REPRESENTATIONS OF METAPLECTIC GROUP

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1. Introduction

This is a study note for the Gan-Savin paper [GS12] on the representations of the metaplectic groups, this is a local Shimura correspondence which extends the well-known result of Waldspurger.

2. Waldspurger's result

Let k be a non-Archimedean local field of characteristic zero with odd residual characteristic p. Let $(W, \langle -, - \rangle)$ be a symplectic vector space of dimension 2 over k with associated symplectic group $\mathrm{Sp}(W)$, the group $\mathrm{Sp}(W)$ has a unique two-fold central extension $\mathrm{Mp}(W)$ which is called the metaplectic group

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1$$

Remark 2.1. For k an archimedean local field, the existence of $\mathrm{Mp}(W)$ follows from that the fundamental group of $\mathrm{SL}_2(\mathbb{R})$ is $\mathbb{Z}$.

Let's recall Waldspurger's result in this case. By studying the theta correspondence for $\mathrm{Mp}(W) \times O(V)$, Waldspurger showed the following result

Theorem 2.2. *Fix an additive character ψ of k.*

- *Given any irreducible representation π of $SO(V)$, the theta lift $\theta_{V,W,\psi}(\pi)$ of π to $Mp(W)$ is irreducible and non-zero.*
- *The previous construction gives a bijection*

$$\Theta_\psi : \ Irr(SO(V^+)) \cup Irr(SO(V^-)) \leftrightarrow Irr(Mp(W))$$

- *$\pi \in Irr(SO(V))$ is a discrete series (resp. tempered) representation if and only if $\Theta_\psi(\pi)$ is a discrete series (resp. tempered) representation.*
- *via the local Langlands correspondence for $SO(V^\pm)$ one has a bijection*

$$\mathscr{L}_\psi : \ Irr(Mp(W)) \leftrightarrow \Phi(Mp(W))$$

It is a consequence of the previous theorem that

- given $\pi \in \mathrm{Irr}(\mathrm{SO}(V))$, exactly one extension π^ϵ of π participates in the theta correspondence.
- given $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$, σ participates in the theta correspondence with exactly one of the $O(V^+)$ or $O(V^-)$.

As a refinement of the previous statements, Waldspurger proved

Theorem 2.3. *We have*

- *Given $\pi \in Irr(SO(V))$, π^ϵ participates in theta correspondence with $Mp(W)$ if and only if*

$$\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi)$$

- *Given $\sigma \in Irr(Mp(W))$ σ participates in theta correspondence if and only if*

$$Z_\psi(\sigma) = \epsilon(V) \cdot \epsilon(1/2, \Theta_\psi(\sigma)) = \epsilon(V) \cdot \epsilon(1/2, \mathscr{L}_\psi(\sigma))$$

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What Waldspurger proved is actually: If $\pi \in \mathrm{Irr}(\mathrm{SO}(V^+))$ has L-parameter ϕ and Jacquet-Langlands transfer $\pi' \in \mathrm{Irr}(\mathrm{SO}(V^-))$, then the L-packet associated to ϕ is

$$\mathscr{L}_{\psi,\phi} = \{\Theta_\psi(\pi),\ \Theta_\psi(JL(\pi))\}$$

The decomposition

$$\mathrm{Irr}(\mathrm{Mp}(W)) = \sqcup_\psi \mathscr{L}_{\psi,\phi}$$

is a canonical decomposition in the sense that it is independent of ψ, however the labelling of the representations in each packet by characters of the component group depends on such a choice, Waldspurger determined how the dependence varies with ψ

Theorem 2.4. *For $a \in k^\times$, let ψ_a be the additive character given by $\psi_a(x) = \psi(ax)$ and let χ_a be the quadratic character associated to the class of $a \in k^\times/k^{\times 2}$ suppose*

$$\mathscr{L}_\psi(\sigma) = (\phi, \eta) \text{ and } \mathscr{L}_{\psi_a}(\sigma) = (\phi_a, \eta_a)$$

then $\phi_a = \phi \otimes \chi_a$ and

$$\eta_a/\eta = \epsilon(1/2, \phi \otimes \chi_a) \cdot \epsilon(1/2, \phi) \cdot \chi_a(-1)$$

The purpose of the Gan-Savin paper is to extend the theorem 2.2, 2.3, 2.4 to the case of higher rank.

