SPHERICAL FUNCTIONS OF HERMITIAN FORMS

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1. Introduction

This is a summary for a series of papers by Hironaka [Hir88a], [Hir89], [Hir88b], where she studies the spherical functions on Hermitian forms.

2. Notation

We denote k a p-adic local field, where the residue field of k is not of char 2, let ℓ be a unramified extension of k of degree 2. For a matrix $A \in M_n(k)$, denote $A^* = \overline{A}^t$. For a positive integer n, let $X = X_n = \{A \in G : A^* = A\}$ and $X(\mathcal{O}) = X \cap M_n(\mathcal{O})$. The group $G = GL_n(k)$ acts on X by $g \cdot x = gxg^*$. For $x \in X$, let $x_{(i)}$ be the upper left i by i block of x and $d_i(x)$ the determinant of x.

Let $C_c^\infty(X)^K = S(K\backslash X)$ be the K-invariant compactly supported smooth function on X. This is naturally a module of $\mathscr{H}(G, K)$.

Now we recall the spherical functions and the spherical transform on $S(K\backslash X)$, let $x \in X$, $s = (s_1, \cdots, s_n) \in \mathbb{C}^n$ and a character $\chi = (\chi_1, \cdots, \chi_n)$ of $(k^*/k^{*2})^n$ for which $\chi_i(\pi) = 1$, we define

$$\zeta(x; z) = \zeta(x; s) = \int_{K'} \prod_{i=1}^{n} |d_i(k \cdot x)|^{s_i}\ dk$$

and

$$F : S(K\backslash X) \longrightarrow \mathbb{C}(q^{z_1}, \cdots, q^{z_n})$$
$$F(\varphi)(z) = \int_X \varphi(z)\zeta(x^{-1}; z)\ dx$$

3. $n = 2$ case

Theorem 3.1. *For $\lambda = (\lambda_1, \lambda_2) \in \Lambda_2$, we have*

$$\zeta(\pi^\lambda, z) = \frac{(-1)^{\lambda_1} q^{-(\lambda_1-\lambda_2)/2}(q^{2z_2} - q^{2z_1-1})}{(1+q^{-2})(q^{2z_2}+q^{2z_1})} \sum_{\sigma \in S_2} \sigma\{q^{\langle \lambda, 2z\rangle} \frac{q^{2z_1}+q^{2z_2-1}}{q^{2z_1}-q^{2z_2}}\}$$

where $\langle \lambda, 2z\rangle = 2\lambda_1 z_1 + 2\lambda_2 z_2$ and σ acts on z_1 and z_2 as permutation.

For $x \in X \cap M_2(\mathcal{O})$, we have

$$\zeta(x; s_1, s_2) = \frac{|\det x|^{s_2}}{1+q^{-2}} \sum_{r \geq 0} \frac{\mu^{pr}(\pi^r, x)}{\mu(\pi^r, \pi^r)} \zeta(\pi^r; s_1)$$

By the formulas for ζ and μ^{pr} we get the formulas for $\zeta(\begin{pmatrix} \pi^{2n} & 0 \\ 0 & 1 \end{pmatrix}; s_1, s_2)$ and $\zeta(\begin{pmatrix} \pi^{2n-1} & 0 \\ 0 & 1 \end{pmatrix}; s_1, s_2)$, and hence the formula for ζ

$$\zeta(\pi^\lambda, z) = \frac{(-1)^{\lambda_1} q^{-(\lambda_1-\lambda_2)/2}(q^{2z_2} - q^{2z_1-1})}{(1+q^{-2})(q^{2z_2}+q^{2z_1})} \sum_{\sigma \in S_2} \sigma\{q^{\langle \lambda, 2z\rangle} \frac{q^{2z_1}+q^{2z_2-1}}{q^{2z_1}-q^{2z_2}}\}$$

We define $\Psi_z(x) = \frac{q^{2z_2}+q^{2z_1}}{q^{2z_2}-q^{2z_1-1}}\zeta(x; z)$, then by theorem 3.1, we see that $\Psi_z(x)$ is an entire function of z in $\mathbb{C}^2$ and satisfies the functional equation $\Psi_{\sigma z}(x) = \Psi_z(x)$ for $\sigma \in S_2$.

