LOCAL LANGLANDS CORRESPONDENCE: THE ARCHIMEDEAN CASE

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1. Introduction

This is a study note for Knapp's article on local Langlands correspondence for archimedean field.

The point of the paper is to given an account of the relevant parts of representation theory that are occurring at the Archimedean places.

2. Tate's method

Let k be $\mathbb{R}$ or $\mathbb{C}$ and let $M_n(k)$ be the n-by-n matrix space over k, let

$$K = \begin{cases} O(n) \text{ if } k = \mathbb{R} \\ U(n) \text{ if } k = \mathbb{C} \end{cases}$$

Let (ρ, V) be an admissible representation of $GL_n(k)$, let $(\tilde{\rho}, \tilde{V})$ be the admissible dual. A K-finite matrix coefficient of ρ is a function

$$c(x) = \langle \rho(x)u, \tilde{u} \rangle$$

The subspace $\mathscr{C}_0$ of the Schwartz space $\mathscr{C}(M_n(k))$ consists of all functions of the form

$$P(x_{ij})\text{exp}(-\pi \sum x_{ij}^2) \qquad \text{if } k = \mathbb{R}$$
$$P(z_{ij}\overline{z}_{ij})\text{exp}(-2\pi \sum z_{ij}\overline{z}_{ij}) \quad \text{if } k = \mathbb{C}$$

For any K finite matrix coefficient c of ρ and any function f in the space $\mathscr{C}_0$, we define

$$\zeta(f, c, s) = \int_{M_n(k)} f(x)c(x)|\text{det } x|_k^s \; d^\times x$$

for s complex. Here $|z|_{\mathbb{R}} = |z|$ and $|z|_{\mathbb{C}} = |z|^2$. The measure $d^\times x = |\text{det}x|_k^{-n} \; dx$.

Assume ρ irreducible, then all the zeta integrals converge for s in a common right half-plane and extend to meromorphic function for s in $\mathbb{C}$. Moreover, there exist finitely many choices of (c, f) such that

$$L(s, \rho) = \sum_i \; \zeta(f_i, c_i, s)$$

For any (c, f)

$$\zeta(f, c, s + \frac{1}{2}(n-1)) = P(f, c, s)L(s, \rho) \tag{2.1}$$

for a polynomial P in s. The function $L(s, \rho)$ is uniquely determined by these properties and is called a local L-factor.

Let ψ be the additive character of k given by

$$\psi(x) = \exp(2\pi i x) \qquad \text{if } k = \mathbb{R}$$
$$\psi(x) = \exp(2\pi i(z + \overline{z})) \quad \text{if } k = \mathbb{C}$$

and define the Fourier transform $\hat{f}$ of a member f of $\mathscr{C}_0$ by

$$\hat{f}(x) = \int_{M_n(k)} f(y)\psi(\text{tr}(xy)) \; dy$$

where dy is the self-dual Haar measure on $M_n(k)$.

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When ρ is irreducible, there exists a meromorphic function $\gamma(s,\rho,\psi)$ independent of f and c such that

$$\zeta(\hat{f},\check{c},1-s+\frac{1}{2}(n-1)) = \gamma(s,\rho,\psi)\zeta(f,c,s+\frac{1}{2}(n-1))$$

in terms of

$$\epsilon(s,\rho,\psi) = \gamma(s,\rho,\psi)\frac{L(s,\rho)}{L(1-s,\tilde{\rho})}$$

the local functional equation reads

$$\frac{\zeta(\hat{f},\check{c},1-s+\frac{1}{2}(n-1))}{L(1-s,\tilde{\rho})} = \epsilon(s,\rho,\psi)\frac{\zeta(f,c,s+\frac{1}{2}(n-1))}{L(s,\rho)}$$