3. Epsilon dichotomy and local Langlands correspondence

Let k be a non-Archimedean local field of characteristic zero with odd residual characteristic p. Let $(W, \langle -, - \rangle)$ be a symplectic vector space of dimension $2n$ over k with associated symplectic group $\mathrm{Sp}(W)$, the group $\mathrm{Sp}(W)$ has a unique two-fold central extension $\mathrm{Mp}(W)$ which is called the metaplectic group

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1$$

Remark 3.1. For k an archimedean local field and $n = 1$, the existence of $\mathrm{Mp}(W)$ follows from that the fundamental group of $\mathrm{SL}_2(\mathbb{R})$ is $\mathbb{Z}$.

In the rest of the section, we will

- obtain a local Langlands correspondence for $\mathrm{Mp}(W)$ and establish some of its expected properties.
- establish a result known as epsilon dichotomy, where certain local root numbers are shown to control the non-vanishing of certain theta lifts.

Remark 3.2. Let's remark that the second property is typical for theta correspondence.

Theorem 3.3. *For each non-trivial additive character $\psi : k \to \mathbb{C}^\times$, there is a bijection*

$$\Theta_\psi : Irr(Mp(W)) \leftrightarrow Irr(SO(V^+)) \cup Irr(SO(V^-))$$

where V^+ (resp. V^-) is the split (resp. non-split) quadratic space of discriminant 1 and dimension $2n+1$. This bijection is given by the theta correspondence with respect to ψ for the group $Mp(W) \times SO(V^\pm)$.

Corollary 3.4. *Assume the local Langlands correspondence for $SO(V^\pm)$, then one obtains a local Langlands correspondence for $Mp(W)$, that is a bijection*

$$\mathscr{L}_\psi : Irr(Mp(W)) \longrightarrow \Phi(Mp(W))$$

where $\Phi(Mp(W))$ is the set of pairs (ϕ, η) such that

- *$\phi : WD_k \to Sp_{2n}(\mathbb{C})$ is a 2n-dimensional symplectic representation of WD_k.*
- *η is an irreducible representation of the component group A_ϕ.*

One may ask whether the local Langlands correspondence satisfies certain typical properties. We have the following result

Theorem 3.5. *Suppose $\pi \in Irr(SO(V))$ and $\sigma \in Irr(Mp(W))$ correspond under Θ_ψ, then the following hold*

(1) *π is a discrete series representation if and only if σ is a discrete series representation.*
(2) *π is tempered if and only if σ is tempered.*

(3) If π and σ are discrete series representations

$$deg(\pi) = deg(\sigma)$$

with suitable normalization of the Haar measure.
(4) If π is a generic representation of $SO(V^+)$, then σ is a ψ-generic representation of $Mp(W)$, if σ is ψ-generic and tempered, then π is generic.

Let's turn to the proof of 3.3, the key steps are the following two statements:

- given an irreducible representation π of $\mathrm{SO}(V)$, exactly one extension of π to $O(V) = \mathrm{SO}(V) \times \{\pm 1\}$ has a non-zero theta lift to $\mathrm{Mp}(W)$.
- given an irreducible representation σ of $\mathrm{Mp}(W)$, σ has non-zero theta lift to $O(V)$ for exactly one V.

