Perverse sheaves

Rui Chen

February 15, 2023

1 Introduction

This is a note written for perverse sheaves, I mainly follow the note by Losev, and most of the background materials are taken from Achar's book [1].

2 Triangulated category

We discuss the derived category of chain complexes as an example of triangulated category.

Definition 2.1. Let $\mathscr{T}$ be an additive category equipped with an automorphism $[1] : \mathscr{T} \to \mathscr{T}$, the **shift functor** and a collection of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

The category $\mathscr{T}$ will be called an triangulated category if a list of axioms hold.

Definition 2.2. A cone of a morphism $f : X \to Y$ is any member of the isomorphism class of objects Z such that there is a distinguished triangle $X \to Y \to Z \to X[1]$.

Definition 2.3. For $\mathscr{A}$ an abelian category, an additive functor $F : \mathscr{T} \to \mathscr{A}$ will be called a **cohomological functor** if it sends distinguished triangles in $\mathscr{T}$ to long exact sequences in $\mathscr{A}$.

Definition 2.4. For $\mathscr{T}$ and $\mathscr{T}'$ two triangulated categories, an additive functor $F : \mathscr{T} \to \mathscr{T}'$ will be called a **triangulated functor**, if it commutes with shift functors and it sends distinguished triangles to distinguished triangles.

Now we use the category of chain complexes as an example to discuss how to build trangulated categories from additive and abelian categories.

Definition 2.5. For $\mathscr{A}$ an additive category, we can define $Ch(\mathscr{A})$ and $K(\mathscr{A})$, the category of chain complexes and the homotopy category of $\mathscr{A}$.

Definition 2.6. Given a chain complex $A = (A^i, d^i_A)_{i\in\mathbb{Z}}$, $A[1]$ will be the chain complex given by $(A[1])^i = A^{i+1}$ with differentials

$$d^i_{A[1]} = -d^{i+1}_A : (A[1])^i \to (A[1])^{i+1}$$

Definition 2.7. Let $f : A \to B$ be a chain map, we can define the chain-map cone $ch(f)$ to be the complex whose terms are $ch(f)^i = A^{i+1} \oplus B^i$, and differential is given by $d^i_{ch(f)} = \begin{pmatrix} -d^{i+1}_A & \\ f^{i+1} & d^i_B \end{pmatrix}$.

there are obvious chain maps $B \to ch(f)$ and $ch(f) \to A[1]$.

Definition 2.8. A diagram $X \to Y \to Z \to X[1]$ in $K(\mathscr{A})$ is called a **distinguished triangle** if for some chain map $\tilde{f} : X \to Y$ in the homotopy class of f, there is a homotopy equivalence $u : Z \to ch(f)$ and the original diagram commute with the diagram given by the chain-map cone for $\tilde{f}$ under the natural connecting homomorphisms.

Definition 2.9. The **derived category** $D(\mathscr{A})$ of an abelian category $\mathscr{A}$ is the Verdier localization of $K(\mathscr{A})$ at the set of quasi-isomorphisms.

Definition 2.10. We define the truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$

$$\tau^{\leq n} : Ch(\mathscr{A}) \to Ch(\mathscr{A}), \quad \tau^{\geq n} : Ch(\mathscr{A}) \to Ch(\mathscr{A})$$

it is defined as

$$(\tau^{\leq n}X)^i = \begin{cases} 0 & if\ i > n \\ ker\ d^i & if\ i = n \\ X^i & if\ i < n \end{cases}$$

we have similar definition for $\tau^{\geq n}$.

by construction, we have natural maps $\tau^{\leq n}X \to X$ and $X \to \tau^{\geq n}X$.

