GLOBAL EULERIAN PAIRS

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1. Introduction

This note globalizes the modified definition of local Eulerian pairs [LE, Definition 3.2.3]. Over a global field F the Weil–Deligne group is replaced by the Langlands group L_F, and a pair $(\mathbf{G}, \mathbf{H})$ is called global Eulerian if, for every X-distinguished global Arthur parameter, the fixed locus on the sliced dual space is a single orbit of the centralizer of the parameter. By the numerical conjecture of [BZSV] this is the statement that the $\mathbf{H}$-period on the corresponding automorphic forms is a constant times one Euler product.

As in the local case the condition is equivalent to a unique lifting property for parameters of $G_X^\vee$ (Proposition 3.2). The global setting adds one thing: a global parameter can have Zariski dense image in $G_X^\vee$, and for such parameters the condition becomes the group-theoretic identity $N_{C_X}(K_X) = K_X \cdot Z_{C_X}(K_X)$ (Theorem 3.4).

Section 5 applies this to the 27 rows of [W, Table 1]. The outcome is:

- global Eulerian: rows 1–4, 8–17, and rows 6, 7 for $m = 1$;
- Eulerian for parameters of general type but not for all parameters: row 18;
- not Eulerian, even for parameters of general type: rows 5 and 19–27;
- outside the definition (T^*X not hyperspherical): rows 6, 7 for $m \geq 2$.

The verdicts for rows 5, 15, 16 and 19–27 rest on dual data computed here, by arguments that are sketched and not fully written out; row 19 is the least certain. See §5.6.

2. Setting and definition

2.1. **Setting.** Let F be a global field with ring of adeles $\mathbb{A}$, and let $\mathbf{G}$ be a split connected reductive group over F. Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a homogeneous affine spherical variety such that $M = T^*X$ is hyperspherical, with dual $M^\vee$, a graded Hamiltonian $G^\vee$-variety, and distinguished morphism $\iota_X : G_X^\vee \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$.

Convention 2.1. $\mathbf{G}$ acts faithfully on $\mathbf{X}$: we replace $\mathbf{G}$ by $\mathbf{G}/\mathbf{N}$, where $\mathbf{N}$ is the largest normal subgroup of $\mathbf{G}$ contained in $\mathbf{H}$. This matches the classical period, which is an integral over $\mathbf{H}(F)(Z_{\mathbf{G}} \cap \mathbf{H})(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})$.

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Let L_F be the Langlands group of F and $|\cdot| : L_F \to \mathbb{R}_{>0}$ the pullback of the idele norm. A *global Arthur parameter* is a homomorphism

$$\phi_A : \ L_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee$$

whose restriction to L_F has bounded image. It is *X-distinguished* if its restriction to SL_2 is conjugate to that of ι_X. Its extended parameter is

$$\phi_E(w) = \Big(\phi_A\big(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})\big),\ |w|^{1/2}\Big) \in G^\vee \times \mathbb{G}_m,$$

which acts on $M^\vee$ through the $G^\vee$-action and the grading. With $\{e, h, f\}$ the $\mathfrak{sl}_2$-triple of ϕ_A and $M^\vee_{\mathrm{slice}} = \mu^{-1}(f + \mathfrak{g}^{\vee, e})$, put

$$\mathrm{Fix}(\phi_A) = \big(M^\vee_{\mathrm{slice}}\big)^{\phi_E(L_F)}, \qquad S_{\phi_A} = Z_{G^\vee}(\mathrm{Im}\,\phi_A).$$

As in [LE, Lemma 3.2.1], S_{ϕ_A} acts on $\mathrm{Fix}(\phi_A)$. Fixed loci are sets of geometric points.

Remark 2.2. The group L_F is hypothetical for number fields. Over a function field one can use the Weil group and ℓ-adic parameters pure of weight zero, as in [BZSV, §14]. The arguments below use only that $|\cdot|$ is non-trivial and that $\phi_A|_{L_F}$ is bounded.

2.2. **The definition.**

Definition 2.3. An X-distinguished global Arthur parameter ϕ_A is *X-Eulerian* if $\mathrm{Fix}(\phi_A)$ is non-empty and S_{ϕ_A} acts transitively on it.

Definition 2.4 (global Eulerian pair)**.** $(\mathbf{G}, \mathbf{H})$ is a *global Eulerian pair* over F if every X-distinguished global Arthur parameter ϕ_A with $\mathrm{Fix}(\phi_A) \neq \emptyset$ is X-Eulerian.

Definition 2.5 (generic version)**.** A parameter $\phi_X : L_F \to G_X^\vee$ is of *general type* if its image is Zariski dense. $(\mathbf{G}, \mathbf{H})$ is *generically global Eulerian* if $\phi_A = \iota_X \circ (\phi_X \times \mathrm{id}_{\mathrm{SL}_2})$ is X-Eulerian for every ϕ_X of general type.

Remark 2.6 (Meaning). For f in an automorphic representation with parameter ϕ_A, [BZSV, Conjecture 14.3.5] expresses the normalized period of f as a sum over the points m of $\mathrm{Fix}(\phi_A)$, assumed finite, of Euler products determined by the action of ϕ_E on $T_m M^\vee_{\mathrm{slice}}$. Points in one S_{ϕ_A}-orbit give the same Euler product. So for an X-Eulerian ϕ_A the period is $|\mathrm{Fix}(\phi_A)|$ times one Euler product, and for a non-Eulerian ϕ_A it is a sum of Euler products that differ in general. When $\mathrm{Fix}(\phi_A)$ is infinite, ϕ_A is not discrete and the statement concerns a continuous family.

