Open spherical-vector periods for the basic split-rank-one list

Research note on Definition 3.4.10 of *Thesis new.pdf*

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Abstract

We apply the coordinate-and-shell method of Proposition 3.6.2 to the thirteen families of Definition 3.4.10. Three explicit formulas below concern the trace-zero $T_1U_1\backslash \operatorname{Res}\mathrm{SL}_2$ family and the elementary T_1U_1 and T_2U_2 subfamilies of SU_3. All three are proved as raw spherical-vector integrals with specified right Haar measures. We distinguish these integrals from the Mellin-transformed values used in Conjecture 4.4.14. We also record the precise uncomputed families, a necessary repair to the cubic representative, and two apparent inconsistencies in the thesis's β definitions. "New" below means a candidate explicit calculation not located in the primary sources checked; it is not a claim of priority.

1 Scope, conventions, and the period being calculated

Let F be a nonarchimedean local field of characteristic zero, $q = |\mathfrak{o}_F/\varpi\mathfrak{o}_F|$, and let L/F be finite unramified of degree d. We use absolute values with $|\varpi|_F = q^{-1}$ and $|\varpi|_L = q^{-d}$. All principal series are normalized and their K-fixed sections ϕ_z satisfy $\phi_z(1) = 1$. The variable z is specified separately for SL_2, SU_3, and the thesis's GL_2 calculation; these three parameter conventions must not be identified.

Choose an integral representative γ of a *rational* principal open B-orbit. The raw open period is

$$P_{H,\gamma}(z) = \int_{(H\cap\gamma^{-1}B\gamma)\backslash H} \phi_z(\gamma h)\, dh_R. \tag{1}$$

The quotient of a *right* Haar measure gives the H-invariant functional for right translation. We normalize its integral compact shell to have volume one, as specified in each calculation. For a nonunimodular H, a G-invariant measure on $H\backslash G$ does not exist; this does not prevent the right-Haar integral (1). The comparison with the thesis's standard eigenmeasure can introduce a factor independent of z, and needs to be checked separately.

For $\operatorname{Res}_{F/k}(H_0\backslash G_0)$, the k-points are the corresponding F-points. Thus an extra unramified restriction F/k replaces the residue cardinality by $q = q_k^{[F:k]}$; it is not a separate local integration problem. A further extension L/F *inside* a coupled T_iU_i subgroup does change the integral.

2 Case ledger for the entire definition

In the table, $Q = |\mathfrak{o}_L/\varpi\mathfrak{o}_L|$, E/F is unramified quadratic for the SU_3 rows, and z always follows the conventions in the indicated proof. A dash means that a generic period of form (1) is not available or has not been evaluated. "Existing" includes results stated in the supplied thesis, even if the precise expression may not have appeared elsewhere.