Lemma 2.11. *Let $\mathscr{A}$ be an abelian category*

- *If $X \in D(\mathscr{A})^{\leq n}$ and $Y \in D(\mathscr{A})^{\geq n+1}$, then $Hom(X, Y) = 0$.*
- *The natural functor $\mathscr{A} \to D(\mathscr{A})$ is fully faithful and its essential image is $D(\mathscr{A})^{\leq 0} \cap D(\mathscr{A})^{\geq 0}$.*

Lemma 2.12. *For any $X \in D(\mathscr{A})$ and $n \in \mathbb{Z}$, there is a unique natural map $\delta : \tau^{\geq n+1}X \to \tau^{\leq n}X[1]$ that gives a natural distinguished triangle*

$$\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \to$$

Proof. In $Ch(\mathscr{A})$, we have an obvious injective map $\tau^{\leq n}X \to X$ and an obvious surjective map $X \to \tau^{n+1}X$, and it factors through

$$X/\tau^{\leq n}X \to \tau^{\geq n+1}X$$

the fancy snake lemma gives us a distinguished triangle $\tau^{\leq n}X \to X \to X/\tau^{\leq n} \to$, we may replace the third term by $\tau^{\geq n+1}X$, then we get a unique natural map $\delta : \tau^{\geq n+1}X \to \tau^{\leq n}X[1]$. □

Definition 2.13. For $X, Y \in \mathscr{A}$, the n-th extension group of X and Y is defined to be

$$Ext^n(X, Y) = Hom_{D(\mathscr{A})}(X, Y[n])$$

Proposition 2.14. Let $\mathscr{A}$ be an abelian category, and let $X, Y \in \mathscr{A}$, there is a natural bijection between $Ext^1_{\mathscr{A}}(X, Y)$ and the equivalence classes of extensions of X by Y.

For map in the one direction, for $0 \to Y \to Z \to X \to 0$ an exact sequence, we get a distinguished triangle $Y \to Z \to X \to$, which gives us a map $X \to Y[1] \in Ext^1_{\mathscr{A}}(X, Y)$.

A t-structure on a trangulated category is an additional structure that allows one to recover the abelian subcategory (the heart) inside, in particular, recovering the abelian category inside the derived category.

Definition 2.15. Let $\mathscr{T}$ be a triangulated category, and let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a pair of strictly full subcategories, for $n \in \mathbb{Z}$, we let

$$\mathscr{T}^{\leq n} = \mathscr{T}^{\leq 0}[-n] \text{ and } \mathscr{T}^{\geq n} = \mathscr{T}^{\geq 0}[-n]$$

We say that the pair $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is a t-structure if the following conditions hold:

- $\mathscr{T}^{\leq -1} \subset \mathscr{T}^{\leq 0}$ and $\mathscr{T}^{\geq 0} \subset \mathscr{T}^{\geq -1}$.
- If $A \in \mathscr{T}^{\leq -1}$ and $B \in \mathscr{T}^{\geq 0}$, then $Hom(A, B) = 0$.
- For any $C \in \mathscr{T}$, there is a distinguished triangle $A \to C \to B \to^{+1}$ with $A \in \mathscr{T}^{\leq -1}$ and $B \in \mathscr{T}^{\geq 0}$.

we define the *heart* of the t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ as $\mathscr{T}^{\mathscr{T}^{\heartsuit}} := \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$.

Example 2.16. Let $\mathscr{A}$ be an abelian category, and $\mathscr{T} := D^b(\mathscr{A})$. Then we set $\mathscr{T}^{\leq 0} = \{M \in \mathscr{T} | H^i(M) = 0, \forall i > 0\}$ and define $\mathscr{T}^{\geq 0}$ in a similar way. Then this is an obvious t-structure with heart identified with $\mathscr{A}$.

Proposition 2.17. Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$, then

- The inclusion $\mathscr{T}^{\leq n} \hookrightarrow \mathscr{T}$ admits a right adjoint ${}^t\tau^{\leq n} : \mathscr{T} \to \mathscr{T}^{\leq n}$.
- The inclusion $\mathscr{T}^{\geq n} \hookrightarrow \mathscr{T}$ admits a left adjoint ${}^t\tau^{\geq n} : \mathscr{T} \to \mathscr{T}^{\geq n}$.

Definition 2.18. For $C \in \mathscr{T}, i \in \mathbb{Z}$, we set

$$H^i(C) := \tau^{\leq 0}\tau^{\geq 0}(C[i]) \in \mathscr{T}^{\heartsuit}$$

Proposition 2.19. Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$. Its heart $\mathscr{T}^{\heartsuit}$ is an abelian category.