Now we want to ask whether it is possible to specify which extensions of $\pi^{\pm}$ participates in the theta correspondence and given a representation σ of $\mathrm{Mp}(W)$ to which $O(V)$ is the theta lift of σ non-zero. We introduce the following notation

$$\epsilon(V) = \begin{cases} +1 \text{ if } V = V^+ \\ -1 \text{ if } V = V^- \end{cases}$$

the sign ϵ in π^ϵ encodes the central character of π^ϵ: $\epsilon = \pi^\epsilon(-1)$.

On the other hand, for an irreducible genuine representation σ of $\mathrm{Mp}(W)$, we can consider its central character ω_σ which is a genuine character of $\tilde{Z}$ the preimage in $\mathrm{Mp}(W)$ of the center Z of $\mathrm{Sp}(W)$. Using the additive character ψ, one can define a genuine character χ_ψ of $\tilde{Z}$, we can define the central sign $z_\psi(\sigma)$ of σ as

$$z_\psi(\sigma) = \omega_\sigma(-1)/\chi_\psi(-1) \in \{\pm 1\}$$

the quotient above is independent of the choice of the preimage of $-1 \in Z$ in $\tilde{Z}$.

Theorem 3.6. *We have the following result*

- *Let π be an irreducible representation of $SO(V)$, then π^ϵ participates in theta correspondence with respect to ψ with $Mp(W)$ if and only if*

$$\epsilon = \epsilon(V) \cdot \epsilon(\frac{1}{2}, \pi)$$

here $\epsilon(s, \pi, \psi)$ is the standard epsilon factor defined by the doubling method.
Let σ be an irreducible representation of $Mp(W)$, then σ has non-zero theta lift with respect to ψ to $O(V)$ if and only if the central character of σ satisfies

$$z_\psi(\sigma) = \epsilon(V) \cdot \epsilon(\frac{1}{2}, \sigma, \psi) = \epsilon(V) \cdot \epsilon(\frac{1}{2}, \Theta_\psi(\sigma))$$

Finally, we investigate how the local Langlands correspondence $\mathscr{L}_\psi$ depends on ψ, for this we have to assume the local Langlands correspondence for $\mathrm{SO}(V^\pm)$ and it satisfies certain expected properties in relation to the theory of endoscopy. To state the result, we recall $\phi : \mathrm{WD}_k \to \mathrm{Sp}_{2n}(\mathbb{C})$ is a symplectic representation of WD_k and if we write $\phi = \oplus_i n_i \cdot \phi_i$ as a direct sum of irreducible representations ϕ_i with some multiplicities n_i, then the component group A_ϕ is given by

$$A_\phi = \prod_{i:\phi_i \text{ symplectic}} \mathbb{Z}/2\mathbb{Z}a_i$$

so that A_ϕ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ with a canonical basis.

We have the following theorem

Theorem 3.7. *For $\sigma \in Irr(Mp(W))$ and $c \in k^\times$, let*

$$\mathscr{L}_\psi(\sigma) = (\phi, \eta) \text{ and } \mathscr{L}_{\psi_c}(\sigma) = (\phi_c, \eta_c)$$

then the following hold

- *$\phi_c = \phi \otimes \chi_c$ where χ_c is the quadratic character associated with $c \in k^\times/k^{\times 2}$ and it follows we have the canonical identification of component groups*

$$A_\phi = A_{\phi_c} = \oplus_i \; \mathbb{Z}/2\mathbb{Z}a_i$$

so it makes sense to compare η and η_c.
- *The characters η and η_c are related by*

$$\eta_c(a_i)/\eta(a_i) = \epsilon(1/2, \phi_i) \cdot \epsilon(1/2, \phi_i \otimes \chi_c) \cdot \chi_c(-1)^{(dim\phi_i)/2} \in \{\pm 1\}$$

It is interesting that the proof of this last theorem uses the Gross-Prasad conjecture for tempered representations of special orthogonal groups.

References

[GS12] Wee Teck Gan and Gordan Savin. Representations of metaplectic groups i: epsilon dichotomy and local langlands correspondence. *Compositio Mathematica*, 148(6):1655–1694, 2012.