LOCAL EULERIAN PAIRS

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1. Introduction

We introduce a notion of local Eulerian pairs.

Definition 2.0.1 below is the original definition: it asks for a parameter whose fixed locus on the sliced dual space is a single point. This turns out to be finer than the classical notion of an Eulerian period. The fixed locus is acted on by the centralizer of the parameter, and fixed points in one orbit contribute the same Euler product to the period; so a period can be Eulerian in the classical sense while the fixed locus has several points, or even positive dimension. For instance the linear periods of Friedberg–Jacquet, which unfold to the Shalika model, fail Definition 2.0.1 (Example 5.0.3).

Section 3 therefore replaces "the fixed locus is a point" by "the fixed locus is a single orbit of the centralizer of the parameter", and replaces "there exists a parameter" by "for every parameter". Section 4 shows that, for the expected shape of $M^\vee$, the new condition says exactly that an X-distinguished Arthur parameter has a unique lift to $G_X^\vee$. Section 5 compares with classical examples, and explains why a local Eulerian pair need not be a Gelfand pair: (G_2, SL_3) is local Eulerian but has a representation with two invariant functionals [C].

2. Local Eulerian pairs

Let k be a p-adic local field of characteristic 0 and let $\mathbf{G}$ be a split connected reductive group over k. Denote by $WD_k = W_k \times \mathrm{SL}_2(\mathbb{C})$ the Weil–Deligne group of k. Given an L-parameter

$$\phi : \ WD_k \longrightarrow G^\vee$$

one can pull back along ϕ and get an action of WD_k on the hyperspherical $G^\vee$-variety $M^\vee$. As $M^\vee$ is equipped with a grading, i.e. a commuting $\mathbb{G}_m$-action, we can twist the action of WD_k by the map

$$\omega_k : \ WD_k \longrightarrow \mathbb{G}_m, \qquad \omega_k(w) = |w|^{1/2},$$

(trivial on the Deligne SL_2), so that WD_k acts on $M^\vee$ via the extended L-parameter

$$\phi_E = \phi \times \omega_k : \ WD_k \longrightarrow G^\vee \times \mathbb{C}^\times.$$

Date: November 2026. Revised draft: Section 2 is the original note; Sections 3–5 are new.

Given an Arthur parameter $\psi : WD_k \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$, with $\psi|_{W_k}$ bounded, one considers its associated L-parameter

$$\phi_\psi(w) = \psi\big(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})\big).$$

The action of the Arthur parameter ψ is then defined to be the action of WD_k on $M^\vee$ by pullback along the extended L-parameter $\phi_{\psi,E}$.

Let $\{e, h, f\}$ be the $\mathfrak{sl}_2$-triple associated to ψ, and set

$$M^\vee_{\mathrm{slice},\psi} = \mu^{-1}(f + \mathfrak{g}^{\vee,e}) \subseteq M^\vee.$$

Throughout, fixed loci are taken as sets of geometric points.

Definition 2.0.1. Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a homogeneous affine spherical variety such that $M = T^*X$ is a hyperspherical variety. We say that $(\mathbf{G}, \mathbf{H})$ is a *local Eulerian pair* if there exists an extended Langlands parameter ϕ_E associated with an X-distinguished Arthur parameter ϕ_A such that

$$M^{\vee,\phi_E}_{\mathrm{slice},\phi_A} = \{m_1\}.$$

Here we say ϕ_A is X-distinguished if it is of the same SL_2-type as the distinguished morphism ι_X of X,

$$\iota_X : \ G_X^\vee \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee.$$

From now on a pair satisfying Definition 2.0.1 is called *strictly* local Eulerian, and "local Eulerian" refers to Definition 3.2.3 below.

3. The modified definition

3.1. **Normalization of the group.** Classically the $\mathbf{H}$-period is taken modulo $Z_\mathbf{G} \cap \mathbf{H}$, and only representations trivial on the part of $\mathbf{G}$ acting trivially on $\mathbf{X}$ can be distinguished.