Proposition 2.20. Let $\mathscr{T}^{\heartsuit}$ be the heart of a t-structure on $\mathscr{T}$, and let $X, Y \in \mathscr{T}^{\heartsuit}$, then there is a bijection between $Hom(X, Y[1])$ and equivalence classes of extensions of X by Y, that is $Ext^1_{\mathscr{T}^{\heartsuit}}(X, Y) = Hom(X, Y[1])$.

Definition 2.21. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two trianglulated categories equipped with t-structures $(\mathscr{T}_1^{\leq 0}, \mathscr{T}_1^{\geq 0})$ and $(\mathscr{T}_2^{\leq 0}, \mathscr{T}_2^{\geq 0})$. A triangulated functor $F : \mathscr{T}_1 \to \mathscr{T}_2$ is said to be left t-exact if $F(\mathscr{T}_1^{\geq 0}) \subset \mathscr{T}_2^{\geq 0}$, and right t-exact if $F(\mathscr{T}^{\leq 0}) \subset \mathscr{T}_2^{\leq 0}$. It is said to be t-exact if it is both left and right t-exact.

We have the following lemma connects the exactness of the sequence and distinguished triangle in the triangulated category.

Lemma 2.22. *Let $\mathscr{T}^{\heartsuit}$ be the heart of a t-structure on $\mathscr{T}$, and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be two morphisms in $\mathscr{T}^{\heartsuit}$. The following conditions are equivalent

- *The sequence*

  $$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

  is a short exact sequence.
- *There exists a morphism $h : Z \to X[1]$ in $\mathscr{T}$, such that*

  $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$$

  is a distinguished triangle.

3 Constructible sheaves

For X, Y complex algebraic varieties and $f : X \to Y$ a smooth morphism.

Proposition 3.1. Suppose $f : X \to Y$ is smooth of relative dimension d, then we have isomorphism $f^! \cong f^*[2d]$.

Example 3.2. When Y is a point, k a field. We apply this isomorphism to constant sheaf $\underline{k} \in Sh(pt, k)$, then the proposition tells us that $f^!\underline{k} = \underline{k_X}[2d]$.

Let $\mathscr{L}$ be a finite rank local system on X, we have

$$Hom_{D(X,k)}(\mathscr{L}, f^!\underline{k}) = Hom_{D(k-\text{mod})}(f_!\mathscr{L}, k) = R\Gamma_c(\mathscr{L})^*$$

On the other hand, for $\mathscr{L}^\vee$ the dual local system of $\mathscr{L}$, we have

$$Hom_{D(X,k)}(\mathscr{L}, f^*\underline{k}[2d]) = Hom_{D(X,k)}(\mathscr{L}, \underline{k_X}[2d]) = R\Gamma(\mathscr{L}^\vee[2d])$$

so the proposition asserts that

$$R\Gamma_c(\mathscr{L})^* = R\Gamma(\mathscr{L}^\vee[2d])$$

which is $H_c^j(X, \mathscr{L}) = H^{2d-j}(X, \mathscr{L}^\vee)^*$ for $j \in \mathbb{Z}$, which is the Poincare duality for local systems. In other words, 3.1 is a relative version of the Poincare duality.

Definition 3.3. A partition $X = \sqcup_{i=0}^k X_i$ is called a *stratification* if

- X_i are smooth connected locally closed subvarieties.
- for each $i, j = 1, \cdots k$, we have $X_i \cap \overline{X_j} = X_i$ or $\emptyset$.

Example 3.4. For U acting on $X = G/P$, we get the parabolic Bruhat strtification.

Definition 3.5. Let $\mathcal{S}$ be a stratification of X, for a stratum X_i, let $h_i : X_i \hookrightarrow X$ be the inclusion. We say that

- $\mathscr{F} \in Sh(X, k)$ *constructible* w.r.t $\mathcal{S}$, if $h_i^*\mathscr{F}$ is a local system of finite type.
- $\mathscr{F} \in D^b(X, k)$ is called *constructible w.r.t* $\mathcal{S}$, if $\mathscr{H}^i(\mathscr{F})$ is constructible w.r.t $\mathcal{S}$ for all i.
- $\mathscr{F} \in D^b(X, k)$ is called *constructible* if it is constructible w.r.t some stratification.

Example 3.6. For the trivial stratification $\mathcal{S}$, we have $Sh_{\mathcal{S}}(X, k) = Loc_{ft}(X, k)$.

