LOCAL LANGLANDS CONJECTURE FOR G_2

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1. Introduction

This is a study note for the Gan-Savin paper[GS23b], it will be abbreviated as the GW paper. Let F be a non-archimedean local field of characteristic zero and residue characteristic p. Let W_F be the Weil group of F and let $WD_F = W_F \times SL_2(\mathbb{C})$ be the Weil-Deligne group. For a connected reductive group G over F, we assume it is split, Langlands conjectured that there is a surjective finite-to-one map from the set $Irr(G)$ of irreducible smooth representations of $G(F)$ to the set $\Phi(G)$ of L-parameters

$$WD_F \longrightarrow G^\vee$$

where $G^\vee$ is the Langlands dual group of G and the homomorphisms are taken up to $G^\vee$-conjugacy. This leads to a partition of the set of equivalence classes of irreducible representations of $G(F)$ into a disjoint union of finite subsets, which are the fibers of the map and are called L-packets. Moreover, one would like to characterize the map

$$\mathcal{L}_G : \ \mathrm{Irr}(G) \longrightarrow \Phi(G)$$

by requiring that it satisfies a number of natural conditions and to have a refined parametrization of its fibers.

The local Langlands conjecture (LLC) has proved for GL_n by Harris-Taylor and Henniart where the map $\mathcal{L}$ is a bijection. Building upon this, the LLC has now been shown for the quasi-split classical groups by the work of Arthur, Moeglin and Mok as a consequence of the theory of twisted endoscopy using the stable twisted trace formula for GL_n and extended to pure inner forms by various authors. It has also been shown for the group GSp_4 and its inner forms using theta correspondence as the main tool. For general G, the recent work of Fargues-Scholze gives a general geometric construction of a semisimplified LLC. In another direction, for general tamely ramified groups, the work of Kaletha constructs an LLC for supercuspidal L-packets.

The purpose of the GW paper is to establish the local Langlands conjecture for the split exceptional group $G = G_2$, more precisely

Theorem 1.1. *There is a natural surjective map*

$$\mathcal{L} : \ Irr(G_2) \longrightarrow \Phi(G_2)$$

with finite fibers satisfying the following properties

- *$\pi \in Irr(G_2)$ is square-integrable if and and only if $\mathcal{L}(\pi)$ does not factor through a proper Levi subgroup of $G_2^\vee = G_2(\mathbb{C})$.*
- *$\pi \in Irr(G_2)$ is tempered if and only if $\mathcal{L}(\pi)(W_F)$ is bounded in $G_2^\vee$.*
- *If $\pi \in Irr(G_2)$ is nontempred, so that π is the unique Langlands quotient of a standard module $Ind_P^{G_2}\tau$ induced from a proper parabolic subgroup $P = MN$, then $\mathcal{L}(\pi)$ is the composite of the L-parameter of τ with the natural inclusion $M^\vee \hookrightarrow G^\vee$.*
- *The map $\mathcal{L}$ fratures in the following two commutative diagrams*

$$\begin{array}{ccc} Irr^{\spadesuit}(G_2) & \longrightarrow & \Phi(G_2) \\ \big\downarrow \theta & & \big\uparrow \iota'_* \\ Irr(PD^\times) & \xrightarrow{L} & \Phi(PGL_3) \end{array}$$

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and

$$\begin{array}{ccc} Irr^{\heartsuit}(G_2) & \xrightarrow{\mathcal{L}} & \Phi(G_2) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \iota_*} \\ Irr(PGSp_6) & & \Phi(PSp_6) \\ \downarrow{\scriptstyle rest} & & \downarrow{\scriptstyle Std_*} \\ Irr(Sp_6)_{/PGSp_6} & \xrightarrow{L} & \Phi(Sp_6) \end{array}$$

For the left hand columns in the two diagrams, the symbol θ refers to an appropriate theta correspondence.

- *The map $\mathcal{L}$ is uniquely characterized by the properties above.*
- *The map $\mathcal{L}$ also features in the following commutative diagram*

$$\begin{array}{ccc} Irr_{gen,ds}(G_2) & \xrightarrow{\mathcal{L}} & \Phi(G_2) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \iota_*} \\ Irr_{gen,ds}(PGSp_6) & & \Phi(PGSp_6) \\ \downarrow{\scriptstyle spin_*} & & \downarrow{\scriptstyle spin_*} \\ Irr(GL_8) & \xrightarrow{L} & \Phi(GL_8) \end{array}$$

here $Irr_{gen,ds}$ refers to the subset of generic discrete series representation in Irr.

- *For each $\phi \in \Phi(G_2)$, the fiber of $\mathcal{L}$ over ϕ is in natural bijection with Irr S_ϕ where*

$$S_\phi = \pi_0(Z_{G_2(\mathbb{C})}(\phi))$$

when $p \neq 3$. Moreover, for tempered ϕ, the trivial character of S_ϕ corresponds to the unique generic element of $\mathcal{L}^{-1}(\phi)$.

