SPHERICAL FUNCTIONS ON A GROUP OF p-ADIC TYPE

RUI CHEN

1. Introduction

This is a summary of the result of Macdonald's book [Mac]. In essence, it is an account of the theory of zonal spherical functions on the group of rational points of a simply-connected simple algebraic group defined over a p-adic field, relatively to a suitably chosen maximal compact group.

2. Plancherel measure

2.1. **The standard case.** Let $\hat{T}$ be the character group of the discrete group T. $\hat{T}$ is the product of ℓ circles, and may be identified with the torus A^*/L, where L is the lattice of linear forms u on A such that $u(\alpha^\vee) \in \mathbb{Z}$ for all $a \in \Sigma_0$. Let ds be Haar measure on the compact group $\hat{T}$, normalized so that the total mass of $\hat{T}$ is 1.

We shall assume that $q_{\alpha/2} \geq 1$ for all $\alpha \in \Sigma_0$ in this section, call this the *standard case*. The exceptional case, where $q_{\alpha/2} < 1$ for some $\alpha \in \Sigma_0$ will be discussed separately.

Theorem 2.1. *Assume that $q_{\alpha/2} \geq 1$ for all $a \in \Sigma_0$, then the Plancherel measure μ on the space Ω^+ of positive definite spherical functions on G relative to K is concentrated on the set $\{\omega_s : s \in \hat{T}\}$ and is given by*

$$d\mu(\omega_s) = \frac{Q(q^{-1})}{|W_0|} \cdot \frac{ds}{|c(s)|^2}$$

where $|W_0|$ is the order of the Weyl group W_0.

2.2. **The exceptional case.** In this section we will consider the case excluded from the considerations of 2.1, namely where $q_{\alpha/2} < 1$ for some $a \in \Sigma_0$. First of all, this will imply that $\frac{a}{2} \in \Sigma_0$ for some $a \in \Sigma_0$, so that the root system Σ_0 is not reduced. Since it is irreducible, it is must be of the type BC_ℓ, there is a basis of Σ_1 consists of

$$\pm e_i,\ (1 \leq i \leq \ell),\ \pm 2e_i,\ (1 \leq i \leq \ell),\ \pm e_i \pm e_j\ (1 \leq i < j \leq \ell)$$

and Σ_0 consists of

$$\pm 2e_i (1 \leq i \leq \ell),\ \pm e_i \pm e_j (1 \leq i < j \leq \ell)$$

We choose the set of simple roots to be

$$\Pi_0 = \{e_1 - e_2,\ e_2 - e_3, \cdots 2e_\ell\}$$

Let

$$q_0 = q_{\pm e_i \pm e_j},\ q_1 = q_{\pm e_i},\ q_2 = q_{\pm 2e_i}$$

so that $q_1 < 1$. Put

$$t_i = t_{2e_i} (1 \leq i \leq \ell)$$

If $s \in S$, put $s_i = s(t_i) \in \mathbb{C}^*$, then $c(s)$ is the product of the factors

$$\frac{(1 + q_1^{-1/2} s_i^{-1})(1 - q_1^{-1/2} q_2^{-1} s_i^{-1})}{1 - s_i^{-2}}$$

for $i = 1 \cdots \ell$ and the factors

$$\frac{1 - q_0^{-1} s_i^{-1} s_j}{1 - s_i^{-1} s_j} \cdot \frac{1 - q_0^{-1} s_i^{-1} s_j^{-1}}{1 - s_i^{-1} s_j^{-1}}$$

Let $\phi(s) = c(s)c(s^{-1})$.

Date: April 2025.

