for $T_{\epsilon,\lambda}$ the characters of $\pi_{\epsilon,\lambda}$, we have*

$$T_{\epsilon,\lambda}(f) = \hat{F}_{f,L}(\epsilon,\lambda)$$

for $f \in C_c^\infty(G)$

We denote $G_{\text{ell}},\ G_{\text{hyp}}$ the open invariant set z of G such that $|\text{tr}(z)| < 2$ (resp. $|\text{tr}(z)| > 2$). Then

$$G' = G_{\text{ell}} \sqcup G_{\text{hyp}}$$

we have maps

$$\varphi_{\text{ell}}:\ G/K \times K' \to G_{\text{ell}},\quad \varphi_{\text{hyp}}:\ G/L \times L' \to G_{\text{hyp}}$$

and

$$\begin{aligned}\varphi_{\text{ell}}^* dG &= |e^{i\theta} - e^{-i\theta}|^2 dG/K\ d\theta\\ \varphi_{\text{hyp}}^* dG &= |e^t - e^{-t}|^2 G/L\ dL\end{aligned}$$

Proposition 3.14. *(Harish-Chandra integration formula) If f is a Borel function on G, then $f \in L^1(G)$ if and only if $F_{|f|,K}$ and $F_{|f|,L}$ exist almost everywhere on B and L respectively and*

$$\int_K |\Delta(u_\theta)| F_{|f|,K}(u_\theta) d\theta < \infty, \ \int_L |e^t - e^{-t}| F_{|f|,L} dL < \infty$$

Then we have

$$\int_G f \ dG = -\int_K \Delta(u_\theta) F_{f,K}(u_\theta) d\theta + \int_{\mathbb{R}} |e^t - e^{-t}| F_{h,L}(a_t) \ dt$$

where $h = (f + f_\gamma)/2$.

We have the following lemma

Lemma 3.15. *Fix $u \in C_c^\infty(\mathbb{R}^2)$ and*

$$U(\theta) = \theta \int_0^\infty u(\theta e^{2t}, \theta e^{-2t})(e^{2t} - e^{-2t}) \ dt, \ \theta \neq 0$$

then

- *$U(0\pm)$ and $\frac{d}{d\theta}U(0\pm)$ exists.*
- *$\frac{d}{d\theta}U$ is continuous at $\theta = 0$ and $\frac{d}{d\theta}U(0) = -u(0,0)$.*
- *$U(0\pm) = \pm\frac{1}{2}\int_0^\infty u(\pm s, 0) \ ds$.*

we can apply this lemma to

$$\psi_{g,\mathfrak{k}}(\theta(X - Y)) = \frac{i\theta}{2}\int_0^\infty \overline{g}(\theta(e^{2t}X - e^{-2t}Y))(e^{2t} - e^{-2t}) \ dt$$

for $\overline{g} = \int g^{u_\theta} \ d\theta$.

Theorem 3.16. *(limit formula) We have*

- *For all $f \in C_c^\infty(G)$, $F'_{f,K}(u_\theta) = \frac{d}{d\theta}F_{f,K}(u_\theta)$ is continuous at $\theta = 0$ and*
$$\frac{1}{i}F'_{f,K}(1) = -\pi f(1)$$
- *For all $g \in \mathcal{S}(\mathfrak{g})$, $\frac{d}{d\theta}\psi_{\mathfrak{g},\mathfrak{k}}(\theta(X - Y))$ is continuous at $\theta = 0$ and*
$$\frac{1}{i}(\frac{d}{d\theta}\psi_{\psi_{\mathfrak{g},\mathfrak{k}}})(0) = -\pi g(0)$$

The orbital integral $F_{f,K}(u_\theta)$ is not continuous at $\theta = 0$ and the jump at $\theta = 0$ is related to the hyperbolic orbital integral $F_{f,L}(1)$.