Family of Definition 3.4.10	Raw normalized period	Status / scope
$\operatorname{Res}\mathrm{SL}_2\backslash\operatorname{Res}\mathrm{SL}_2$	Point evaluation 1	Type G; no generic invariant $B\backslash G$ integral.
$\operatorname{Res}T\backslash\operatorname{Res}\mathrm{SL}_2$, split	$(1+Q^{-1}z)/(1-Q^{-1}z)$	Classical toric shell integral.
$\operatorname{Res}T\backslash\operatorname{Res}\mathrm{SL}_2$, unramified anisotropic	1	Compact T in an integral embedding; ramified tori not treated.
$N(\operatorname{Res}T)\backslash\operatorname{Res}\mathrm{SL}_2$	Same as the corresponding torus, under compact-quotient normalization	Type N; outside Conj. 4.4.14.
$U\backslash\operatorname{Res}\mathrm{SL}_2$	$(1-Q^{-1}z)/(1-z)$	Standard untwisted unipotent integral.
$\operatorname{Res}_{F/k}T_1U_1\backslash\operatorname{Res}_{L/k}\mathrm{SL}_2$, $d=[L:F]$	$\boxed{(1+q^{-1}z)/(1-q^{d-2}z)}$	Derived here for every $d\geq 2$; candidate new explicit family.
$\operatorname{Res}_{F/k}\mathrm{SL}_2\backslash\operatorname{Res}_{L/k}\mathrm{SL}_2$, $d=2$	$(1+q^{-1}z)/(1-z)$	Thesis Prop. 3.5.2 / JLR Prop. 37, in its GL_2 parameter.
Same, $d=3$	$(1+q^{-3/2}z)/(1-q^{-1/2}z)$	Thesis Prop. 3.6.2; require a stronger choice of i.
$\operatorname{Res}\mathrm{SU}_3\backslash\operatorname{Res}\mathrm{SU}_3$	Point evaluation 1	Type G.
$\operatorname{Res}\mathrm{SO}_3\backslash\operatorname{Res}\mathrm{SU}_3$	—	Type N, not computed here; outside Conj. 4.4.14.
$U\backslash\operatorname{Res}\mathrm{SU}_3$	$(1-q^{-2}z)(1+q^{-1}z)/(1-z^2)$	Rank-one Gindikin–Karpelevich integral, odd residual characteristic in our coordinate proof.
$\operatorname{Res}\mathrm{SU}_2\backslash\operatorname{Res}\mathrm{SU}_3$	$(1+q^{-1}z)/(1-z)$	Already thesis §3.7.
$\operatorname{Res}_{F/k}T_1U_1\backslash\operatorname{Res}_{L/k}\mathrm{SU}_3$	$\boxed{(1+q^{-1}z)/(1-qz)}$ for $L=F$	Derived here for the $d=1$ subfamily, $p\neq 2$; coupled $d>1$ open.
$\operatorname{Res}_{F/k}T_2U_2\backslash\operatorname{Res}_{L/k}\mathrm{SU}_3$	$\boxed{(1+q^{-1}z)/(1-z)}$ for $L=F$	Derived here for the $d=1$ subfamily, $p\neq 2$; odd $d>1$ open.

The two type N rows should still be studied if a complete period table, rather than a test of Conjecture 4.4.14, is desired. Definition 3.4.10 says "maximal torus" without specifying an unramified integral model. Accordingly the toric numerical values in this note do not include ramified maximal tori.

3 A uniform new calculation for $T_1U_1\backslash\operatorname{Res}_{L/F}\mathrm{SL}_2$

Put $G=\mathrm{SL}_2(L)$ and $K=\mathrm{SL}_2(\mathfrak{o}_L)$. Write

$$t(a)=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix},\quad u(x)=\begin{pmatrix}1&x\\0&1\end{pmatrix},\quad w=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad H=\{u(x)t(a):a\in F^\times,\ x\in\ker\operatorname{Tr}_{L/F}\}.$$

This realizes the (a-A-2-d) row in the thesis's geometric description; for $d=1$ it specializes to a split torus. Let $\chi(t(\varpi))=z=q^{-ds}$ and put $R=q^{-d}z$. If the maximum norm of the bottom row of g equals q^{dj}, then $\phi_z(g)=R^j$. Normalize dx so $\operatorname{vol}(\ker\operatorname{Tr}\cap\mathfrak{o}_L)=1$ and $d^\times a$ so $\operatorname{vol}(\mathfrak{o}_F^\times)=1$.

Proposition 1. *Choose $i \in \mathfrak{o}_L$ with $\mathrm{Tr}_{L/F}(i) \in \mathfrak{o}_F^\times$ and let $\gamma = wu(i) \in K$. The open stabilizer is $H \cap \gamma^{-1}B\gamma = \mu_2$. With $h = u(x)t(a)$ and right Haar $dh_R = dx\, d^\times a$, normalized after the finite quotient, one has*

$$P_{T_1U_1,\gamma}(z) = \frac{1+q^{-1}z}{1-q^{d-2}z}, \qquad |q^{d-2}z| < 1. \tag{2}$$

It continues rationally in z. The integral identity holds without an odd-residue assumption; comparisons invoking a smooth integral μ_2-model may need one.

Proof. The trace on integral rings is onto for an unramified extension, so the chosen i exists and is a unit. Stabilizing the open line under $u(x)t(a)$ gives $x = (a^2-1)i$ up to a harmless sign convention. Taking traces forces $a^2 = 1$ and $x = 0$, hence the stabilizer is μ_2. The group law in the $u(x)t(a)$ coordinates proves that $dx\, d^\times a$ is right Haar.