Example 3.7. For the Bruhat stratification on $\mathbb{P}^1$. All the strata are simply-connected, hence the local systems are determined by the vector spaces on the stalks. Set $W_0 = \mathscr{F}_{[1:0]}$, $W_1 := \mathscr{F}_{[0:1]}$. The restriction map from the neighborhood of 0 to the punctual neighborhood $U^\times$ gives us a transition map

$$\varphi : W_0 = \Gamma(U, \mathscr{F}) \longrightarrow \Gamma(U^\times, \mathscr{F}) = W_1$$

the data (W_0, W_1, φ) gives us the data that needed to define a finite dimensional A_2 quiver representation. In fact, we have an equivalence between $Sh_{\mathcal{S}}(X, \mathbb{C})$ and the category of finite dimensional representations of A_2 quiver.

The functors we constructed preserves the structure of the constructible derived categories.

Theorem 3.8. *The following are true*

- *for* $\mathscr{F} \in D_c^b(X, k)$, *we have* $f_*\mathscr{F}, f_!\mathscr{F} \in D_c^b(Y, k)$.
- *for* $\mathscr{G} \in D_c^b(Y, k)$, *we have* $f^*\mathscr{G}, f^!\mathscr{G} \in D_c^b(X, k)$.

- *for* $\mathscr{F}_1, \mathscr{F}_2 \in D^b_c(X,k)$, *we have* $R\mathscr{H}om(\mathscr{F}_1, \mathscr{F}_2) \in D^b_c(X,k)$.

Definition 3.9. We define the dualizing sheaf ω_X to be $a_X^! \underline{k}$, where $a_X : X \to pt$.

so we have

$$Hom_{D^b(X,k)}(\mathscr{F}, \omega_X) = Hom_{D^b(k-\mathrm{mod})}(R\Gamma_c(\mathscr{F}), k)$$

Definition 3.10. The Verdier duality functor is

$$\mathbb{D}(\bullet) := R\mathscr{H}om(\bullet, \omega_X) : D^b_c(X,k) \to D^b_c(X,k)^{opp}$$

The following theorem summarizes the properties of $\mathbb{D}$

Theorem 3.11. *The following claims are true:*

- $\mathbb{D}$ *is an equivalence, moreover* $\mathbb{D}^2$ *is naturally isomorphic to id.*
- *We have* $f_! \circ \mathbb{D}_X = \mathbb{D}_Y \circ f_*$ *and* $f_* \circ \mathbb{D}_X = \mathbb{D}_Y \circ f_!$.
- *We have* $f^! \circ \mathbb{D}_X = \mathbb{D}_Y \circ f^*$ *and* $f^* \circ \mathbb{D}_X = \mathbb{D}_Y \circ f^!$.
- *We have* $R\mathscr{H}om(\mathscr{F}_1, \mathscr{F}_2) = R\mathscr{H}om(\mathbb{D}\mathscr{F}_2, \mathbb{D}\mathscr{F}_1)$.

We have the following theorem which will be useful for us to do many calculations.

Theorem 3.12. *Let* $i \hookrightarrow X$ *be a closed embedding, and let* $j : U \hookrightarrow X$ *be the complementary open embedding, then*

- *We have* $i^* \circ j_! = 0$, $i^! \circ j_* = 0$, *and* $j^* \circ i_* = 0$.
- *For any* $\mathscr{F} \in D^+(X, \mathbb{C})$, *there is a natural distinguished triangle*

$$j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to \tag{3.1}$$

- *For any* $\mathscr{F} \in D^+(X, \mathbb{C})$, *there is a natural distinguished triangle*

$$i_* i^! \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F} \to \tag{3.2}$$

For the first statement, since $i^*, j_!, j^*$ and i_* come from exact functors , the claims $i^* \circ j_! = 0$ and $j^* \circ i_* = 0$ can be checked at the level of abelian categories, and from the adjoint relations, $i^! \circ j_*$ is right adjoint to $j^* \circ i_! \cong j^* \circ i_*$, so it also vanishes.