Theorem 3.17. *(Harish-Chandra jump relation) For all $f \in C_c^\infty(G)$*

$$[\frac{1}{i}F_{f,K}(u_\theta)]_{\theta=0^-}^{\theta=0^+} = F_{f,L}(1)$$

For all $g \in \mathcal{S}(\mathfrak{g})$

$$[\frac{1}{i}\psi_{g,\mathfrak{k}}(\theta(X - Y))]_{\theta=0^-}^{\theta=0^+} = \psi_{\mathfrak{g},\mathfrak{a}}(0)$$

Proof. We only need to prove this at the Lie algebra level, this follows from

$$\psi_{g,\mathfrak{k}}(0\pm) = \pm\frac{i}{4}\int_0^\infty \overline{g}(\pm sX) \ ds$$

and

$$\psi_{g,\mathfrak{a}}(0) = \frac{1}{4}\int_{-\infty}^\infty \overline{g}(sX) \ ds$$

hence

$$\frac{1}{i}\psi_{g,\mathfrak{k}}(0+) - \frac{1}{i}\psi_{g,\mathfrak{k}}(0-) = \frac{1}{4}\int_{-\infty}^\infty \overline{g}(sX) = \psi_{g,\mathfrak{a}}(0)$$

□

Proposition 3.18. *For all $f \in C_c^\infty(G)$, we have*

$$\int \Theta_m f dG = sgn(m) \int e^{im\theta} F_{f,K}(u_\theta) d\theta + \int_0^\infty e^{-|m|t} F_{h,L}(a_t) dt$$

Proof. By the Harish-Chandra integral formula 3.14 and the character formula for discrete series 3.4, we obtain

$$\int_G \Theta_m f dG = \mathrm{sgn}(m) \int e^{im\theta} F_{f,K}(u_\theta) d\theta + \int_0^\infty e^{-|m|t} F_{h,L}(a_t) dt$$

□

Corollary 3.19. *For Θ_m characters of π_m, we have*

$$\int_G \Theta_m \Omega f dG = m^2 \int_G \Theta_m f dG$$

The boundary terms at $\theta = 0, \pi, t = 0$ gets cancelled because of the limit formula and jump relations.

3.3. **Proof of Plancherel formula.** Recall that we have characters Θ_m, $T_{\epsilon,\lambda}$, $\epsilon = 0, 1$ and $\lambda \in i\mathbb{R}$.

Lemma 3.20. *For any integer $r \geq 1$ there is a continuous seminorm μ on $C_c^\infty(G)$ such that*

$$|\Theta_m(f)| \leq m^{-2}\mu(f)$$
$$|T_{\epsilon,\lambda}(f)| \leq (1 + |\lambda|^2)^{-r}\mu(f)$$

for all $f \in C_c^\infty(G)$ and $m \neq 0$, all $\lambda \in i\mathbb{R}$.

This lemma assures the convergence of the series and integrals that we shall encounter. Put

$$\hat{F}_{f,K}(m) = \int F_{f,K}(u_\theta) e^{im\theta} d\theta$$

Proposition 3.21. *For all $f \in C_c^\infty(G)$, we have*

$$\int_G \Theta_m f \ dG = sgn(m) \int e^{im\theta} F_{f,K}(u_\theta) \ d\theta + \int_0^\infty e^{-|m|t} F_{h,L}(a_t) dt$$

here $h = f + (-1)^{m-1} f_\gamma$.

Proof. From proposition 3.18, we have

$$\hat{F}_{f,K}(m) = \mathrm{sgn}(m)\Theta_m(f) - \mathrm{sgn}(m) \int_0^\infty e^{-|m|t} F_{h,L}(a_t) dt$$

use integration by part to $e^{-|m|t} F_{h,L}$, we get the result. □

We can now calculate the Fourier transform of $F'_{f,K}$

Proposition 3.22. *For all $f \in C_c^\infty(G)$, $m \in \mathbb{Z}$ writing $'$ for $d/d\theta$ and d/dt, then we have*

$$(-iF'_{f,K}(m))\widehat{} = -|m|\Theta_m(f) + \int_0^\infty e^{-|m|t} F'_{h,L}(a_t) dt$$

Proof. We have

$$\begin{aligned}(-iF'_{f,K}(m))\widehat{} &= -i \int F'_{f,K}(u_\theta) e^{im\theta} d\theta \\ &= -\frac{1}{i}[F_{f,K}(u_\theta)]_{0-}^{0+} - \frac{1}{i}[F_{f,K}(u_\theta)e^{im\theta}]_{\pi^-}^{\pi^+} - m\hat{F}_{f,K}(m)\end{aligned}$$

apply the result from previous proposition 3.21, and note

$$F_{f,K}(u_{\theta+\pi}) = -F_{f_\gamma,K}(u_\theta)$$

we get

$$(-iF'_{f,K}(m))\widehat{} = -|m|\Theta_m(f) + \int_0^\infty e^{-|m|t}F'_{h,L}(a_t)dt$$
$$+ \{F_{f,L}(1) - \frac{1}{i}[F_{f,K}(u_\theta)]_{0-}^{0+}\}$$
$$+ (-1)^{m-1}\{F_{f_\gamma,L}(1) - \frac{1}{i}[F_{f_\gamma,K}(u_\theta)]_{0-}^{0+}\}$$

the expressions within {} are zero by Harish-Chandra jump relation, hecne the result. □