3. Langlands classification for $GL_n(\mathbb{R})$

Let $K = O(n)$ be the maximal compact subgroup of $G = GL_n$, let $SL_m^{\pm}(\mathbb{R})$ be the subgroup of elements g of $GL_m(\mathbb{R})$ with $|\det g| = 1$. We shall specify certain irreducible representations of $SL_m^{\pm}(\mathbb{R})$ for the cases $m = 1$ and $m = 2$. For $m = 1$, there are only two representations, we write 1 for the trivial and sgn for the nontrivial sign representation. For $m = 2$, the representations of interest are the discrete series, denoted by D_ℓ for integers $\ell \geq 1$, there representations are induced from $SL_2(\mathbb{R})$ as

$$D_\ell = \mathrm{ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})}(D_\ell^+)$$

here D_ℓ^+ acts in the space of analytic function f in the upper half plane with

$$||f||^2 = \int\int |f(z)|^2 y^{\ell-1}\ dx\ dy$$

the representations D_ℓ of $SL_2^{\pm}(\mathbb{R})$ are irreducible unitary, and their matrix coefficients are square integrable.

The building blocks for irreducible admissible representations of $GL_n(\mathbb{R})$ are the representations of $GL_1(\mathbb{R})$ and $GL_2(\mathbb{R})$ obtained by tensoring the above representations on $SL^{\pm}$ with a quasicharacter $a \to |\det a|_{\mathbb{R}}^t$ on the positive scalar matrices of size 1 or 2. Thus the building blocks will be $1 \otimes |\cdot|_{\mathbb{R}}^t$, $\mathrm{sgn} \otimes |\cdot|_{\mathbb{R}}^t$ for $GL_1(\mathbb{R})$ and $D_\ell \otimes |\det(\cdot)_{\mathbb{R}}^t|$ for $GL_2(\mathbb{R})$.

To any partition of n into 1's and 2's, say $(n_1, n_2, \cdots, n_r)$ with each n_i equal to 1 or 2, and $\sum n_i = n$, we associate the block diagonal subgroup

$$D = GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_r}(\mathbb{R})$$

for each j with $1 \leq j \leq r$ let σ_j be a representation of $GL_{n_j}(\mathbb{R})$ we introduced for $GL_1(\mathbb{R})$ and $GL_2(\mathbb{R})$ and write t_j for t. Then $(\sigma_1, \cdots, \sigma_r)$ defines by tensor product a representation of the block diagonal subgroup, and we extend this representation to the corresponding block diagonal subgroup and we extend this representation to the corresponding upper triangular subgroup $Q = DU$ by making it to be the identity on the block strictly upper triangular subgroup U. We set

$$I(\sigma_1, \cdots, \sigma_r) = \mathrm{ind}_Q^G(\sigma_1, \cdots, \sigma_r)$$

using unitary induction.

Theorem 3.1. *For $G = GL_n(\mathbb{R})$*

- *if the parameters $n_j^{-1} t_j$ of $(\sigma_1, \cdots, \sigma_r)$ satisfy*

$$n_1^{-1} Re\ t_1 \geq n_2^{-1} Re\ t_2 \geq \cdots \geq n_r^{-1} Re\ t_r \tag{3.1}$$

  then $I(\sigma_1, \cdots, \sigma_r)$ has a unique irreducible quotient $J(\sigma_1, \cdots, \sigma_r)$.
- *the representations $J(\sigma_1, \cdots, \sigma_r)$ exhaust the irreducible admissible representations of G, up to infinitesimal equivalence.*
- *two such representations $J(\sigma_1, \cdots, \sigma_r)$ and $J(\sigma_1', \cdots, \sigma_r')$ are infinitesimally equivalent if and only if $r' = r$ and there exists a permutation $j(i)$ of $\{1, \cdots, r\}$ such that $\sigma_i' = \sigma_{j(i)}$ for $1 \leq i \leq r$.*

The constituent $J(\sigma_1, \cdots, \sigma_r)$ of $I(\sigma_1, \cdots, \sigma_r)$ is defined as the image of a certain standard intertwining operator on $I(\sigma_1, \cdots, \sigma_r)$.