Date: March 2025.

We define for each $\varphi \in S(K\backslash X)$

$$\hat{\varphi}(z) = \int_X \varphi(x)\Psi_z(x^{-1})\ dx$$

Let $\mathfrak{a}^* = \{i\mathbb{R}/(\pi/(\log q)\ \mathbb{Z})\}^2$ and denote $d\mu(z)$ the measure on $\mathfrak{a}^*$ by

$$d\mu(z) = \frac{(1+q^{-2})^2}{2(1-q^{-1})} \cdot \frac{dz}{|c(z)|^2}$$

where dz is the measure on $\mathfrak{a}^*$ normalized by $\int_{\mathfrak{a}^*} dz = 1$, $z = (z_1, z_2) \in \mathfrak{a}^*$ and

$$c(z) = \frac{q^{2z_1} + q^{2z_2-1}}{q^{2z_1} - q^{2z_2}}$$

Theorem 3.2. *(Plancherel formula) For any $\varphi, \psi \in S(K\backslash X)$, the following identity holds:*

$$\int_X \varphi(x)\overline{\psi(x)}dx = \int_{\mathfrak{a}^*} \hat{\varphi}(z)\overline{\hat{\psi}(z)}d\mu(z)$$

For $\varphi \in S(K\backslash X)$, define $\check{\varphi} \in S(K\backslash X)$ by $\check{\varphi}(x) = \varphi(x^{-1}),\ x \in X$. To prove theorem 3.2, it suffices to show that the identity holds for $(\mathrm{ch}_\lambda)^\vee$ and $(\mathrm{ch}_\mu)^\vee$, let $v(K\cdot\pi^\lambda) = \int_X \mathrm{ch}_\lambda(x)\ dx$, it is easy to see

$$\int_X (\mathrm{ch}_\lambda)^\vee(x)\overline{(\mathrm{ch}_\mu)^\vee(x)}\ dx = \delta_{\lambda\mu}v(K\cdot\pi^\lambda)$$

4. General case

X has an open P-orbit X' and finite P-orbit decomposition $X' = \sqcup_u X_u$, $\{d_i(x) \mid 1 \le i \le r\}$ forms a set of basic relative P-invariants defined over k and k-rank of $(P) = \mathrm{rank}(X^*(P))$.

For $x \in \Omega, g \in G$ and $s \in \mathbb{C}^r$ with $\mathrm{Re}(s_i) \ge 0$ we put

$$d_u^s(g;x) = 1_{X_u}(g\cdot x)\prod_{i=1}^r |d_i(g\cdot x)|^{s_i}$$

we set

$$\omega_u^s(x) = \int_K d_u^s(k;x)\ dk$$

then we have

$$\omega(x;s) = \sum_u \omega_u^s(x)$$

Proposition 4.1. *For $x \in \Omega$ generic s and $\chi = \chi_s$, we have*

$$(\omega_u^\chi(x))_{u\in\mathscr{U}} = \frac{1}{Q}\sum_{\sigma\in W}\gamma(\sigma\chi)B_\sigma(\chi)(\mathscr{P}_B(d_u^{\sigma\chi}(;x))(1))_{u\in\mathscr{U}}$$

and $B_\sigma(\chi)$ is the invertible matrix determined by

$$(\omega_u^\chi(x))_{u\in\mathscr{U}} = B_\sigma(\chi)(\omega_u^{\sigma\chi}(x))_{u\in\mathscr{U}}$$

