# MULTIPLICITY ONE SPHERICAL PAIRS

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## 1. Introduction

We let $k$ be a $p$-adic local field of characteristic 0 and $\mathfrak{o}$ its integer ring, let $\mathbf{G}$ be a split connected reductive group over $k$ with a fixed Borel subgroup $\mathbf{B}$ and a maximal torus $\mathbf{A} \subset \mathbf{B}$, we denote $K = \mathbf{G}(\mathfrak{o})$. We assume $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ is a homogeneous affine spherical variety and $\mathbf{A}_X$ its maximal torus. In the Book [SV17], the following assumption is needed when formulating [SV17, Conjecture 17.4.1].

**Assumption 1.0.1.** *We assume*

$$dim\ Hom_G(\pi^\infty, C^\infty(X)) \leq 1$$

*for all $\pi$ an irreducible unitary representation of $G$.*

This assumption is very difficult to check for a given spherical variety $\mathbf{X}$, and we want to formulate a weaker multiplicity one assumption which is easy to check. We first want to look at the spherical varieties $\mathbf{X}$ that has a multiplicity one property for the unramified spectrum.

Let's recall the following generic multiplicity formula from [Sak08]: Denote $P(X)$ the associated parabolic subgroup of $X$,

**Theorem 1.0.2.** *([Sak13, Theorem 3.2.2]) Let $\pi$ be an irreducible unramified representation of $G$. Then a necessary condition for $\pi$ to be $X$-distinguished is that $\pi$ is the unramified subquoteint for $I_{P(X)}(\chi)$ for some $\chi \in \overline{A_X^*}$. For a generic $\chi$ and $\pi$ is the irreducible unramified quotient of $I_{P(X)}(\chi)$ then:*

$$dim\ Hom_G(\pi, C^\infty(X)) = g \cdot a$$

*here $g$ is the degree of the generic fiber of the map: $\overline{A_X^*}/W_X \to A^*/W$ and $a$ is the number of the open Borel orbit of $X$.*

**Definition 1.0.3.** For a spherical pair $(\mathbf{G}, \mathbf{H})$ we say that $(\mathbf{G}, \mathbf{H})$ is a spherical pair of generic multiplicity one if $g = 1$ and $a = 1$.

For a symmetric pair $(\mathbf{G}, \mathbf{H})$, we have $g = 1$, and hence $(\mathbf{G}, \mathbf{H})$ is a spherical pair of generic multiplicity one if $a = 1$, that is to say $\mathring{X}$ consists of a single $B$-orbit.

In the work [AG09], stability of a symmetric pair is introduced to study the Gelfand property of a symmetric pair. In [Car19], Carmeli introduced the notion of $p$-stability for a symmetric pair. A stable symmetric pair is $p$-stable but the converse is not true for a split group $\mathbf{G}$. I realized that the $p$-stability for a symmetric pair $(\mathbf{G}, \mathbf{H})$ with $\mathbf{G}$ a connected split group over $k$ is equivalent to the generic multiplicity one property (Definition 1.0.3).

Since the $p$-stability and stability are not equivalent in general. We therefore want to study the uniform multiplicity one symmetric pairs.

We introduce the following restriction map: Denote $\mathbf{A}_X$ the maximal torus of $\mathbf{X}$,

$$\text{Res:} \quad \mathbb{C}[A^*]^W \longrightarrow \mathbb{C}[\delta_{(X)}^{1/2} A_X^*]^{W_X} \tag{1.1}$$

We have the following theorem of Sakellaridis:

**Theorem 1.0.4.** *([Sak13, Corollary 8.0.4]) Suppose the simple spherical roots of $\boldsymbol{X}$ satisfy the statements [Sak13, Statement 6.3.1] and [Sak13, Statement 7.1.5] and assume that the restriction map (1.1) is surjective. Then for every irreducible unramified representation $\pi$ we have*

$$dim\ Hom_G(\pi, C^\infty(X)) \leq 1$$

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*Remark* 1.0.5. The combinatorial statements in this theorem can be checked directly case by case.