Convention 3.1.1. $\mathbf{G}$ acts faithfully on $\mathbf{X}$. This can always be arranged, without changing $\mathbf{X}$, by replacing $\mathbf{G}$ with $\mathbf{G}/\mathbf{N}$, where $\mathbf{N}$ is the largest normal subgroup of $\mathbf{G}$ contained in $\mathbf{H}$ (in particular $\mathbf{N} \supseteq Z_\mathbf{G} \cap \mathbf{H}$). The hypersphericity of $M = T^*X$ is required for this quotient.

The convention matters: a finite central subgroup of $\mathbf{G}$ inside $\mathbf{H}$ lies in every stabilizer on T^*X and can make the generic stabilizer disconnected for $\mathbf{G}$ but not for $\mathbf{G}/\mathbf{N}$ (the basic example is $\mathbb{G}_m\backslash\mathrm{SL}_2$ versus $\mathbb{G}_m\backslash\mathrm{PGL}_2$).

3.2. **The centralizer action.** For an Arthur parameter ϕ_A put

$$S_{\phi_A} = Z_{G^\vee}(\mathrm{Im}\,\phi_A), \qquad \mathrm{Fix}(\phi_A) = M^{\vee,\phi_E}_{\mathrm{slice},\phi_A}.$$

Lemma 3.2.1. *The group S_{ϕ_A} acts on* $\mathrm{Fix}(\phi_A)$.

Proof. An element $z \in S_{\phi_A}$ centralizes the image of the Arthur SL_2, hence fixes f and preserves $\mathfrak{g}^{\vee,e}$; since μ is $G^\vee$-equivariant, z preserves $M^\vee_{\mathrm{slice},\phi_A}$. Moreover z commutes with $\phi_A(WD_k \times \mathrm{SL}_2)$, which contains the image of the associated L-parameter, and the $G^\vee$-action commutes with the grading. So z commutes with the action of WD_k through ϕ_E and preserves its fixed points. □

Definition 3.2.2. An X-distinguished Arthur parameter ϕ_A is called *X-Eulerian* if $\mathrm{Fix}(\phi_A)$ is non-empty and S_{ϕ_A} acts transitively on it.

Definition 3.2.3 (modified definition)**.** Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a homogeneous affine spherical variety satisfying Convention 3.1.1 and such that $M = T^*X$ is hyperspherical. We say that $(\mathbf{G}, \mathbf{H})$ is a *local Eulerian pair* if every X-distinguished Arthur parameter ϕ_A with $\mathrm{Fix}(\phi_A) \neq \emptyset$ is X-Eulerian.

Remark 3.2.4 (Why this is the classical notion). Classically a period is Eulerian if, on each distinguished $\pi = \otimes_v \pi_v$, it is a constant times a product of local functionals, as happens after unfolding to a unique model. In [BZSV, Conjecture 14.3.5] the normalized period attached to ϕ_A is a sum over the points of $\mathrm{Fix}(\phi_A)$, the term at m being an Euler product determined by the action of ϕ_E on the tangent space $T_m M^\vee_{\mathrm{slice}}$. By Lemma 3.2.1 two points in one S_{ϕ_A}-orbit have isomorphic tangent representations, hence equal terms. So for an X-Eulerian parameter the sum is one Euler product times the number of fixed points, while distinct orbits give, in general, distinct Euler products. This is the reason for asking for one orbit rather than one point.

Remark 3.2.5 (Why the quantifier changes). With transitivity in place of "one point", the existential quantifier of Definition 2.0.1 would be vacuous: by Lemma 4.1.1 below the parameter $\phi_A = \iota_X|_{\mathrm{SL}_2}$, trivial on WD_k, is X-Eulerian for every $\mathbf{X}$. The classical notion is a statement about all distinguished representations, whence "every".