The following theorem is proven based on the Casselman-Shalika method

Theorem 4.2. *For each $\lambda \in \Lambda_n$, we have*

$$\omega(\pi^\lambda;z) = (-1)^{\sum_{i=1}^n i\lambda_i}q^{-\sum_{i=1}^n(n-2i+1)\lambda_i/2}\prod_{i=1}^n\frac{1-q^{-2}}{1-q^{-2i}}\prod_{1\le i<j\le n}\frac{q^{z_j}-q^{z_i-1}}{q^{z_j}+q^{z_i}}$$
$$\times\sum_{\sigma\in S_n}\sigma(q^{\langle z,\lambda\rangle}\prod_{1\le i<j\le n}\frac{q^{z_i}+q^{z_j-1}}{q^{z_i}-q^{z_j}})$$

where $\sigma \in S_n$ acts on $z = (z_1,\cdots,z_n)$ as $\sigma(z) = (z_{\sigma(1)},\cdots,z_{\sigma(n)})$.

Let $\chi = (\chi_1, \cdots, \chi_n) \in \{(k^\times/N(\ell^\times))^\wedge\}^n$. Define

$$\begin{aligned} L(x;\chi;s) &= L(x;\chi;z) \\ &= \int_K \prod_{i=1}^n |d_i(k\cdot x)|^{s_i} \chi_i(d_i(k\cdot x))\ dk \end{aligned}$$

then for $x \in \Omega_j$, we have

$$L(x;\chi;s) = \sum_{u\in\{0,1\}^n} \chi(u)\omega_u^s(x)$$

We define the spherical Fourier transform on $S(K\backslash X)$ as

$$\begin{aligned} \wedge:\ S(K\backslash X) &\longrightarrow \mathbb{C}(q^{z_1}, \cdots, q^{z_n}) \\ \varphi &\longmapsto \hat{\varphi}(z) = \int_X \varphi(x)\Psi_z(x^{-1})\ dx \end{aligned}$$

Theorem 4.3. *The spherical Fourier transform $\wedge$ gives an $\mathscr{H}(G,K)$-module isomorphism*

$$S(K\backslash X) \cong \mathbb{C}[q^{\pm z_1}, \cdots, q^{\pm z_n}]^{S_n}$$

where the right hand side is regarded as the $\mathscr{H}(G,K)$-module.

Using the explicit expression of spherical functions in theorem 1, we can prove the following theorem on Plancherel formula by the same argument used in the Macdonald formula

Theorem 4.4. *(Plancherel formula) Let $\mathfrak{a}^* = \{i(\mathbb{R}/\frac{2\pi}{logq}\mathbb{Z})\}^n$ and define the measure $d\mu(z)$ on $\mathfrak{a}^*$ by*

$$d\mu(z) = \frac{1}{n!}\frac{\omega_n(-q^{-1})}{(1+q^{-1})^n}\cdot\frac{dz}{|c(z)|^2}$$

where dz is the Haar measure on $\mathfrak{a}^*$ normalized by $\int_{\mathfrak{a}^*} dz = 1$ and

$$c(z) = \prod_{1\le i<\le j\le n} \frac{q^{z_i} + q^{z_j - 1}}{q^{z_i} - q^{z_j}}$$

then for any $\varphi, \psi \in S(K\backslash X)$ we have

$$\int_X \varphi(x)\overline{\psi(x)}\ dx = \int_{\mathfrak{a}^*} \hat{\varphi}(z)\overline{\hat{\psi}(z)}\ d\mu(z)$$

References

[Hir88a] Yumiko Hironaka. Spherical functions of hermitian and symmetric forms I. *Japanese journal of mathematics. New series*, 14(1):203–223, 1988.

[Hir88b] Yumiko Hironaka. Spherical functions of hermitian and symmetric forms III. *Tohoku Mathematical Journal, Second Series*, 40(4):651–671, 1988.

[Hir89] Yumiko Hironaka. Spherical functions of hermitian and symmetric forms II. *Japanese journal of mathematics. New series*, 15(1):15–51, 1989.