4. Local Langlands correspondence for $GL_n(\mathbb{R})$

The Weil group of $\mathbb{R}$, denoted by $W_{\mathbb{R}}$ is the nonsplit extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z}$ by

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$$

where $j^2 = -1$ and $jcj^{-1} = \bar{c}$, here bar denotes complex conjugation. We shall be interested in the set of equivalence classes of n-dimensional complex representations of $W_{\mathbb{R}}$ whose image consist of semisimple elements.

The one-dimensional representations are parametrized by a sign and a complex parameter $t = 2\mu$

$$(+,t): \ \varphi(z) = |z|_{\mathbb{R}}^t \text{ and } \varphi(j) = +1$$
$$(-,t): \ \varphi(z) = |z|_{\mathbb{R}}^t \text{ and } \varphi(j) = -1$$

now we describe the two dimensional representations of $W_{\mathbb{R}}$, let u, v be a basis in which $\varphi(\mathbb{C}^\times)$ is diagonal, put $u' = \varphi(j)u$. The equivalence class of φ is classified by a pair (ℓ, t) there exists a basis u, u' such that

$$(\ell,t): \varphi(re^{i\theta})u = r^{2t}e^{i\ell\theta}u, \qquad \varphi(j)u = u'$$
$$\varphi(re^{i\theta})u' = r^{2t}e^{-i\ell\theta}u', \quad \varphi(j)u' = (-1)^\ell u$$

Lemma 4.1. *Every finite dimensional representation φ of $W_{\mathbb{R}}$ is fully reducible and each irreducible representation has dimension one or two.*

Now let φ be an n-dimensional semisimple complex representation of $W_{\mathbb{R}}$, then φ is fully reducible. If we list the dimensions of the irreducible constituents in any order, we can regard the result as a partition of n into 1's and 2's, say $(n_1, \cdots, n_r)$ with each n_j equal to 1 or 2 and with $\sum n_j = n$. Let φ_j be the corresponding irreducible constituent of φ, to φ_j we associate a representation σ_j as follows

$$\begin{aligned} (+,t) \ &\to 1 \otimes |\cdot|_{\mathbb{R}}^t \\ (-,t) \ &\to \mathrm{sgn} \otimes |\cdot|_{\mathbb{R}}^t \\ (\ell,t) \ &\to D_\ell \otimes |\det(\cdot)|_{\mathbb{R}}^t \end{aligned}$$

in this way, we associate a tuple $(\sigma_1, \cdots, \sigma_r)$ of representations of φ, if the complex numbers $t_1, \cdots, t_r$ don't satisfy.

Using theorem 3.1, we can make the assignment

$$\varphi \longrightarrow \rho_{\mathbb{R}}(\varphi) = J(\sigma_1, \cdots, \sigma_r) \tag{4.1}$$

and we have the following conclusion

Theorem 4.2. *The association* (4.1) *is a well defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{R}}$ and the set of all equivalence classes of irreducible admissible representations of $GL_n(\mathbb{R})$.*

To each finite-dimensional semisimple complex representation φ of the Weil group over a local field, Weil has associated a local L factor with certain nice properties.

In the case $W_{\mathbb{R}}$ if φ is irreducible, the formula is

$$L(s,\varphi) = \begin{cases} \pi^{-(s+t)/2}\Gamma(\frac{s+t}{2}) & \text{if } \varphi \text{ is given by } (+,t) \\ \pi^{-(s+t+1)/2}\Gamma(\frac{s+t+1}{2}) & \text{if } \varphi \text{ is given by } (-,t) \\ 2(2\pi)^{-(s+t+\frac{l}{2})}\Gamma(s+t+\frac{l}{2}) & \text{if } \varphi \text{ is given by } (l,t) \end{cases}$$

We can define local factors $L(s,\rho)$ for each irreducible admissible representations of $GL_n(\mathbb{R})$ as

$$\text{if } \rho = \rho_{\mathbb{R}}(\varphi) \text{as in 4.2} = \begin{cases} L(s,\rho) = L(s,\varphi) \\ \epsilon(s,\rho,\varphi) = \epsilon(s,\varphi,\psi) \end{cases} \tag{4.2}$$

Theorem 4.3. *The two definitions* (2.1) *and* (4.2) *agree.*