Remark 3.2.6 (Multiplicity one). Definition 3.2.3 does not imply multiplicity one, that is,

$$\dim \mathrm{Hom}_{\mathbf{H}(k)}(\pi, \mathbb{C}) \leq 1 \quad \text{for every irreducible } \pi.$$

See §5.2.

4. Reformulation: unique lifting to $G_X^\vee$

4.1. **The fixed locus.** Write $\epsilon = \iota_X|_{1\times \mathrm{SL}_2}$ and

$$K_X = \iota_X(G_X^\vee \times 1), \qquad C_X = Z_{G^\vee}(\epsilon(\mathrm{SL}_2)), \qquad K_X \subseteq C_X.$$

After conjugation an X-distinguished parameter is $\phi_A(w, s) = \rho(w)\epsilon(s)$ with $\rho : WD_k \to C_X$ bounded on W_k, and then $S_{\phi_A} = Z_{C_X}(\rho)$.

We assume in this section that $M^\vee$ is the Whittaker induction of a symplectic $G_X^\vee$-representation S_X along ι_X, as in [BZSV, §4], and that ι_X is injective on $G_X^\vee$.

Lemma 4.1.1. *There is a bijection, equivariant for $S_{\phi_A} = Z_{C_X}(\rho)$ acting by left translation,*

$$\mathrm{Fix}(\phi_A) \;\simeq\; (C_X/K_X)^{\rho(WD_k)} = \{\, cK_X :\; c^{-1}\rho(WD_k)c \subseteq K_X \,\}.$$

Sketch. This is [BZSV, §14.3 and §14.6.4]; see also [N, Lemma 1.1]. Boundedness of $\rho|_{W_k}$ and the weights of Frobenius force a fixed point of ϕ_E to be fixed by the one-parameter group $t \mapsto (\epsilon(\mathrm{diag}(t, t^{-1})), t)$. This group

contracts the slice $f + \mathfrak{g}^{\vee,e}$ to f and scales S_X with weight one, so its fixed points on $M^\vee_{\text{slice}}$ are the classes $[c, 0, 0]$ with $c \in C_X$, i.e. C_X/K_X. On these, ϕ_E acts through left translation by ρ. □

Note that S_X and the field k play no role in the right-hand side.

4.2. **Orbits and lifts.**

Proposition 4.2.1. *Let $\phi_A = \rho \cdot \epsilon$ be X-distinguished. The map $cK_X \mapsto c^{-1}\rho c$ induces a bijection between*

(a) *the S_{ϕ_A}-orbits on* $\mathrm{Fix}(\phi_A)$*, and*
(b) *the $G_X^\vee$-conjugacy classes of parameters $\phi_X : WD_k \to G_X^\vee$ such that $\iota_X \circ (\phi_X \times \mathrm{id}_{\mathrm{SL}_2})$ is $G^\vee$-conjugate to ϕ_A.*

The stabilizer in S_{ϕ_A} of the point corresponding to ϕ_X is $S_{\phi_X} = Z_{G_X^\vee}(\mathrm{Im}\,\phi_X)$.

Proof. Two Arthur parameters $\rho\epsilon$ and $\rho'\epsilon$ are $G^\vee$-conjugate if and only if ρ, ρ' are C_X-conjugate, so (b) is the set of K_X-conjugacy classes of homomorphisms into K_X that are C_X-conjugate to ρ; these are the $c^{-1}\rho c$ with $cK_X \in (C_X/K_X)^\rho$. For two such cosets, $c'^{-1}\rho c' = k^{-1}c^{-1}\rho ck$ with $k \in K_X$ if and only if $z = ckc'^{-1} \in Z_{C_X}(\rho)$, i.e. if and only if $c'K_X = z^{-1}cK_X$. Finally $zcK_X = cK_X$ if and only if $c^{-1}zc \in Z_{K_X}(c^{-1}\rho c)$. □

Corollary 4.2.2. (1) *ϕ_A is X-Eulerian if and only if it factors through ι_X and its lift ϕ_X to $G_X^\vee$ is unique up to $G_X^\vee$-conjugacy. In that case* $\mathrm{Fix}(\phi_A) \simeq S_{\phi_A}/S_{\phi_X}$.