Define a function $\phi_J(s)$ inductively as follows

$$\phi_\emptyset(s) = \phi(s) = c(s)c(s^{-1})$$

also

$$\phi_J(s) = \phi_{j_1,\cdots,j_r}(s) = \lim_{s_{j_1}\to -q_0^{r-1}q_1^{1/2}} \frac{\phi_{j_2\cdots j_r}(s)}{1+q_0^{1-r}q_1^{-1/2}s_{j_1}}$$

The final form of the Plancherel measure will depend on how many of the numbers

$$q_1^{1/2}, q_0 q_1^{1/2}, \cdots, q_0^{\ell-1}q_1^{1/2}$$

are less then 1. So we define ϵ_r $0 \le r \le \ell$ as follows $\epsilon_0 = 1$ and

$$\epsilon_r = \begin{cases} 1 \text{ if } q_0^{r-1}q_1^{1/2} < 1 \\ 0 \text{ otherwise} \end{cases}$$

we may define

$$\phi_r(\omega_s) = \phi_J(s) \text{ if } |J| = r$$

Theorem 2.2. *The Plancherel measure μ on the space of positive definite spherical functions is concentrated on the sets Ω_r such that $\epsilon_r = 1$ where ϵ_r is defined as in. On Ω_r, μ is given by*

$$d\mu(\omega) = \frac{Q(q^{-1})}{\omega_r}\frac{d\omega}{\phi_r(\omega)}$$

where $\omega_r = 2^{\ell-r}(\ell-r)!$ is the order of the normalizer in W_0 of any $\hat{T}$ such that $|J| = r$.

2.3. **Comparison with the real and complex cases.** Let k be a local field, that is to say k is $\mathbb{R}$, $\mathbb{C}$ or a p-adic field, and the additive group of k is self-dual. Associated canonically with k there is a meromorphic function $\gamma_k(s)$ of a complex variable s, sometimes called the gamma-function of k.

If f is any well-behaved function on k, let $\hat{f}$ be its Fourier transform with respect to the additive group structure, since k^+ is self-dual, $\hat{f}$ is a function on k^+. If $Re(s) > 0$, define

$$\zeta(f,s) = \int_{k^\times} f(x)\ ||x||^s d^\times x$$

Then $\zeta(f,s)$ has a functional equation

$$\zeta(f,s) = \gamma_k(s)\zeta(\hat{f}, 1-s)$$

$k = \mathbb{R}$: take $f(x) = e^{-2\pi|x|^2}$ ($|x|$ the ordinary absolute value on $\mathbb{C}$), then again $\hat{f} = f$, we find that

$$\gamma_{\mathbb{R}}(s) = \frac{\pi^{-s/2}\Gamma(\frac{s}{2})}{\pi^{(s-1)/2}\Gamma(\frac{1-s}{2})}$$

so that

$$\gamma_{\mathbb{R}}(s)\gamma_{\mathbb{R}}(-s) = B(\frac{1}{2}, \frac{1}{2}s)B(\frac{1}{2}, -\frac{1}{2}s)$$

$k = \mathbb{C}$: take $f(x) = e^{-2\pi|x|^2}$, then again $\hat{f} = f$ and we have

$$\gamma_{\mathbb{C}}(s) = \frac{(2\pi)^{-s}\Gamma(s)}{(2\pi)^{s-1}\Gamma(1-s)}$$

so that

$$\gamma_{\mathbb{C}}(s)\gamma_{\mathbb{C}}(-s) = \frac{-4\pi^2}{s^2}$$

k a p-adic field: taking f to be the characteristic function of the ring of integer of k, then we find that

$$\gamma_k(s) = d^{s-\frac{1}{2}}\frac{1-q^{s-1}}{1-q^{-s}}$$

here q is the number of elements in the residue field. In this case, we have

$$\gamma_k(s)\gamma_k(-s) = d^{-1}\frac{1-q^{-1-s}}{1-q^{-s}}\frac{1-q^{-1+s}}{1-q^s}$$

Now let G be a universal Chevalley group $G(\Sigma_0, k)$ where k is any local field. If $k = \mathbb{R}$ or $\mathbb{C}$, let K be the maximal compact subgroup of G. If k is p-adic, let K be the maximal compact subgroup of G.