For the second statement, one can check that we have the short exact sequence in $Sh(X, \mathbb{C})$ by checking the exactness at stalks

$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0$$

then the 2.22 gives us a distinguished triangle $j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to$. Note

$$\mathrm{Hom}(j_! j^* \mathscr{F}, i_* i^* \mathscr{F}[-1]) \cong \mathrm{Hom}(i^* j_! j^* \mathscr{F}, i^* \mathscr{F}[-1]) = 0$$

so the third map in the triangle is unique.

For the third statement, similarly one show that there is a short exact sequence

$$0 \to i_* i^! \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F} \to 0$$

and then 2.22 will give us the distinguished triangle.

Example 3.13. For $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$, we have $j_!\underline{\mathbb{C}}$ and $j_*\underline{\mathbb{C}}$ are constructible with respect to the Bruhat strtification $\mathbb{C} = \mathbb{C}^\times \cup \{0\}$, we want calculate the stalks of cohomology local systems on $x \in \mathbb{C}^\times$ and 0, here $\mathscr{F}_x := i_x^* \mathscr{F}$, for $i_x : \{x\} \hookrightarrow \mathbb{C}$.

From that $i^* j_! = 0$, we see the stalk cohomology of $j_!\underline{\mathbb{C}}$ at 0 is zero. For $x \in \mathbb{C}$, we use the formula

$$H^k((j_!\mathscr{F})_x) = lim_{x\in U} H^k(U, \mathscr{F}|_U)$$

and we get $H^0((j_!\underline{\mathbb{C}})_x) = \mathbb{C}$ and $H^1((j_!\underline{\mathbb{C}})_x) = 0$.

For the stalks of cohomology local systems of $j_*\underline{\mathbb{C}}$, note we have

$$\begin{aligned}\mathscr{F} &= lim_{x\in U} R\Gamma_*(U, \mathscr{F}|U)\\ &= lim_{x\in U} R\Gamma_*(U \cap \mathbb{C}^\times, \mathscr{F}|_{U\cap\mathbb{C}^\times})\end{aligned}$$

where the second equality is by open base change. we get for $x \in \mathbb{C}^\times$ $H^0((j_*\underline{\mathbb{C}})_x) = \mathbb{C}$ and $H^1((j_*\underline{\mathbb{C}})_x) = 0$, and $H^0((j_*\underline{\mathbb{C}})_0) = \mathbb{C}$ and $H^1((j_*\underline{\mathbb{C}})_0) = \mathbb{C}$.

Later we will check that $j_!\underline{\mathbb{C}}[1]$ and $j_*\underline{\mathbb{C}}[1]$ are perverse sheaves and determine their composition factors.

4 Perverse sheaves

We have the tautological t-structure $({}^\tau D_c^b(X,k)^{\leq 0}, {}^\tau D_c^b(X,k)^{\geq 0})$ on $D_c^b(X,k)$ (as it is the derived category of the constructible sheaves). However this is not the best t-structure, for example, it doesn't behave well under the Verdier duality.

For example, if $\iota : Z \hookrightarrow X$ is a closed inclusion of a smooth connected subvariety of X and $\mathscr{L}_Z$ is a finite type local system on Z, then we have $\mathbb{D}(\mathscr{L}_Z) \cong \mathscr{L}_Z^\vee[2dimZ]$.

We would like to have a t-structure which is compatible with $\mathbb{D}$, meaning that $\mathbb{D}$ sends the ≤ 0 part to ≥ 0 part. We define

$$ {}^p D_c^b(X,k)^{\leq 0} := \{\mathscr{F} \in D_c^b(X,k) | \text{ dim supp } \mathscr{H}^i(\mathscr{F}) \leq -i, \forall i \in \mathbb{Z}\} \tag{4.1}$$

$$ {}^p D_c^b(X,k)^{\geq 0} := \{\mathscr{F} \in D_c^b(X,k) | \text{ dim supp } \mathscr{H}^i(\mathbb{D}\mathscr{F}) \leq -i, \forall i \in \mathbb{Z}\} \tag{4.2}$$

Definition 4.1. We set

$$\text{Perv}(X,k) := {}^p D_c^b(X,k)^{\leq 0} \cap {}^p D_c^b(X,k)^{\leq 0}$$

the objects of $\text{Perv}(X,k)$ are called *Perverse sheaves*.

Example 4.2. Let $\iota : Z \hookrightarrow X$ be a closed embedding, then $\mathscr{L}_Z[dim\ Z] \in \text{Perv}(X,k)$.