(2) *$(\mathbf{G}, \mathbf{H})$ is a local Eulerian pair if and only if the map*

$$\phi_X \longmapsto \iota_X \circ (\phi_X \times \mathrm{id}_{\mathrm{SL}_2})$$

from bounded parameters of $G_X^\vee$ to Arthur parameters of $G^\vee$ is injective on conjugacy classes.

(3) $\mathrm{Fix}(\phi_A)$ *is a single point, as in Definition 2.0.1, if and only if ϕ_A is X-Eulerian and $S_{\phi_A} = S_{\phi_X}$.*

(4) *If $C_X = K_X \cdot Z_{C_X}(K_X)$, then $(\mathbf{G}, \mathbf{H})$ is a local Eulerian pair. This holds in particular when $C_X = K_X$.*

Proof. (1)–(3) are immediate from Proposition 4.2.1. For (4), write $c = kz$ with $k \in K_X$ and z centralizing $K_X \supseteq \mathrm{Im}\,\rho$; then $c^{-1}\rho c = k^{-1}\rho k$. □

When $\mathrm{Fix}(\phi_A)$ is finite, the index $[S_{\phi_A} : S_{\phi_X}]$ in (1) is the constant in front of the Euler product in Remark 3.2.4.

Remark 4.2.3 (Relation between the two definitions). Neither definition implies the other. Example 5.0.3 is local Eulerian but not strictly so. Conversely a pair can have one parameter with a single fixed point and another parameter with two non-conjugate lifts; see row 18 of Table 1.

Remark 4.2.4 (A generic variant). If $\mathrm{Im}\,\phi_X$ is Zariski dense in $G_X^\vee$, then ϕ_A is X-Eulerian if and only if

$$N_{C_X}(K_X) = K_X \cdot Z_{C_X}(K_X).$$

A parameter of WD_k rarely has dense image, so locally this is only a heuristic; it is the condition one expects to govern a general global parameter. One may call $(\mathbf{G}, \mathbf{H})$ *generically* local Eulerian when it holds. It is implied by Definition 3.2.3 whenever some ϕ_X has dense image.

5. Examples

Example 5.0.1 (Whittaker and strongly tempered periods)**.** If $G_X^\vee = G^\vee$ and the SL_2 is trivial, then $C_X = K_X = G^\vee$ and $\mathrm{Fix}(\phi_A)$ is a point for every ϕ_A. The pair is local Eulerian in both senses.

Example 5.0.2 (The group case)**.** For $\mathbf{X} = \mathbf{H}\backslash(\mathbf{H}\times\mathbf{H})$ one has $C_X/K_X \simeq H^\vee$ with $\rho = (\phi, \phi^c)$ acting by conjugation, so $\mathrm{Fix}(\phi_A) = Z_{H^\vee}(\phi)$, a single orbit of S_{ϕ_A}. The pair is local Eulerian, in agreement with the factorization of the Petersson norm, and it is strictly so only for parameters with trivial centralizer. Compare [BZSV, §14].

Example 5.0.3 (Linear periods)**.** Let $\mathbf{G} = \mathrm{PGL}_{2n}$ and $\mathbf{H}$ the image of $\mathrm{GL}_n \times \mathrm{GL}_n$. Then $G^\vee = \mathrm{SL}_{2n}$, $K_X = \mathrm{Sp}_{2n}$, the SL_2 is trivial and $C_X = \mathrm{SL}_{2n}$, so $\mathrm{Fix}(\phi_A)$ is the set of ρ-invariant symplectic forms of Pfaffian one. Two homomorphisms into Sp_{2n} that are conjugate in GL_{2n} are conjugate in Sp_{2n} [GGP, Theorem 8.1], so the pair is local Eulerian by Corollary 4.2.2(2). This matches [FJ], where the period unfolds to the Shalika model. For irreducible ρ one has $S_{\phi_A} = \mu_{2n}$ and $S_{\phi_X} = \mu_2$, so there are n fixed points: the pair is not strictly local Eulerian for $n \geq 2$.