Since $F'_{f,K}$ is continuous at $\theta = 0$, we have

$$2\pi(\frac{1}{i}F'_{f,K}(u_\theta))_{\theta=0} = \sum_m (\frac{1}{i}F'_{f,K}(m))\widehat{}$$

by the limit formula 3.16

$$\frac{1}{i}F'_{f,K}(1) = -\pi f(1)$$

hence by proposition 3.22, we get

$$-2\pi^2 f(1) = \sum_m (\frac{1}{i}F'_{f,K}(m))\widehat{}$$
$$= -|m|\Theta_m(f) + \int_0^\infty e^{-|m|t}F'_{h,L}(a_t))dt$$

We can separate the cases $f = \pm f_\gamma$

Proposition 3.23. *For $f = f_\gamma$, we have*

(3.1)
$$2\pi^2 f(1) = \sum_{m \ odd} |m|\Theta_m(f) + \frac{1}{2}\int_0^\infty \mu tanh(\pi\mu/2) \cdot T_{0,i\mu}(f)d\mu$$

Proof. For $f = f_\gamma$, $f + (-1)^{m-1}f_\gamma = 0$ for m even and $2f$ for m odd.
we have

$$2\pi^2 f(1) = \sum_{m \text{ odd}} |m|\Theta_m(f) - 2\sum_{m\text{odd}} \int_0^\infty e^{-|m|t}F'_{f,L}(a_t)dt$$
$$= \sum_{m \text{ odd}} |m|\Theta_m(f) - 2\int_{-\infty}^\infty \frac{t}{e^t - e^{-t}}F'_{f,L}(a_t)/t \ dt$$

For $f = f_\gamma$, we have

$$\int_{-\infty}^\infty F_{f,L}(a_t)e^{i\mu t}dt = \frac{1}{2}\hat{F}_{f,L}(0, i\mu)$$

We now apply the Plancherel formula over $\mathbb{R}$ to the last term,

$$(\frac{t}{e^t - e^{-t}})\tilde{}(\mu) = \frac{d}{d\mu}\tanh\frac{\pi\mu}{2}$$

for $\tilde{u}(\mu) = \int_{-\infty}^\infty u(t)e^{i\mu t} \ dt$. We get

$$\int_{-\infty}^\infty \frac{t}{e^t - e^{-t}}F'_{f,L}(a_t)/t \ dt = -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\pi}{2}\tanh(\frac{\pi\mu}{2})\frac{\mu}{2}T_{0,i\mu}(f)d\mu$$

remember that $T_{0,i\mu} = T_{0,-i\mu}$, put all things together, we get

$$2\pi^2 f(1) = \sum_{m \text{ odd}} |m|\Theta_m(f) + \frac{1}{2}\int_0^\infty \mu\tanh(\frac{\pi\mu}{2}) \cdot T_{0,i\mu}(f)d\mu$$

□

For $f = -f_\gamma$, we have

$$2\pi^2 f(1) = \sum_{m \text{ even}} |m|\Theta_m(f) + \frac{1}{2}\int_0^\infty \mu \coth(\frac{\pi\mu}{2}) T_{1,i\mu}(f) d\mu \tag{3.2}$$

Combining equations (3.1) and (3.2) together, we get

Theorem 3.24. *(Plancherel formula) Let the distribution characters be computed with respect to the Riemannian Haar measure d G, then for all $f \in C_c^\infty(G)$*

$$2\pi^2 f(1) = \sum_{m\neq 0} \Theta_m(f) + \frac{1}{2}\int_0^\infty \mu \ tanh(\pi\mu/2) \cdot T_{0,i\mu}(f) \ d\mu + \frac{1}{2}\int_0^\infty \mu \ coth(\pi\mu/2) \cdot T_{1,i\mu}(f) \ d\mu$$