KOSZUL DUALITY

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1. Introduction

In this note, we first state the result from [GKM98] on Koszul duality and then we explain a formal way to under this result as David Ben-Zvi explained to me.

2. Statement of Koszul duality

This is a summary of the Koszul duality from the paper section 11.2.

Suppose a compact torus K acts on a reasonable space X, our object now is to treat the ordinary cohomology $H^*(X, \mathbb{R})$ and the equivariant cohomology $H_K^*(X, \mathbb{R})$ in a completely parallel manner. The equivariant cohomology is a module over the symmetric algebra $S(\mathfrak{k}^*) \cong H^*(BK, \mathbb{R})$. The ordinary cohomology $H^*(X)$ is a module over the exterior algebra $\Lambda = \Lambda(\mathfrak{k}) \cong H^*(K, \mathbb{R})$.

There is a beautiful relation between modules over the symmetric algebra and the exterior algebra given by the Koszul duality. One might expect that the S-module $H_K^*(X)$ and the Λ-module $H^*(X)$ determines each other under the Koszul duality, but this turns out to be false, however the corresponding statement is true at the cochain level up to quasi-isomorphisms. It is possible to lift the action of the exterior algebra to an appropriate model of the cochain complexes $C^*(X, \mathbb{R})$ in such a way the elements of $\mathfrak{k}$ lower degree by one. Similarly, it is possible to lift the action of the symmetric algebra S to an appropriate model for the equivariant cochain complex $C_K^*(X, \mathbb{R})$, in such a way that the elements $x \in \mathfrak{k}^* \subset S(\mathfrak{k}^*)$ raise degree by two.

Let k be a field and let $P = \oplus_{j \in \mathbb{Z}} P_j$ be a graded vectorspace over k with homogeneous components of odd positive degrees only and let $\Lambda = \wedge P$ denote the exterior algebra on P. Let $\tilde{P}^*$ denote the dual vectorspace $P^* = \mathrm{Hom}_k(P, k)$, graded by homogeneous components of even degrees, let $S = S(\tilde{P}^*)$.

Let $K_+(\Lambda)$ be the bounded below differential graded Λ-modules and the derived category $D_+(\Lambda)$ obtained by localizing the category $K_+(\Lambda)$. Let $K_+^f(\Lambda)$ denote the homotopy category whose objects are differential graded Λ-modules N which are bounded from below such that the cohomology $H^*(N)$ is a finitely generated Λ-module and homotopy classes of maps. Let $D_+^f(\Lambda)$ denote the corresponding derived category obtained by inverting quasi-isomorphisms. Similarly, we can define the categories $K_+(S),\ D_+(S),\ K_+^f(S),\ D_+^f(S)$.

The first Koszul duality functor $h : K_+(S) \to K_+(\Lambda)$ assigns to any complex (M, d_M) of S modules the following complex of Λ-modules

$$h(M) = \mathrm{Hom}_k(\Lambda, M)$$

The second Koszul duality functor $t : K_+(\Lambda) \to K_+(S)$ assigns to any complex (N, d_N) of Λ modules the following complex of S-modules

$$t(N) = S \otimes_k N$$

Theorem 2.1. *(Koszul duality theorem) The Koszul duality functors h and t pass to functors $h : D_+(S) \to D_+(\Lambda)$ and $t : D_+(\Lambda) \to D_+(S)$ where they become quasi-inverse equivalences of categories.*

There is a equivariant global section functor

$$G :\ D_K^b(pt) \longrightarrow D_+(S)$$

there is also a functor of ordinary global section

$$E :\ D_K^b(pt) \longrightarrow D_+(\Lambda)$$

Date: June 2024.

Theorem 2.2. *The functors G and E are equivalences of categories and they are related by the Koszul duality: there are natual isomorphism of functors*

$$hG \cong E \quad \text{and} \quad G \cong tE$$

these functors restrict to equivalences of the full subcategories

$$\begin{tikzcd} & D^f_+(S) \arrow[dd, shift left, "h"] \\ D^b_{K,c}(pt) \arrow[ur, "G"] \arrow[dr, "E"'] & \\ & D^f_+(\Lambda) \arrow[uu, shift left, "t"] \end{tikzcd}$$

If X is a compactifiable K space and $c : X \to pt$ is the map to a point then we have the following diagram

$$\begin{tikzcd} & & D^f_+(S) \arrow[r, "H"] \arrow[dd, shift left, "h"] & S - mod \\ D^b_{K,c}(X) \arrow[r] & D^b_{K,c}(pt) \arrow[ur, "G"] \arrow[dr, "E"'] & & \\ & & D^f_+(\Lambda) \arrow[r, "H"'] \arrow[uu, shift left, "t"] & \Lambda - mod \end{tikzcd}$$

the composition across the top is the equivariant cohomology, and the composition across the bottom is the ordinary cohomology.

3. A formal explanation for Koszul duality

Let X be a space with a S^1-action, the action $S^1 \times X \to X$ makes $H^*(S)$ into a $\Lambda = k[y]/y^2$ module with y cohomological degree -1. We consider the Borel construction $X//S^1 := ES^1 \times^{S^1} X$, this is a fibration over BS^1 with fiber X, the equivariant cohomology $H^*_{S^1}(X) := H^*(ES^1 \times^{S^1} X)$ is a module over the symmetric algebra $S = H^*(BS^1)$.

Now let's think X as $X = pt \times_{pt/G} X//S^1$, we get

$$H^*(X) = H^*_{S^1}(X) \otimes_S k$$

think $X//S^1$ as X^{S^1}, we get

$$H^*_{S^1}(X) \cong \mathrm{Hom}_\Lambda(k, H^*(X))$$

here we recall that suppose we have an algebraic group G acting on a vector space V, then we have $V^G = \mathrm{Hom}_G(k, V)$.

It will be interesting to understand this formal explanation from the TQFT point of view and to use this formal explanation to understand the derived geometric Satake equivalence, that is to say, in the set up for equivariant cohomology for algebraic groups.

References

[GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, koszul duality, and the localization theorem. *Inventiones mathematicae*, 131(1):25–84, 1998.