Example 5.0.4 (Rows 1 and 2 of Wan's table)**.** Let $\mathbf{G} = \mathrm{PGL}_N$ and $\mathbf{H}$ the image of $\mathrm{GL}_a \times \mathrm{GL}_{N-a}$ with $r = N - 2a \geq 1$. Then $K_X = \mathrm{Sp}_{2a}$, the SL_2 is principal in GL_r, and $C_X = \{(g, z) \in \mathrm{GL}_{2a} \times \mathrm{GL}_1 : \det g \cdot z^r = 1\}$. For $\phi_A = \rho_1 \oplus S_r$ the fixed locus is an r-fold cover of the set of ρ_1-invariant symplectic forms on $\mathbb{C}^{2a}$ (the pairs (ω, z) with z^r prescribed by the Pfaffian of ω). It is never a point, but it is a single S_{ϕ_A}-orbit by the same argument as in Example 5.0.3. The pair is local Eulerian. Here ϕ_A is not discrete and the orbit is one-dimensional.

Example 5.0.5 (A non-example)**.** Let $\mathbf{G} = \mathrm{PSp}_{2n}$, $n \geq 3$, and $\mathbf{H}$ the image of $\mathrm{Sp}_{2n-2} \times \mathrm{GL}_1$. The spherical roots are α_1 and $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$, which gives $K_X = \mathrm{SL}_2 \times \mathrm{SL}_2 = \mathrm{Spin}_4 \subset \mathrm{Spin}_{2n+1}$, while C_X is the preimage of O_4, containing K_X with index two and exchanging its two factors. For $\phi_X = (A, B)$ the lifts of ϕ_A are (A, B) and (B, A), which are not conjugate in K_X unless $A \simeq B$. So the pair is not local Eulerian, and not generically so either. (The dual data here were computed by hand and should be double-checked.)

5.1. **Wan's Table 1.** Table 1 records what Definition 3.2.3 gives for the rows of [W, Table 1], using the groups $K_X \subseteq C_X$ computed in [N]. The groups in [N] are those printed in [W]. This is harmless for the positive entries: if the map of Corollary 4.2.2(2) is injective for $\mathbf{G}$, it is injective

for $\mathbf{G}/\mathbf{N}$, because K_X is unchanged and conjugacy in the dual group of $\mathbf{G}/\mathbf{N}$ implies conjugacy of the images in $G^\vee$. Rows 5 and 13 are taken after normalization. The entries rely on the dual data of [N] and should be regarded as provisional.

In row 18 one has $K_X = \{(d_1, d_2) \in \mathrm{SO}_4 \times \mathrm{SO}_4 : \ j(d_1) = j(d_2)\}$, the image of SL_2^3 with the middle factor shared. Take $\phi_X = [A, B, A]$ with $A \not\simeq B$, for instance A trivial and B the standard representation of the Deligne SL_2. Then $[B, A, B]$ is a second lift of the same ϕ_A, not conjugate to the first; the two images are exchanged by the element swapping the factors of each SO_4.

Rows	Reason	Definition 3.2.3
3, 4, 8, 10, 11, 12, 13, 14	$C_X = K_X$	Eulerian (row 12: see §5.2)
1, 2	Example 5.0.4	Eulerian
9	$C_X = K_X \times \mathrm{SO}_2$	Eulerian
6, 7 with $m = 1$	$C_X = K_X \times \{\pm 1\}$	Eulerian
17	unique lift from SO_4	Eulerian
18	$N_{C_X}(K_X) = K_X$; two lifts for special ρ	generically Eulerian only
5	Example 5.0.5	not Eulerian
6, 7 with $m \geq 2$	T^*X not hyperspherical (type N)	not defined
15, 16, 19–27	dual data not yet computed after Conv. 3.1.1	open

TABLE 1. Status of Wan's Table 1 under the modified definition.