We make the following definition of uniform multiplicity one spherical pairs:

**Definition 1.0.6.** For a spherical pair $(\mathbf{G}, \mathbf{H})$, we say that $(\mathbf{G}, \mathbf{H})$ is a spherical pair of uniform multiplicity one if for $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$, we have $\mathbf{X}$ doesn't have a type $N$ roots over algebraic closure, and all the simple spherical roots of $\mathbf{X}$ satisfy the statements [Sak13, Statement 6.3.1] and [Sak13, Statement 7.1.5], also $\mathring{X}$ consists of a single $B$-orbit and the restriction map

$$\mathbb{C}[A^*]^W \longrightarrow \mathbb{C}[\delta_{(X)}^{1/2} A_X^*]^{W_X}$$

is surjective.

We make the following conjecture:

**Conjecture 1.0.7.** *For $(\boldsymbol{G}, \boldsymbol{H}, \theta)$ a spherical symmetric pair of uniform multiplicity one, then we have $(\boldsymbol{G}, \boldsymbol{H}, \theta)$ is a stable symmetric pair.*

We have the following conjecture:

**Conjecture 1.0.8.** *For $(\boldsymbol{G}, \boldsymbol{H})$ a spherical pair of uniform multiplicity one, then we have $\boldsymbol{X} = \boldsymbol{H}\backslash\boldsymbol{G}$ satisfies Assumption 1.0.1.*

## 2. Symmetric subgroups in simple algebraic groups

In this section, we let $G$ be a simple algebraic group defined over an algebraically closed $k$ of characteristic 0. We summarize the classification of connected symmetric subgroups $H$ in $G$ from [KR15].

**Theorem 2.0.1.** *([KR15, Main theorem]) Let $G$ be a simple connected reductive group and $H \subset G$ be a closed connected reductive group of $G$. Then $H$ is a symmetric subgroup in $G$ if and only if $H$ is one of the groups in the following list:*
*Classical groups:*

- $(G, H) = (SL_n, SO_n),\ n \geq 2.$
- $(G, H) = (SL_{m+n}, S(GL_m \times GL_n)),\ m \geq n \geq 1.$
- $(G, H) = (SL_{2n}, Sp_{2n}),\ n \geq 2.$
- $(G, H) = (Sp_{2n}, GL_n),\ n \geq 1.$
- $(G, H) = (Sp_{2m+2n}, Sp_{2m} \times Sp_{2n}),\ m \geq n \geq 1.$
- $(G, H) = (SO_{2n}, GL_n),\ n \geq 2$ *even (there are two conjugacy classes of subgroups swapped by an outer automorphism of $G$).*
- $(G, H) = (SO_{2n}, GL_n),\ n \geq 3$ *odd.*
- $(G, H) = (SO_{m+n}, SO_m \times SO_n),\ m \geq n \geq 1,\ m + n = 3$ *or* $m + n \geq 5.$

*Exceptional groups:*

- $(G, H) = (G_2, A_1 \times \tilde{A}_1).$
- $(G, H) = (F_4, B_4).$
- $(G, H) = (F_4, C_3 \times A_1).$
- $(G, H) = (E_6, C_4).$
- $(G, H) = (E_6, F_4).$
- $(G, H) = (E_6, \mathbb{G}_m \cdot D_5).$
- $(G, H) = (E_6, A_5 \times A_1).$
- $(G, H) = (E_7, \mathbb{G}_m \cdot E_6).$
- $(G, H) = (E_7, A_7).$
- $(G, H) = (E_7, D_6 \times A_1).$
- $(G, H) = (E_8, D_8).$
- $(G, H) = (E_8, E_7 \times A_1).$

We expect that our Conjecture 1.0.7 is true for all the uniform multiplicity one symmetric pairs which are $k$-forms of symmetric pairs in the List 2.0.1, and this can be checked case by case.