Theorem 4.3. *The pair* $({}^p D_c^b(X,k)^{\leq 0}, {}^p D_c^b(X,k)^{\geq 0})$ *is a t-structure, hence Perv*(X,k) *is an abelian category.*

Recall that there is a morphism $h_! \to h_*$, we get a morphism

$$ {}^p\mathscr{H}^0(h_{!\bullet}) \to {}^p\mathscr{H}^0(h_{*\bullet}) \tag{4.3}$$

of functors $\text{Perv}(Z,k) \to \text{Perv}(X,k)$.

For $Z \subset X$ a smooth irreducible and locally closed subvariety, $h : Z \hookrightarrow X$ and inclusion.

Definition 4.4. The intermediate extension functor $h_{!*} : \text{Perv}(Z,k) \to \text{Perv}(X,k)$ is the image of (4.3).

For $\mathscr{F} \in \text{Perv}(Z)$, the object $h_{!*}(\mathscr{F})$ is the unique $\mathscr{G} \in \text{Perv}(X)$ such that

- *Supp*$(\mathscr{G}) \subset \overline{Z}$.

- $\mathscr{G}|_Z \cong \mathscr{F}$.
- $\mathscr{G}$ has no subs and quotients supported on $\overline{Z}\backslash Z$.

because of this, sometimes $h_{!*}$ is called the *minimal extension*.

Definition 4.5. Let $Z \subset X$ be a smooth, irreducible and loally closed subvariety, $h : Z \hookrightarrow X$ be an inclusion, and $\mathscr{L}$ a local system on Z, we define the intersection cohomology sheaf $IC(Z, \mathscr{L})$ to be $h_{!*}(\mathscr{L}[dim Z])$.

Theorem 4.6. *The following claims are true*

- *if $\mathscr{L}$ is irreducible, then $IC(Z, \mathscr{L})$ is simple in $Perv(X, k)$.*
- *every simple in $Perv(X, k)$ is isomorphic to $IC(Z, \mathscr{L})$ for some $Z, \mathscr{L}$.*
- *we have $IC(Z_1, \mathscr{L}_1) \cong IC(Z_2, \mathscr{L}_2)$ if and only if $Z_1 \cap Z_2$ is open in both Z_1, Z_2 and $\mathscr{L}_1|_{Z_1 \cap Z_2} = \mathscr{L}_2|_{Z_1 \cap Z_2}$.*
- *We have $\mathbb{D}IC(Z, \mathscr{L}) = IC(Z, \mathscr{L}^\vee)$.*

Suppose $\overline{Z} = X$, then from $R\Gamma_c \circ \mathbb{D}_X = \mathbb{D}_{pt} \circ R\Gamma = R\Gamma^*$, we get

$$R\Gamma_c(IC(Z, \mathscr{L}^\vee)) = R\Gamma(IC(Z, \mathscr{L}))^*$$

this is a generalization of the Poincare duality to singular varieties.

Example 4.7. For the open embedding $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$, we have the sheaves $j_!\underline{\mathbb{C}}[1], j_*\underline{\mathbb{C}}[1] \in D^b_c(\mathbb{C})$.

From the calculation of the stalks, we see that $j_!\underline{\mathbb{C}}[1]$ and $j_*\underline{\mathbb{C}}[1] \in {}^pD^{\leq 0}$, and note from the Verdier duality $\mathbb{D}j_!\underline{\mathbb{C}}[1] = j_*\underline{\mathbb{C}}[1]$, we see both $j_!\underline{\mathbb{C}}[1]$ and $j_*\underline{\mathbb{C}}[1]$ are perverse sheaves.