The bottom row of $\gamma u(x)t(a)$ is $(a, (i+x)a^{-1})$. Since $\mathrm{Tr}(i+x)$ is a unit, $|i+x|_L = \max(1, |x|_L)$; the equality also follows by reducing the trace modulo ϖ for the unit shell. Let $c = q^{d-1}$. The trace-zero unit ball has volume one, while its shell $v_L(x) = -r$, $r \geq 1$, has volume $(1-c^{-1})c^r$. For $v_F(a) = t$, the section value is $R^{\max(-t,t+r)}$, with $r = 0$ for the unit ball. Therefore

$$P = \sum_{t\in\mathbb{Z}} \sum_{r\geq 0} v_r R^{\max(-t,t+r)}, \qquad v_0 = 1, quad v_r = (1-c^{-1})c^r \ (r \geq 1).$$

One way to sum without parity cases is to fix t. For $t \geq 0$ the x-integral equals R^tC, where

$$C = 1 + (1-c^{-1})\sum_{r\geq 1}(cR)^r = \frac{1-R}{1-cR}.$$

For $t = -j < 0$, the ball $r \leq 2j$ has volume c^{2j} and the remaining shells form another geometric tail. The result is $C(c^2R)^j$. Hence

$$P = C\left(\sum_{t\geq 0} R^t + \sum_{j\geq 1}(c^2R)^j\right) = \frac{1+cR}{1-c^2R} = \frac{1+q^{-1}z}{1-q^{d-2}z}.$$

The stated disk ensures absolute convergence of both sums and of the inner x-integral. □

For example $d = 2$ gives $(1+q^{-1}z)/(1-z)$, while $d = 3$ gives $(1+q^{-1}z)/(1-qz)$. The z here is the value of an $\mathrm{SL}_2(L)$ torus character on $t(\varpi)$; it is *not* the thesis's GL_2 parameter $z_{\mathrm{GL}} = \chi_{\mathrm{GL}}(\varpi)$ for the diagonal Galois varieties. Under restriction of $\mathrm{Ind}(\chi_{\mathrm{GL}}, \chi_{\mathrm{GL}}^{-1})$, the SL_2 parameter is z_{GL}^2.

4 Two elementary new unitary subgroup periods

Let E/F be unramified quadratic, $p \neq 2$, and take $G = \mathrm{SU}_3(F)$ in the thesis's antidiagonal Hermitian model, $K = G \cap \mathrm{SL}_3(\mathfrak{o}_E)$. Put $z = q^{-2s}$ and

$$\phi_z(g) = M(g)^{-s-1}, \quad M(g) = \max_j |g_{3j}|_E, \quad |\varpi|_E = q^{-2}. \tag{3}$$

Choose $\eta \in \mathfrak{o}_E^\times$ with $\eta + \bar\eta = -1$ and residue outside $\mathbb{F}_q$; the latter strengthens the choice in the thesis and ensures $|1+\eta x|_E = \max(1, |x|_F^2)$ for $x \in F$. Take the thesis's $\gamma \in K$ with bottom row $(\eta, 1, 1)$. All additive unit balls and compact torus unit groups below have volume one. The following results concern the $[L : F] = 1$ members of the two SU_3 families; in Definition 3.4.10 an additional $\mathrm{Res}_{F/k}$ is harmless.

Proposition 2. *For the* (a-A-5-1) *realization*

$$H_2 = T(E^\times)U_{2\alpha}, \quad t(a) = \mathrm{diag}(a, \bar{a}/a, \bar{a}^{-1}), \quad u(y) = I + yE_{13}, \quad \mathrm{Tr}_{E/F}\, y = 0,$$

the chosen γ has finite open stabilizer $Z(G) = \mu_3$. For $h = t(a)u(y)$ the right Haar measure is $|\,\mathrm{N}_{E/F}(a)|_F\, d^\times a\, dy$, normalized on the compact quotient. Then