## 3. Symmetric pairs and Gelfand pairs

In this section, let $k$ be a $p$-adic local field of characteristic 0, we summarize the results of [AG09] on Gelfand properties for symmetric pairs.

### 3.1. **Distributions.**

**Definition 3.1.1.** A $B$-analytic manifold is an analytic manifold over a non-Archimedean field $k$.

Let $V$ be a $k$-rational finite dimensional representation of a reductive group $G$. We set

$$Q_G(V) := Q(V) := (V/V^G)(k)$$

since $G$ is reductive, we have a canonical embedding $Q(V) \hookrightarrow V(k)$. Let $\pi : V(k) \to (V/G)(k)$ be the natural map. We set

$$\Gamma_G(V) := \Gamma(V) := \pi^{-1}(\pi(0))$$

Note that $\Gamma(V) \subset Q(V)$, we set

$$R_G(V) = R(V) := Q(V) - \Gamma(V)$$

**Definition 3.1.2.** For a $B$-analytic manifold $M$, we define the space of Schwartz distributions $\mathcal{S}^*(M)$ by $\mathcal{S}^*(M) := \mathcal{S}(M)^*$, here $\mathcal{S}(M)^*$ is the dual space of $\mathcal{S}(M)$.

### 3.2. **Symmetric pairs.**

**Definition 3.2.1.** A **symmetric pair** is a triple $(G, H, \theta)$ where $H \subset G$ are reductive groups, and $\theta$ is an involution of $G$ such that $H = G^\theta$. We call a symmetric pair **connected** if $H\backslash G$ is connected.

For a symmetric pair $(G, H, \theta)$, we define an anti-involution $\sigma$ as

$$\sigma(g) = \theta(g^{-1})$$

we introduce the symmetrization map

$$s : \ G \longrightarrow G^\sigma, \quad s(g) = g\sigma(g) = g\theta(g^{-1})$$

Denote $\mathfrak{g} = \mathrm{Lie}(G)$, we denote

$$\mathfrak{g}^\sigma = \{a \in \mathfrak{g} \mid \sigma(a) = a\}$$

**Definition 3.2.2.** ([AG09, Definition 7.1.6]) A symmetric pair $(G, H, \theta)$ is called stable if for all closed $H(k) \times H(k)$ orbit $\mathcal{O} \subset G(k)$, we have $\sigma(\mathcal{O}) = \mathcal{O}$.

**Definition 3.2.3.** A symmetric pair $(G, H, \theta)$ is called a **GK pair** if

$$\mathcal{S}^*(G(k))^{H(k)\times H(k)} \subset \mathcal{S}^*(G(k))^\sigma$$

### 3.3. **Descendants of symmetric pairs.** Let $k$ be a local field of characteristic 0, we let

$$(G, H, \theta), \quad H = G^\theta$$

**Definition 3.3.1.** For $g \in G(k)$, such that the $H \times H$-orbit $HgH$ is closed. Such an element $g$ is called $H \times H$ semisimple.

Set $x = s(g)$ and define

$$G_x = Z_G(x), \quad H_x = H \cap G_x$$

we say that

$$(G_x, H_x, \theta_x = \theta|_{G_x})$$

is a **descendant** of $(G, H, \theta)$.

3.4. **Regular symmetric pairs.**

**Definition 3.4.1.** Let $(G, H, \theta)$ be a symmetric pair. We call an element $g \in G(k)$ **admissible** if

- $\mathrm{Ad}(g)$ commutes with $\theta$.
- $\mathrm{Ad}(g)|_{\mathfrak{g}^\sigma}$ is $H$-admissible.