From the distinguished triangle (4.3) apply to $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ and $i : \{0\} \hookrightarrow \mathbb{C}$ and $\mathscr{F} = \underline{\mathbb{C}}$, we get a distinguished triangle

$$j_!\underline{\mathbb{C}} \to \underline{\mathbb{C}} \to \underline{\mathbb{C}}_0 \to$$

and from 2.22, we get a short exact sequence of perverse sheaves

$$0 \to \underline{\mathbb{C}}_0 \to j_!\underline{\mathbb{C}}[1] \to \underline{\mathbb{C}}[1] \to 0$$

Similarly, from the distinguished triangle (3.2) note $i^! = i^*[-2]$, we get a distinguished triangle

$$\underline{\mathbb{C}}_0[-2] \to \underline{\mathbb{C}} \to j_*\underline{\mathbb{C}} \to$$

by 2.22, we get the short exact sequence of perverse sheaves

$$0 \to \underline{\mathbb{C}}[1] \to j_*\underline{\mathbb{C}}[1] \to \underline{\mathbb{C}}_0 \to 0$$

Note from the definition of the intersection cohomology sheaf as $Im(j_! \to j_*)$, we have $IC(\mathbb{C}^\times) = \mathbb{C}[1]$ and $IC(\{0\}) = \underline{\mathbb{C}}_0$.

Example 4.8. We now describe the category $\mathrm{Perv}_{\mathcal{S}}(\mathbb{P}^1, \mathbb{C})$, where $\mathcal{S}$ is the standard Bruhat stratification. For $X_0 = \{[1:0]\}$, $X_1 = \mathbb{A}^1$.

Since both strata are connected, we have two simple objects $L_0 := i_*\underline{\mathbb{C}}$, $L_1 := IC(X_1, \underline{\mathbb{C}}_{X_1}) = \underline{\mathbb{C}}_{\mathbb{P}^1}[1]$. For $j : X_1 \hookrightarrow \mathbb{P}^1$, we have the objects $\nabla_1 := j_*\underline{\mathbb{C}}_{X_1}[1]$, $\Delta_1 := j_!\underline{\mathbb{C}}_{X_1}[1]$ which are also perverse.

Note that for any $\mathscr{F} \in D^b_c(\mathbb{P}^1, \mathbb{C})$,

$$Hom_{D^b_c(\mathbb{P}^1, \mathbb{C})}(\mathscr{F}, L_0) = H^0(\mathscr{F}_0)^*$$

recall that $H^0(j_*\underline{\mathbb{C}}_0) = H^1(j_*\underline{\mathbb{C}}_0) = 1$, we get

$$\dim Hom_{Perv}(\nabla_1, L_0) = \dim Ext^1_{Perv}(\nabla_1, L_0) = 1$$

From $\dim Ext^1_{Perv}(\nabla_1, L_0) = 1$, let P_0 denote the universal extension

$$0 \longrightarrow L_0 \longrightarrow P_0 \longrightarrow \nabla_1 \longrightarrow 0$$

we have $\dim Ext^1_{Perv}(L_0, L_1) = \dim Ext^1_{Perv}(L_1, L_0) = 1$, and from the exact triangles obtained from $j : X^1 \hookrightarrow \mathbb{P}^1$, we get short exact sequences of perverse sheaves

$$0 \longrightarrow L_1 \longrightarrow \nabla_1 \longrightarrow L_0 \longrightarrow 0$$
$$0 \longrightarrow L_0 \longrightarrow \Delta_1 \longrightarrow L_1 \longrightarrow 0$$

we claim that $Ext^1_{Perv}(P_0, L_1) = 0$, equivalently, $Ext^1_{Perv}(L_1, P_0) = 0$. For this we use the exact sequence

$$0 \longrightarrow L_0 \longrightarrow P_0 \longrightarrow \nabla_1 \longrightarrow 0$$

we consider $Hom(L_1, \cdot)$, the relevant terms are

$$Hom(L_1, \nabla_1) \to Ext^1(L_1, L_0) \to Ext^1(L_1, P_0) \to Ext^1(L_1, \nabla_1)$$

the first two spaces are $\mathbb{C}$ and the homomorphism between them is an isomorphism as ∇_1 realizes a nontrivial extension between L_0 and L_1, so the last homomorphism is injective, then we note that $Ext^1(L_1, \nabla_1) = 0$, and thus we have $Ext^1_{Perv}(P_0, L_1) = 0$.

There are five indecomposable perverse sheaves, up to isomorphism

$$IC(X_1), IC(X_0), j_!\underline{\mathbb{C}}[1], j_*\underline{\mathbb{C}}[1], P_0$$

References

[1] Pramod N Achar. *Perverse sheaves and applications to representation theory*, volume 258. American Mathematical Soc., 2021.