$$P_{H_2,\gamma}(z) = \frac{1 + q^{-1}z}{1 - z}, \qquad |z| < 1. \tag{4}$$

Proof. Write $n = v_E(a)$ and identify $\ker \mathrm{Tr}_{E/F}$ with F by a trace-zero unit. The bottom row of $\gamma t(a)u(y)$ is $(\eta a, \bar{a}/a, \eta a y + \bar{a}^{-1})$. The trace of $1/\eta$ is $-1/\,\mathrm{N}(\eta)$, a unit, so the last two summands cannot cancel on their equal-valuation shell. Thus, exactly,

$$M(\gamma t(a)u(y)) = \max\{q^{2|n|}, q^{-2n}|y|_F^2\}.$$

Put $r = q^{-2}z$ and $C = (1 - r)/(1 - qr)$. The shell sum over y at $n = 0$ is C. Without the right-Haar weight the $n \geq 0$ integral equals $z^n C$, and the $n = -j < 0$ integral equals $r^j C$. The weight $|\,\mathrm{N}(a)|_F = q^{-2n}$ exchanges the two tails. Therefore

$$P = C\left(\sum_{n\geq 0} r^n + \sum_{j\geq 1} z^j\right) = \frac{1 - q^{-2}z^2}{(1 - q^{-1}z)(1 - z)} = \frac{1 + q^{-1}z}{1 - z}.$$

For the orbit assertion, the lower triangular entries of $\gamma t(a)u(y)\gamma^{-1}$ vanish only if $y = 0$ and $\bar{a} = a^2$, whence $a^3 = 1$. The finite stabilizer has dimension zero, and $\dim H_2 + \dim B = \dim G$. $\square$

Proposition 3. *For the* (a-A-4-1) *realization let $\tau \in \mathfrak{o}_E^\times$ have trace zero and put*

$$H_1 = T_1 U_1, \quad t(a) = \mathrm{diag}(a, 1, a^{-1}),\ a \in F^\times, \quad u(x, v) = \begin{pmatrix} 1 & x & \eta x^2 + \tau v \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix},\ x, v \in F.$$

The chosen γ has trivial open B-stabilizer. In coordinates $h = u(x, v)t(a)$ the right Haar measure is $dx\, dv\, d^\times a$. Then

$$P_{H_1,\gamma}(z) = \frac{1 + q^{-1}z}{1 - qz}, \qquad |qz| < 1. \tag{5}$$

Proof. The bottom row of $\gamma u(x, v)t(a)$ is

$$(\eta a,\ 1 + \eta x,\ a^{-1}P(x, v)), \quad P = 1 - x + \eta^2 x^2 + \eta\tau v.$$

Put $L_x = 1 + \eta x$. Since $\eta + \bar{\eta} = -1$, we have $P = \mathrm{N}(L_x) + \eta\tau(v + cx^2)$ for a constant $c \in F$ (write $\eta - \bar{\eta} = c\tau$). Because $p \neq 2$, 1 and $\eta\tau$ form an integral F-basis of $\mathfrak{o}_E$ after a unit change of coordinates. Translating v by cx^2 therefore yields

$$|P(x, v)|_E = \max\{|L_x|_E^2, |v'|_F^2\}.$$

For $j = \max(0, -v_F(x))$ and $n = v_F(a)$, $|L_x|_E = q^{2j}$ and

$$M = \max\{q^{-2n}, q^{2j}, q^{2n}\max(q^{4j}, |v'|_F^2)\}.$$

Scale $v' = \varpi^{-2j} w$ and shift $n' = n + j$. This extracts q^{2j} from M and q^{2j} from dv'. The remaining sum over n' and w is exactly the shell sum of Proposition 2, namely $J(z) = (1 + q^{-1}z)/(1 - z)$. The shell $j = 0$ has x-volume one, whereas $j \geq 1$ has volume $(1 - q^{-1})q^j$. Thus

$$P = J(z)\left[1 + (1 - q^{-1})\sum_{j\geq 1}(q^3 r)^j\right] = J(z)\frac{1 - q^2 r}{1 - q^3 r} = \frac{1 + q^{-1}z}{1 - qz}, \quad r = q^{-2}z.$$

To check openness, $\gamma h \gamma^{-1} \in B$ implies the row $(\eta, 1, 1)h$ is a scalar multiple of $(\eta, 1, 1)$. Its first two entries give $x = (a - 1)/\eta$; since $a, x \in F$ and $\eta \notin F$, this forces $a = 1$, $x = 0$, then $v = 0$. Also $\dim H_1 + \dim B = \dim G$. $\square$