5.2. **Multiplicity one is a different condition.** A local Eulerian pair need not satisfy $\dim \mathrm{Hom}_{\mathbf{H}(k)}(\pi, \mathbb{C}) \leq 1$ for all irreducible smooth π. Row 12, the pair (G_2, SL_3), is an example.

Here $\Sigma_X = \{\gamma\}$ with $\gamma = 2\alpha + \beta$, and $P(X) = P_\beta$ [C]. Since $\gamma \perp \beta$, the groups K_X and $\epsilon(\mathrm{SL}_2)$ are the two commuting root SL_2's of $G_2^\vee$ attached to $\gamma^\vee$ and $\beta^\vee$, and $C_X = K_X$. So the pair is local Eulerian, and also strictly so. On the other hand, by [C, Theorem 1.5], for p odd:

(1) every irreducible unramified σ of $G_2(k)$ has $\dim \mathrm{Hom}_{\mathrm{SL}_3(k)}(\sigma, \mathbb{C}) \leq 1$;
(2) the unitarizable representation $\pi = \tau^\vee$, with $\tau = J_{P_2}(\mathrm{St}_{\mathrm{GL}_2}, 1/2)$, has $\dim \mathrm{Hom}_{\mathrm{SL}_3(k)}(\pi, \mathbb{C}) \geq 2$.

Remark 5.2.1 (Where the second functional comes from). The two functionals in (2) come from the theta lifts of the even and the odd Weil representations $\omega_\psi^\pm$ of $\widetilde{\mathrm{SL}}_2$, both of which contain τ after restriction to G_2 [GG], [C, §3]. In terms of parameters this reads as follows.

- ω_ψ^- lies in the Waldspurger packet of the Steinberg representation. It leads to the X-distinguished parameter $\phi_A = \rho \cdot \epsilon$ in which ρ maps the Deligne SL_2 isomorphically onto K_X. Its fixed locus is one point.

- ω_ψ^+ is not tempered: its parameter is trivial on WD_k and non-trivial on the Arthur SL_2 of $G_X^\vee$. It leads to an Arthur parameter of G_2 that is trivial on WD_k and whose SL_2 is diagonal in $K_X \times \epsilon(\mathrm{SL}_2)$. This is not of the SL_2-type of ι_X, so it is not X-distinguished.

Thus $\tau^\vee$ lies in two Arthur packets and each contributes one functional. Definition 3.2.3 tests one parameter at a time, and only X-distinguished ones, so it does not see the coincidence. A hypothesis implying multiplicity one would have to control the intersections of X-distinguished Arthur packets with packets of other SL_2-types; it cannot be read off $\mathrm{Fix}(\phi_A)$ for a single ϕ_A.

Remark 5.2.2 (Unramified multiplicity one and unique lifting). The proof of (1) rests on the surjectivity of the restriction map

$$\mathbb{C}[A^*]^W \longrightarrow \mathbb{C}[\delta_{(X)}^{1/2} A_X^*]^{W_X}$$

[C, Proposition 2.20]. Equivalently $\delta_{(X)}^{1/2} A_X^*/W_X \to A^*/W$ is a closed embedding, in particular injective. So an unramified ϕ_X is determined up to conjugacy by the L-parameter of $\iota_X \circ (\phi_X \times \mathrm{id}_{\mathrm{SL}_2})$, hence by this Arthur parameter. This is Corollary 4.2.2(2) restricted to unramified parameters: for (G_2, SL_3), unramified multiplicity one and the Eulerian property at unramified parameters have the same source.

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