A HARISH-CHANDRA HOMOMORPHISM FOR REDUCTIVE GROUP ACTIONS

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1. Introduction

This is a study note for Knop's paper[Kno94].

For a semisimple complex Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$, in order to study unitary representations of semisimple Lie groups, Harish-Chandra extablished an isomorphism between the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$ and the algebra of invariant polynomials $\mathbb{C}[\mathfrak{t}^*]^W$, here $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subspace and W the Weyl group of $\mathfrak{g}$. Later Harish-Chandra found a similar isomorphism for symmetric space $X = G/K$, here he considered the algebra $\mathcal{D}(X)^G$ of invariant differential operators on X and showed that it is isomorphic to the ring of invariants of the little Weyl group W_X.

Knop generalizes Harish-Chandra homomorphism to smooth affine variety X with a connected reductive group action, and he showed that $\mathcal{Z}(X)$ the center of the invariant operators admits an Harish-Chandra type isomorphism $\mathcal{Z}(X) \cong k[\rho + \mathfrak{a}_X^*]^{W_X}$.

2. Main result

Let X be a smooth G-variety, then the G-action induces a homomorphism $\mathfrak{g} \to \mathcal{T}_X(X)$ and which extends to a homomorphism

$$\phi : \mathfrak{U} \to \mathcal{D}(X)$$

both algebras have a natural filtration and ϕ is compatible with them, hence there is a homomorphism between the associated graded objects

$$\overline{\phi} : \text{ gr } \mathfrak{U} \longrightarrow \text{gr } \mathcal{D}(X)$$

By the Poincare-Birkhoff-Witt theorem, gr $\mathfrak{U}$ is isomorphic to $k[\mathfrak{g}^*]$ and gr$\mathcal{D}(X)$ is isomorphism to a subalgebra of $k[T_X^*]$, where $\pi : T_X^* \to X$ is the cotangent bundle of X and $\overline{\phi}$ is induced by the moment map $\Phi : T_X^* \to \mathfrak{g}^*$, Knop proved that there is a canonical factorization

$$T_X^* \longrightarrow M_X \longrightarrow \mathfrak{g}^*$$

of the moment map, where the first map has connected general fibers and the second map is finite. The ring of invariants $k[\mathfrak{g}^*]^G$ by Chevalley's theorem is isomorphic to $k[\mathfrak{t}^*]^W$ where $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subspace and W its Weyl group or in the geometric terms

$$\mathfrak{t}^*/W \cong \mathfrak{g}^*//G$$

A similar thing happens for M_X, from [Kno90], we know that there is a certain subspace $\mathfrak{a}_X^* \subseteq \mathfrak{t}^*$ and a subquotient W_X of W and a Chevalley isomorphism $\mathfrak{a}_X^*/W_X \cong M_X//G$ such that

$$\begin{array}{ccc} M_X & \longrightarrow & M_X//G \cong \mathfrak{a}_X^*/W_X \\ \downarrow & & \downarrow \\ \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^*//G \cong \mathfrak{t}^*/W \end{array}$$

we will denote the common quotient $M_X//G = \mathfrak{a}^*/W_X$ by L_X and call W_X the *little Weyl group* of X. Knop proved that this is a finite reflection group and hence L_X is an affine space and $k[M_X]^G$ is a polynomial ring.

The main purpose of Knop's paper is to carry these result to the non-commutative case. Knop constructed an algebra $\mathfrak{U}(X)$ as the noncommutative analog of M_X which is a set of differential operators with a certain behavior at infinity.

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Let X be a smooth G-variety and $\mathcal{U}_X := \mathcal{O}_X \otimes \mathfrak{U}$, this carries a canonical algebra structure, the homomrphism $\mathfrak{U} \to \mathcal{D}(X)$ induces a localized homomophism between the sheaves of algebras $\mathcal{U}_X \to \mathcal{D}_X$, we will denote its image by $\mathfrak{U}_X$.

We will denote $\tilde{T}_X$ the closure of the image of the canonical morphism

$$\pi \times \Phi : \ T_X^* \longrightarrow X \times \mathfrak{g}^*$$

where $\pi : T_X^* \to X$ is the projection and $\Phi : T_X^* \to \mathfrak{g}^*$ is the moment map. It has the following property: the projection $\pi : \tilde{T}_X \to X$ is an affine morphism.

Definition 2.1. A G-variety X is called *pseudo-free* if $\tilde{T}_X \to X$ is a vector bundle.