Remark 1. *The two unitary periods are numerically quite different although P_{H_2} coincides with the existing $\mathrm{SU}_2 \backslash \mathrm{SU}_3$ period. In particular P_{H_1} has a pole at $z = q^{-1}$ and requires the smaller convergence disk $|z| < q^{-1}$. A shared rational period does not identify the corresponding orbit geometry or the L-group representation.*

5 Remaining direct calculations and existing anchor cases

For $G = \mathrm{SL}_2(L)$ let $Q = q^d$ and use the same $z = \chi(t(\varpi))$ as in Proposition 1. For split T choose $wu(1) \in K$. Its period is $\sum_{n\in\mathbb{Z}}(Q^{-1}z)^{|n|} = (1 + Q^{-1}z)/(1 - Q^{-1}z)$, valid for $|Q^{-1}z| < 1$. For an unramified anisotropic torus embedded in K, $T(L)$ is compact and $\phi_z(\gamma h) = 1$ for $\gamma, h \in K$, so its normalized period is one. An integral normalizer coset has a representative in K; after normalizing the compact quotient to volume one, the normalizer period agrees with its torus period. The rational finite stabilizer must be included in that quotient normalization. For U, the bottom row of $wu(b)$ has norm $\max(1, |b|_L)$, giving

$$P_{U\backslash \mathrm{SL}_2}(z) = 1 + (1 - Q^{-1})\sum_{j\geq 1} z^j = \frac{1 - Q^{-1}z}{1 - z}, \qquad |z| < 1.$$

These are ordinary *untwisted* open integrals; they are not Whittaker integrals with a nontrivial character.

For $U\backslash \mathrm{SU}_3$ the usual upper-unipotent coordinates are (x, y) with $x \in E$ and $y = -x\bar{x}/2 + y_0$, $\mathrm{Tr}\, y_0 = 0$. With unit-ball volumes one and an integral opposite Weyl representative,

$$M(wu) = \max(1, |x|_E^2, |y_0|_E).$$

Put $R = q^{-2}z$. The set on which the exponent of this maximum is at most N has volume $C_N = q^{N+2\lfloor N/2\rfloor}$. Summation by shells gives

$$P_{U\backslash \mathrm{SU}_3}(z) = (1 - R)\sum_{N\geq 0} C_N R^N = \frac{(1 - q^{-2}z)(1 + q^{-1}z)}{1 - z^2}, \qquad |z| < 1.$$

This agrees with the rank-one unramified intertwining factor in thesis §3.3.

For the diagonal Galois SL_2 varieties, the thesis first integrates a GL_2 principal series. Its $z_{\mathrm{GL}} = \chi_{\mathrm{GL}}(\varpi)$

and $q = |\mathbb{F}_F|$ give

$$P_{\text{quad}}(z_{\text{GL}}) = \frac{1+q^{-1}z_{\text{GL}}}{1-z_{\text{GL}}} \qquad \text{(thesis Prop. 3.5.2; [1, §20, Prop. 37]),}$$
$$P_{\text{cubic}}(z_{\text{GL}}) = \frac{1+q^{-3/2}z_{\text{GL}}}{1-q^{-1/2}z_{\text{GL}}} \qquad \text{(thesis Prop. 3.6.2, after the representative repair below),}$$
$$P_{\mathrm{SU}_2\,\backslash\,\mathrm{SU}_3}(z) = \frac{1+q^{-1}z}{1-z} \qquad \text{(thesis §3.7, } z = q^{-2s}\text{).}$$

For the last formula, the SU_2 double coset $K_H \operatorname{diag}(\varpi^n, 1, \varpi^{-n})K_H$ has volume $q^{2n} + q^{2n-1}$ for $n \geq 1$ and section value $q^{-2n(s+1)}$; their sum is the displayed expression. The original cubic formula has convergence $\Re\lambda > -1/3$ with $z_{\text{GL}} = q^{-3\lambda/2}$.