- *The LLC for G_2 satisfies the following global-local compatibility. Suppose that Π is globally generic regular algebraic cuspidal automorphic representation of G_2 over a totally real number field k with a Steinberg component, suppose that*

$$\rho_\Pi : \ Gal(\overline{k}/k) \longrightarrow GL_7(\overline{\mathbb{Q}}_\ell)$$

is a Galois representation attached to Π, then ρ_Π factors through $G_2(\overline{\mathbb{Q}}_\ell)$, the restriction of ρ_Π to the local Galois group $Gal(\overline{k}_v/k_v)$ corresponds to the local L-parameter $\mathcal{L}(\Pi_v)$.

2. Theta correspondence

One has the dual pairs

- $(PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2 \subset E_6 \rtimes \mathbb{Z}/2\mathbb{Z}$.
- $PD^\times \times G_2 \subset E_6^D$.
- $G_2 \times PGSp_6 \subset E_7$.

where D denotes a cubic division F-algebra. One can consider the restriction of the minimal representation of E to the relevant dual pair and obtain a local theta correspondence, in particular for a representation π of one member of a dual pair, one has a big theta lift $\Theta(\pi)$ on the other member of the dual pair, and its maximal semisimple quotient $\theta(\pi)$. In [GS23a], the following theorem was shown

Theorem 2.1. *We have*

(1)(Howe duality) $\Theta(\pi)$ has finite length and its maximal simple quotient $\theta(\pi)$ is irreducible or zero.
(2)(Theta dichotomy) Let $\pi \in Irr(G_2)$, then π has a nonzero lift to exactly one of $PD^\times$ or $PGSp_6$, in particular, one has a decomposition

$$Irr(G_2) = Irr^{\heartsuit}(G_2) \sqcup Irr^{\spadesuit}(G_2)$$

where $Irr^{\heartsuit}(G_2)$ consists of those irreducible representations which participate in theta correspondence for $PGSp_6$ and $Irr^{\spadesuit}(G_2)$ consists of those which participate in $PD^{\times}$.
(3) The theta correspondence for $PD^{\times} \times G_2$ defines an injective map

$$\theta_D : \ Irr^{\spadesuit}(G_2) \hookrightarrow Irr(PD^{\times})$$

which is bijective if $p \neq 3$ and $Irr^{\spadesuit}(G_2)$ is contained in the subset $Irr_{ds}(G_2)$ of discrete series representations.

The theta correspondence for $G_2 \times PGSp_6$ defines an injection

$$\theta : \ Irr^{\heartsuit}(G_2) \hookrightarrow Irr(PGSp_6)$$

the map theta carries tempered representations to tempered representations.

The theta correspondence for $(PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2$ defines an injective map

$$\theta_{M_3} : \ Irr^{\clubsuit}(G_2) \hookrightarrow Irr(PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z})$$

where $Irr^{\clubsuit}(G_2) \subset Irr^{\heartsuit}(G_2)$ is the subset of representations which participates in theta correspondence with $PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}$. The map θ_{M_3} respects tempered (resp. discrete series) representations.

So we have a further decomposition

$$\mathrm{Irr}^{\heartsuit}(G_2) = \mathrm{Irr}^{\diamondsuit}(G_2) \sqcup \mathrm{Irr}^{\clubsuit}(G_2)$$

where $\mathrm{Irr}^{\clubsuit}(G_2)$ consists of those representations which participate in theta correspondence with $PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Irr}^{\diamondsuit}(G_2)$ consists of those which participate exclusively in the theta correspondence with $PGSp_6$. So we have the following trichotomy result for discrete series representations

Proposition 2.2. *Each irreducible discrete series representation of G_2 has a nonzero discrete series theta lift to exactly one of $PD^{\times}$, $PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ or $PGSp_6$. Setting $Irr^{\bullet}_{ds}(G_2) = Irr_{ds}(G_2) \cap Irr^{\bullet}(G_2)$, then we have the disjoint union*

$$Irr_{ds}(G_2) = Irr^{\spadesuit}_{ds}(G_2) \sqcup Irr^{\clubsuit}_{ds}(G_2) \sqcup Irr^{\diamondsuit}_{ds}(G_2)$$

3. Parametrization of the fiber of $\mathcal{L}$

Proposition 3.1. *For $\phi \in \Phi^{\diamondsuit}_{ds}(G_2)$, there is a unique generic representation in $\mathcal{L}^{-1}(\phi)$.*

If not, we can denote $\sigma = \theta(\pi)$, $\sigma' = \theta(\pi')$, then we have σ and σ' are distinct generic discrete series representations of $PGSp_6$.