**Definition 3.4.2.** We say that a symmetric pair $(G, H, \theta)$ **regular** if for any admissible $g \in G(k)$ such that $\mathcal{S}^*(R(g^\sigma))^{H(k)} \subset \mathcal{S}^*(R(\mathfrak{g}^\sigma))^{\mathrm{Ad}(g)}$ we have

$$\mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{H(k)} \subset \mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{\mathrm{Ad}(g)}$$

We have the following theorem on the criterion for a GK pair,

**Theorem 3.4.3.** *([AG09, Theorem 7.4.5]) Let $(G, H, \theta)$ be a stable symmetric pair such that all its descendants are regular. Then it is a GK pair.*

We recall the conjecture on the regularity of symmetric pairs from the AG paper.

**Conjecture 3.4.4.** *([AG09, Conjecture 4]) Any symmetric pair is regular.*

This conjecture was verified for most classical symmetric pairs and several exceptional ones.

3.5. **Gelfand properties.** We introduce the various Gelfand properties in this section.

**Definition 3.5.1.** ([AG09, Definition 8.1.2]) Let $H \subset G$ be a pair of reductive groups.

- We say that $(G, H)$ satisfy **GP1** if for any irreducible admissible representation $\pi$ of $G$ we have
$$\dim \mathrm{Hom}_{H(k)}(\pi, \mathbb{C}) \leq 1$$
- We say that $(G, H)$ satisfy **GP2** if for every irreducible admissible representation $\pi$ of $G$, the following property holds:
$$\dim(\mathrm{Hom}_{H(k)}(\pi, \mathbb{C})) \cdot \dim(\mathrm{Hom}_{H(k)}(\tilde{\pi}, \mathbb{C})) \leq 1$$
- We say that $(G, H)$ satisfy **GP3** if for any irreducible unitary representation $\pi$ of $G(k)$ on a Hilbert space $\mathcal{H}$ we have
$$\dim \mathrm{Hom}_{H(k)}(\mathcal{H}^\infty, \mathbb{C}) \leq 1$$
here $\mathcal{H}^\infty$ is the space of smooth vectors of the Hilbert space.

We have $GP1 \Rightarrow GP2 \Rightarrow GP3$ and we have the following criterion for GP2,

**Proposition 3.5.2.** *Let $H \subset G$ be reductive groups and let $\tau$ be an involutive anti-automorphism of $G$ and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(k)$-invariant Schwartz distributions $\xi$ on $G(k)$. Then $(G, H)$ satisfies GP2.*

**Proposition 3.5.3.** *([AG09, Corollary 8.1.6]) Any symmetric GK pair satisfies GP2.*

This proposition is true because we have $\tau(g) = \theta(g^{-1})$ is an involutive anti-automorphism of $G$ and $\tau(H) = H$, from the definition of the GK-pair (Definition 3.2.3), we have $\tau(\xi) = \xi$ for all bi $H(k)$-invariant Schwartz distributions $\xi$ on $G(k)$ and we may apply Proposition 3.5.2.

3.6. **$p$-stable symmetric pairs.** In this section we let $k$ be a $p$-adic local field of characteristic 0 and $G$ the $k$-points of a connected split reductive group $\mathbf{G}$ over $k$. We let $\mathbf{P}$ be a $k$-parabolic subgroup of $\mathbf{G}$ and $P = \mathbf{P}(k)$, let $\mathbf{M}_P$ denote a Levi factor of $\mathbf{P}$ and $X^*_{u,ur}(M_P)$ the set of unitary unramified characters of $M_P$.

We let $\mathbf{T} \subseteq \mathbf{P}$ be a $\theta$-stable maximal $k$-split torus and $\mathbf{A} = \mathbf{T}^-$ its symmetric part, then $\mathbf{A}$ is a maximal $(\theta, k)$-split torus. The Levi subgroup of $\mathbf{P}$ can be identified as $\mathbf{M}_P = Z_G(\mathbf{A})$.