Definition 2.2. Let X be any G-variety and let $\phi : \tilde{X} \to X$ be equivariant, birational, proper with $\tilde{X}$ smooth and pseudo-free, then let $\overline{\mathfrak{U}}_X := \phi_* \mathfrak{U}_{\tilde{X}} \subseteq \mathcal{D}_X$, for any equivariant completion $X \hookrightarrow \overline{X}$ let $\mathfrak{U}(X) := H^0(\overline{X}, \overline{\mathfrak{U}}_{\overline{X}})$ be the algebra of *completely regular* differential operators.

Theorem 2.3. *Let X be a normal G-variety and $\mathfrak{U}(X)$ its algebra of completely regular differential operators with its natural filtration, then there are canonical isomorphisms*

$$gr\mathfrak{U}(X) \cong k[M_X], \ gr\mathfrak{U}(X)^G \cong k[L_X]$$

For X smooth complete and pseudo-free, we have $\mathfrak{U}(X) = \mathfrak{U}_X(X)$ and $k[M_X] = k[\tilde{T}_X]$.

Corollary 2.4. *Let X be a spherical variety, then $\mathcal{Z}(X) = \mathcal{D}(X)^G$, every G-invariant differential operator on X is completely regular.*

This is because we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{gr}\mathcal{Z}(X) & \longrightarrow & \mathrm{gr}\mathcal{D}(X)^G \\ \downarrow & & \downarrow \\ k[L_X] & \longrightarrow & k[T_X^*]^G \end{array}$$

for a spherical variety we have $k[L_X] \cong k[T_X^*]^G$.

Theorem 2.5. *Let X be a smooth affine G-variety, then:*

- *The algebras $k[M_X]$ and $k[T_X^*]^G$ are the commutants of each other inside $k[T_X^*]$.*
- *Their intersection $k[L_X]$ is the center of both $k[M_X]$ and $k[T_X^*]^G$.*
- *The algebras $k[T_X^*]$, $k[T_X^*]^G$ and $k[M_X]$ are free $k[L_X]$-modules.*

Knop proved the following noncommutative analog

Theorem 2.6. *Let X be a smooth affine G-variety then*

- *The algebras $\mathfrak{U}(X)$ and $\mathcal{D}(X)^G$ are the commutants of each other inside $\mathcal{D}(X)$.*
- *Their intersections $\mathcal{Z}(X)$ is the center of both $\mathfrak{U}(X)$ and $\mathcal{D}(X)^G$.*
- *The algebras $\mathcal{D}(X)$, $\mathcal{D}(X)^G$ and $\mathfrak{U}(X)$ are free $\mathcal{Z}(X)$-modules.*

In fact, we have the following connection between various commutative and noncommutative rings:

$$\mathrm{gr}\mathcal{D}(X) = k[T_X^*], \ \mathrm{gr}\mathcal{D}(X)^G = k[T_X^*]^G, \ \mathrm{gr}\mathfrak{U}(X) = k[M_X], \ \mathrm{gr}\mathcal{Z}(X) = k[L_X]$$

the first follows from that X is affine, the last two is our previous result.

Corollary 2.7. *Let X be a smooth affine G-variety, then the center $\mathcal{D}(X)^G$ is a polynomial ring in rk X generators.*

We will see examples in the next section where we can make the result of this corollary more explicit.

3. Examples

We consider $G = SO_{2n+1}$ and $H = GL_n \subset G$ embedded as the Levi subgroup of the last maximal parabolic subgroup, we consider $X = G/H$, let $V = k^n$ be the standard representation of H, then we have as an H-representation

$$\mathfrak{h} = V \oplus V^* \oplus \wedge^2 V \oplus \wedge^2 V^*$$

the general isotropy group of G in T^*X is trivial, hence $\mathfrak{a}_X^* = \mathfrak{t}^*$ and it follows that the general isotropy group of B on X is trivial. This implies that

$$k[L_X] = k[\mathfrak{h}^\perp]^H$$

and this ring is generated by functions of certain degree, the only reflection subgroup of $W = BC_n$ having these degrees are $W_X = BC_m \oplus BC_m$ for $n = 2m$ and $W_X = BC_m \oplus BC_{m+1}$ for $n = 2m + 1$. The ring extension $\mathcal{Z}(G) \hookrightarrow \mathcal{Z}(X)$ has degree $\binom{n}{m}$.

The ring of invariant differential operators for multiplicity free representations has been worked out by Howe-Umeda [HU91].

References

[HU91] Roger Howe and Toru Umeda. The capelli identity, the double commutant theorem, and multiplicity-free actions. *Mathematische Annalen*, 290(1):565–619, 1991.

[Kno90] Friedrich Knop. Weylgruppe und momentabbildung. *Inventiones mathematicae*, 99(1):1–23, 1990.

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