6 Comparison with Conjecture 4.4.14: values, gaps, and repairs

The raw period P does *not* equal the thesis's β automatically. In the three existing calculations, the thesis uses the group-dependent Mellin factor $M(z)$ from $R_\chi\Phi_K = M(z)\phi_z$ and obtains $\mathrm{ev}_{x_0} \Delta_\chi(\Phi_K) = C\, M(z)P(z)$, where C is independent of z. The verified anchors are

Case	Mellin factor, ignoring a constant	$M(z)P(z)$
Quadratic GL_2	$(1-z^2)/(1-q^{-2}z^2)$	$(1+z)/(1-q^{-1}z)$
Cubic GL_2	$(1-z^2)/(1-q^{-3}z^2)$	$(1-z^2)/[(1-q^{-1/2}z)(1-q^{-3/2}z)]$
$\mathrm{SU}_2\setminus\mathrm{SU}_3$	$(1-z^2)/[(1-q^{-2}z)(1+q^{-1}z)]$	$(1+z)/(1-q^{-2}z)$

The last line uses thesis §3.7. For the two new elementary unitary calculations, *if* the standard eigenmeasure restricts at the chosen base point to a z-independent multiple of our right Haar, the same universal SU_3 Mellin factor gives the following *period-implied candidates*:

$$A_{T_1}(z) = \frac{1-z^2}{(1-q^{-2}z)(1-qz)}, \qquad A_{T_2}(z) = \frac{1+z}{1-q^{-2}z}. \tag{6}$$

For $T_1U_1\backslash \operatorname{Res}_{L/F}\mathrm{SL}_2$, the corresponding *conditional* expression in the present SL_2 parameter is

$$A_{\text{trace}}(z) = \frac{(1-z)(1+q^{-1}z)}{(1-q^{-d}z)(1-q^{d-2}z)},$$

using the $\mathrm{SL}_2(L)$ Mellin factor $(1-z)/(1-q^{-d}z)$. These expressions are formulas to test, not identifications with an L-factor. For the two untwisted U rows, the same group Mellin factor cancels the raw Gindikin–Karpelevich period, leaving a constant, as expected from the type U normalization.

There are four separate obligations before a proof of Conjecture 4.4.14 follows: (i) identify its $\widetilde{\chi}$ and $\delta(X)^{1/2}$ shift with each z; (ii) fix the eigenmeasure/right-Haar scalar and rational orbit; (iii) compute the Frobenius weights and grading of V_X^+ and the denominator prescribed in the conjecture; and (iv) show that the Fourier functional equation acts by the claimed scalar on the selected open-orbit line. A ratio $B_\omega(z) = \beta(z)/\beta(z^{-1})$ determines β only up to a Weyl-invariant multiplier.

There is also a *definitional* obstacle: thesis Definitions 4.4.9–4.4.10 define V_X and V_X^+ only for homogeneous *affine* varieties (and assume Conjecture 4.4.7), whereas Conjecture 4.4.14 quantifies over every basic variety without type N. The U and T_iU_i rows have nonreductive stabilizers and are not homogeneous affine in characteristic zero. Their $L(\widetilde{\chi}, V_X^+)$ is therefore not defined by those definitions.

A formal product of positive torus weights, with a specified Frobenius action, is one possible repair; it must be stated and verified. The text of Proposition 4.4.15 gives a one-line case list, not an independent computation of these weights. Sakellaridis's L-value formula has its own hypotheses and an additional Weyl-invariant factor in the non-affine setting [2, Thm. 7.2.1].

A necessary repair to thesis Proposition 3.6.2. The written choice $i \in \mathfrak{o}_L^\times \setminus \mathfrak{o}_F^\times$ in the cubic case does not imply $|i-b|_L = 1$ for every $b \in \mathfrak{o}_F$. For example $i = 1+\varpi\theta$ with $\bar{\theta} \notin \mathbb{F}_q$ is allowed, but $i-1$ is divisible by ϖ. The shell $a=\varpi$, $b \in 1+\varpi\mathfrak{o}_F$ then contradicts the claimed valuation split. Choose instead $\bar{i} \notin \mathbb{F}_q$; the proof's shell computation and displayed rational function are then valid. More generally, if i approaches F to positive valuation, the integral acquires a nonconstant parameter-dependent factor. The quadratic thesis choice $\mathrm{Tr}(i)=0$ and $i \in \mathfrak{o}_L^\times$ already has nonrational residue when $p \neq 2$.