If $k = \mathbb{R}$ or $\mathbb{C}$, then the zonal spherical function on G relative to K are parametrized by the $\mathbb{R}$-linear mappings $s : A \to \mathbb{C}$, and the Plancherel measure for the positive definite spherical functions is supported on the space of pure imaginary s. If ds is a Euclidean measure on this space, then the Harish-Chandra measure μ is of the form

$$d\mu(\omega_s) = \kappa \frac{ds}{|c(s)|^2}$$

where κ is a constant and

$$c(s) = \prod_{a \in \Sigma_0^+} B(\frac{1}{2}, \frac{1}{2} s(a^\vee))$$

we have

$$c(s)c(-s) = \prod_{a \in \Sigma_0} \gamma_k(s(a^\vee))$$

On the other hand, if k is p-adic, then we have seen the support of the Plancherel measure is the character group $\hat{T}$ of T, to bring out the analogy with the real and complex case we shall replace the multiplicative parametrization of the spherical functions, if $s_0 \in S = \mathrm{Hom}(T, \mathbb{C}^*)$, we define

$$s : A \longrightarrow \mathbb{C}/(\frac{2\pi i}{\log q})\mathbb{Z}$$

by the rule

$$s_0(t_a) = q^{-s(a^\vee)}$$

where as before q is the number of elements in the residue field of k. Then from 2.1, the Plancherel measure μ is of the form

$$d\mu(\omega_s) = \kappa \frac{ds}{|c(s)|^2}$$

where ds is the Euclidean measure on the space of pure imaginary s, κ is a constant and

$$c(s) = \prod_{a \in \Sigma_0^+} \frac{1 - q^{-1-s(a^\vee)}}{1 - q^{-s(a^\vee)}}$$

we have

$$c(s)c(-s) = \prod_{a \in \Sigma_0} \gamma_k(s(a^\vee))$$

so sum up

Theorem 2.3. *If k is any local field and $G = G(\Sigma_0, k)$ is a universal Chevalley group, then the Plancherel measure on the space of positive definite spherical functions on G relative to the maximal compact subgroup K is of the form*

$$d\mu(\omega_s) = \frac{\kappa \cdot ds}{\prod_{a \in \Sigma_0} \gamma_k(s(a^\vee))}$$

Also for non-split groups, there is a strong resemblance between the Plancherel measure in the real case and in the p-adic case.

In the work of Harish-Chandra, the Plancherel measure μ is given by

$$d\mu(\omega_\lambda) = \kappa \frac{d\lambda}{|c(\lambda)|^2}$$

where κ is a constant and $c(\lambda)$ is a product of beta-functions, namely

$$c(\lambda) = \prod_{b \in \Sigma_1^+} B(\frac{1}{2} m_b, \frac{1}{4} m_{b/2} + \frac{1}{2} \lambda(b^\vee))$$

let

$$\zeta_{\mathbb{R}}(s) = \pi^{-\frac{1}{2}s} \Gamma(\frac{1}{2}s)$$

which is the local zeta function $\zeta(f,s)$ for $f(x)=e^{-\pi x^2}$, we have

$$c(\lambda)=\kappa\prod_{b\in\Sigma_1^+}\frac{\zeta_{\mathbb{R}}(\frac{1}{2}m_{b/2}+\lambda(b^\vee))}{\zeta_{\mathbb{R}}(m_b+\frac{1}{2}m_{b/2}+\lambda(b^\vee))}$$

where κ is independent of A.

Now we turn to the case k is p-adic. If the number of elements in the residue field of k is q, then each index q_b $(b\in\Sigma_1)$ is a power of q. We shall write

$$q_b=q^{m_b}\ (b\in\Sigma_1)$$

and call m_b the formal multiplicity of the root $b\in\Sigma_1$.

Theorem 2.4. *For k real or p-adic, the Plancherel measure is*

$$d\mu(\omega_\lambda)=\kappa\cdot\frac{d\lambda}{|c(\lambda)|^2}$$

where κ is a constant and

$$c(\lambda)=\prod_{b\in\Sigma_1^+}\frac{\zeta_k(\frac{1}{2}m_{b/2}+\lambda(b^\vee))}{\zeta_k(\frac{1}{2}m_{b/2}+m_b+\lambda(b^\vee))}$$

the functions ζ_k are defined in (5.3.5) and (5.3.7).

Remark 2.5. There is one important difference between the real and p-adic cases, in the p-adic case the formal multiplicity m_b can be negative if $2b$ is also the root, this is precisely the ”exceptional case” dealt before.

References

[Mac] Ian G Macdonald. Spherical functions on a group of p-adic type. Ramanujan Institute, Centre for Advanced Study in Mathematics, University of Madras, Madras, 1971. *Publications of the Ramanujan Institute*, (2):9.