Consider the space of unramified unitary characters $X^*_{u,ur}(A)$ and $X^*_{u,ur}(T)$, we have $X^*_{u,ur}(A) \cong X^*_{u,ur}(T)^\sigma$. We have inclusions

$$A \subseteq T \subseteq M_P \subseteq P$$

and therefore we have the restriction map,

$$r_T^{M_P} \ : \ X^*_{u,ur}(M_P) \longrightarrow X^*_{u,ur}(T)$$

Since $A \subseteq M_P$ and the intersection $A \cap [M_P, M_P]$ is finite, hence all the elements of $X^*_{u,ur}(A)$ are trivial on $[M_P, M_P] \cap A$. Thus they can be viewed as a character of $M_P$ via $(r_T^{M_P})^{-1}$ and we will make this identification from now on.

Let $\delta_P$ denote the modulus character of $P$, and $\delta_P^{1/2}$ its square root. For $\psi \in X^*_{u,ur}(M_P)$, we define the unitary parabolic induction of $\psi$ by

$$I_P^u(\psi) = \{f \in C^\infty(G, \mathbb{C}) \ : \ f(pg) = \psi(p)\delta_P(g)^{1/2} f(g)\}$$

**Definition 3.6.1.** ([Car19, Definition 5.1]) Let $(G, H, \theta)$ be a symmetric pair. We say that $(G, H, \theta)$ is **p-stable** if $H$ has a single open orbit in $G/B$.

*Remark* 3.6.2. It is known that the total number of $\mathbf{H}$-orbits in $\mathbf{G}/\mathbf{B}$ is finite.

The stability implies the $p$-stability.

**Theorem 3.6.3.** *([Car19, Theorem 5.27]) A stable symmetric pair is p-stable.*

*Remark* 3.6.4. For split groups the $p$-stability is weaker than the stability.

We have the following theorem:

**Theorem 3.6.5.** *([Car19, Theorem 7.2]) Let $(G, H, \theta)$ be a symmetric pair. If $(G, H)$ is a Gelfand pair, then $(G, H, \theta)$ is p-stable.*

Theorem 3.6.5 follows from the following proposition,

**Proposition 3.6.6.** *([Car19, Proposition 7.10]) Let $(G, H, \theta)$ be a symmetric pair, $P$ a minimal $\theta$-split parabolic subgroup and $A$ a minimal $(\theta, k)$-split torus of $G$ contained in $P$. There exists an open dense subset $V \subseteq X^*_{u,ur}(A)$ such that for every $\psi \in V$, the representation $I_P^u(\psi)$ is irreducible and admits at least many $H$-invariant functionals as the number of open $H(k)$-orbits in $P(k)\backslash G(k)$.*

## 4. Uniform multiplicity one symmetric pairs

In this section, we let $k$ be a $p$-adic local field of characteristic 0, we let $\mathbf{G}$ be a connected split reductive group over $k$.

### 4.1. Uniform multiplicity one symmetric pairs.

We let $(\mathbf{G}, \mathbf{H})$ be a symmetric pair,

**Theorem 4.1.1.** *Let $(\boldsymbol{G}, \boldsymbol{H})$ be a spherical symmetric pair of uniform multiplicity one. Assume the Conjecture 3.4.4 for $(\boldsymbol{G}, \boldsymbol{H})$, and the Conjecture 1.0.7 for $(\boldsymbol{G}, \boldsymbol{H})$. For $\boldsymbol{X} = \boldsymbol{H}\backslash\boldsymbol{G}$, then we have $(\boldsymbol{G}, \boldsymbol{H}, \theta)$ is a symmetric pair with GP2 property.*

*Proof.* We have $(\mathbf{G}, \mathbf{H}, \theta)$ is stable from Conjecture 1.0.7 and all its descendants are regular from Conjecture 3.4.4. We get $(\mathbf{G}, \mathbf{H}, \theta)$ is a GK pair from Proposition 3.4.3.

From Proposition 3.5.2, we have $(\mathbf{G}, \mathbf{H}, \theta)$ satisfy GP2.

□

## References


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