Two apparent β inconsistencies on the displayed branch. In §3.5 the thesis's GL_2 computation has $\beta_{\mathrm{GL}}(z) = (1+z)/(1-q^{-1}z)$ (Definition 3.5.5), but Definition 3.5.7 writes $(1-u)/(1-q^{-1}u)$ for the SL_2 transfer. On the stated distinguished branch $u = \widetilde{\chi}_1(\varpi) = z$, the Weyl ratios differ by a sign: $(1-u)/(1-u^{-1}) = -u$, while $(1+u)/(1+u^{-1}) = u$. Indeed Proposition 3.5.4 and the displayed multiplier $-e^{\check{\alpha}} = -z^2$ require the plus numerator. In §3.6 Definition 3.6.9 writes $(1-u)/[(1-q^{-1/2}u)(1-q^{-3/2}u)]$, whereas Definitions 3.6.6 and Proposition 3.6.5 on the displayed branch require $1-u^2$ in the numerator. These are algebraic inconsistencies in the shown coordinate unless an unstated change of parameter or Weyl action intervenes. Resolve that coordinate before using Propositions 3.5.8 or 3.6.10 to declare the conjecture proved. The cubic thesis statement singles out distinguished unramified characters; do not silently extend a particular scalar to every rational orbit or character branch.

7 Evidence and next calculations

The proofs of Propositions 1–3 use exact valuation partitions and convergent geometric series. As a diagnostic independent of the final rational simplification, finite shell sums at $q=3, z=1/20$ give $61/57$ for the T_2U_2 formula up to an error below $1.7 \cdot 10^{-34}$ (torus shells $|n| \leq 25$, unipotent shells $j \leq 30$), and $61/51$ for the T_1U_1 formula up to $5.6 \cdot 10^{-14}$ (also truncating its outer x shell at 15). These finite checks are diagnostic, not proofs of the infinite identities.

The remaining genuine calculations are the coupled $[L:F] > 1$ unitary T_1U_1 and odd-degree T_2U_2 families, the type N $\mathrm{SO}_3 \setminus \mathrm{SU}_3$ period, and ramified torus forms if intended by Definition 3.4.10. The two coupled unitary subgroups impose trace/sum relations across field embeddings, so their periods cannot be obtained by replacing q in Propositions 3–2. A useful route is to write a trace-compatible integral representative, verify its finite open stabilizer, then count joint valuations in the constrained unipotent coordinates. For Conjecture 4.4.14, the decisive next step is a table of $\delta(X)$, the actual V_X^+ weights, and Frobenius on those weights for each new case.

Prior-art boundary. Jacquet–Lapid–Rogawski explicitly calculated the quadratic GL_2 open period [1, §20, Prop. 37]. Sakellaridis treated the split ambient group's rank-one U, T, and N functional equations and the split triple-product example [2, §§5–7]; his theorem is not, by itself, a cyclic field or nonsplit unitary period calculation. Wan–Zhang treat a related split trilinear relative character [3, §2.3.2], not the precise cubic open spherical-section integral here. The cubic and SU_2 periods are already in this thesis. We did not identify the three new explicit T_iU_i period formulas in these primary sources. This limited comparison supports labeling them *candidate new direct evaluations*; a systematic bibliography search is still needed before a publication priority assertion.

References

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[2] Y. Sakellaridis, *Spherical functions on spherical varieties*, Amer. J. Math. **135** (2013), 1291–1381; arXiv:0905.4244v3. https://arxiv.org/pdf/0905.4244.

[3] C. Wan and L. Zhang, *Periods of automorphic forms associated to strongly tempered spherical varieties*, arXiv:2102.03695, §2.3.2. https://arxiv.org/pdf/2102.03695.

[4] Supplied *Thesis new.pdf*, Definition 3.4.10 (p. 109), Propositions 3.5.2, 3.6.2, 3.7.2 (pp. 128, 130–131, 136), and Conjecture 4.4.14 (p. 151), version supplied 29 September 2026.