We can pick out a unique Xu's packet contained in $\tilde{\Pi}_{\phi^b}$, namely

$$\tilde{\Pi}^{X_*}_{\phi^b} := \text{Xu's packet containing the unique generic representation } \theta(\pi) \text{ with } \mathcal{L}(\pi) = \phi$$

We have

Proposition 3.2. *Let $\phi \in \Phi^{\diamondsuit}_{ds}(G_2)$, the local theta correspondence defines a bijection*

$$\mathcal{L}^{-1}(\phi) \leftrightarrow \tilde{\Pi}^{X_*}_{\phi^b}$$

The proof of this proposition is based on the following lemma-one in all in

Lemma 3.3. *Let $\tilde{\Pi}^{X}_{\phi^b} \subset \tilde{\Pi}_{\phi^b}$ be any Xu's packet, then either all elements of $\tilde{\Pi}^{X}_{\phi^b}$ have nonzero theta lift to G_2 or none of them has.*

Now let's discuss the proof of the main theorem on the fiber of $\mathcal{L}$ assuming 3.2. The inclusion

$$\iota : G_2 \longrightarrow \mathrm{Spin}_7$$

gives rise to an isomorphism

$$S_{\phi} \cong S_{\iota\circ\phi}/Z(\mathrm{Spin}_7)$$

and thus a bijection

$$\mathrm{Irr}(S_{\iota\circ\phi}/Z(\mathrm{Spin}_7)) \longleftrightarrow \mathrm{Irr}(S_{\phi})$$

on the other hand, the result from the endoscopy classification gives a bijection

$$\tilde{\Pi}^{X_*}_{\phi^b} \longleftrightarrow \mathrm{Irr}(S_{\iota\circ\phi}/Z(\mathrm{Spin}_7))$$

combining this with proposition 3.2, we get a bijection

$$\mathcal{L}^{-1}(\phi) \leftrightarrow \mathrm{Irr}(S_\phi)$$

For $\phi \in \Phi_{ds}^{\diamondsuit}(G_2)$, we define the L-parameter of the distinguished Xu's packet $\tilde{\Pi}_{\phi^b}^{X_*}$ to be $\iota \circ \phi$. This definition is not really ad hoc, the distinguished Xu's packet $\tilde{\Pi}_{\phi^b}^{X_*}$ is the unique one on $\tilde{\Phi}_{\phi^b}$ for which the Langlands-Shahidi Spin L-function of its unique generic member is equal to the local L-factor for $\mathrm{Spin} \circ \iota \circ \phi$ and hence has a pole at $s = 0$. Now with this definition of the L-packet for $PGSp_6$

Theorem 3.4. *We have*
(a) For $\tau \in Irr(PGL_3)$ with L-parameter ϕ_τ, the local theta lifts of τ to G_2 is the set of $\pi \in Irr(G_2)$ whose enhanced L-parameter (ϕ, η) satisfies

$$\phi = \iota' \circ \phi_\tau, \ \eta \circ \iota'_* = 1$$

where we recall that

$$\iota' : \ SL_3(\mathbb{C}) \longrightarrow G_2(\mathbb{C})$$

is the natural embedding which induces a map of component groups

$$\iota'_* : \ \pi_0(Z_{SL_3}(\phi_\tau)) \longrightarrow S_{\iota' \circ \phi_\tau} = \pi_0(Z_{G_2}(\iota' \circ \phi_\tau))$$

(b) If τ_D denote the Jacquet-Langlands lift of τ to $D^\times$, then the set of local theta lifts of τ_D and $\tau_D^\vee$ to G_2 consists of those $\pi \in Irr(G_2)$ whose enhenced L-parameter (ϕ, η) satisfies

$$\phi = \iota' \circ \phi_\tau, \quad \eta \circ \iota'_* \neq 1$$

(c) Suppose $\pi \in Irr(G_2)$ has enhanced L-parameter (ϕ, η), then its local theta theta correspondence $\theta(\pi) \in Irr(PGSp_6)$ has enhanced L-parameter (ϕ', η') satisfying

$$\phi' = \iota \circ \phi, \quad \eta = \eta' \circ \iota_*$$

where we recall that

$$\iota : \ G_2(\mathbb{C}) \longrightarrow Spin_7(\mathbb{C})$$

is the natural embedding which induces a map of component groups

$$\iota_* : \ S_\phi \longrightarrow S_{\iota \circ \phi}$$

Thus, we have come full circle.

References

[GS23a] Wee Teck Gan and Gordan Savin. Howe duality and dichotomy for exceptional theta correspondences. *Inventiones mathematicae*, 232(1):1–78, 2023.

[GS23b] Wee Teck Gan and Gordan Savin. The local langlands conjecture for G_2. In *Forum of Mathematics, Pi*, volume 11, page e28. Cambridge University Press, 2023.