Remark 2.7 (Local multiplicity one is not required). A global Eulerian pair need not be a Gelfand pair at every place. Row 12, (G_2, SL_3), is global Eulerian, although some π_v carry two invariant functionals [C, Theorem 1.5]. The prediction is that the global period, an element of $\otimes_v \mathrm{Hom}_{\mathbf{H}(F_v)}(\pi_v, \mathbb{C})$, is nevertheless a pure tensor.

3. Reformulation

Write $\epsilon = \iota_X|_{1\times \mathrm{SL}_2}$, $K_X = \iota_X(G_X^\vee \times 1)$ and $C_X = Z_{G^\vee}(\epsilon(\mathrm{SL}_2))$. An X-distinguished ϕ_A is conjugate to $\rho\cdot\epsilon$ with $\rho : L_F \to C_X$ bounded, and $S_{\phi_A} = Z_{C_X}(\rho)$. We assume that $M^\vee$ is the Whittaker induction of a symplectic representation S_X of $G_X^\vee$ along ι_X, and that ι_X is injective on $G_X^\vee$.

Lemma 3.1. $\mathrm{Fix}(\phi_A) \simeq (C_X/K_X)^{\rho(L_F)} = \{cK_X : c^{-1}\rho(L_F)c \subseteq K_X\}$, *equivariantly for* S_{ϕ_A}.

Sketch. As in [LE, Lemma 4.1.1] and [BZSV, §14.3, §14.6.4]. Choose $w \in L_F$ with $|w| \neq 1$. On the weight j part of an equivariant embedding of $M^\vee$, the element $\phi_E(w)$ acts by $\rho(w)|w|^{j/2}$ and $\rho(w)$ has eigenvalues of absolute value one, so fixed points have weight zero for $t \mapsto (\epsilon(\mathrm{diag}(t, t^{-1})), t)$. The fixed points of this group on the slice are the classes $[c, 0, 0]$, $c \in C_X$. □

Proposition 3.2. *The* S_{ϕ_A}*-orbits on* $\mathrm{Fix}(\phi_A)$ *are in bijection with the* $G_X^\vee$*-conjugacy classes of* $\phi_X : L_F \to G_X^\vee$ *such that* $\iota_X \circ (\phi_X \times \mathrm{id})$ *is* $G^\vee$*-conjugate to* ϕ_A*. The stabilizer of the point attached to* ϕ_X *is* S_{ϕ_X}*. Hence* ϕ_A *is* X*-Eulerian if and only if it has exactly one lift to* $G_X^\vee$ *up to conjugacy, and then* $\mathrm{Fix}(\phi_A) \simeq S_{\phi_A}/S_{\phi_X}$.

The proof is that of [LE, Proposition 4.2.1], which uses nothing about the source group.

Definition 3.3. The inclusion $K_X \subseteq C_X$ has *unique lifting* if, for every group Γ, two homomorphisms $\Gamma \to K_X$ that are conjugate under C_X are conjugate under K_X.

Theorem 3.4. (1) *If* ϕ_X *is of general type, then* $\iota_X \circ (\phi_X \times \mathrm{id})$ *is* X*-Eulerian if and only if*

$$N_{C_X}(K_X) = K_X \cdot Z_{C_X}(K_X). \tag{3.1}$$

So, if parameters of general type exist, $(\mathbf{G}, \mathbf{H})$ *is generically global Eulerian if and only if* (3.1) *holds.*

(2) *If* $K_X \subseteq C_X$ *has unique lifting, then* $(\mathbf{G}, \mathbf{H})$ *is global Eulerian over every global field, and local Eulerian over every local field.*

(3) *Unique lifting holds if* $C_X = K_X \cdot Z_{C_X}(K_X)$*, in particular if* $C_X = K_X$.

(4) *If* $n \in N_{C_X}(K_X)$ *induces on* K_X *an automorphism that is not inner, then* (3.1) *fails, and for* ϕ_X *of general type* ϕ_X *and* $n^{-1}\phi_X n$ *are two non-conjugate lifts of the same* ϕ_A.

Proof. (1) If $\rho(L_F)$ is dense in K_X, then $c^{-1}\rho c \subseteq K_X$ if and only if $c \in N_{C_X}(K_X)$, and $Z_{C_X}(\rho) = Z_{C_X}(K_X)$. So the fixed locus is $N_{C_X}(K_X)/K_X$ and the orbit of the base point is $Z_{C_X}(K_X)K_X/K_X$. (2) and (3) follow from Proposition 3.2 and [LE, Corollary 4.2.2]. (4) If $n = kz$ with z centralizing K_X, then n acts on K_X as the inner automorphism of k. □

Remark 3.5 (Local and global). Unique lifting is a property of the complex groups $K_X \subseteq C_X$ and gives the local and the global statement at once. Without it the two notions are not comparable in an obvious way, because the images of local and of global parameters are different classes of subgroups of K_X. Criterion (3.1) is genuinely global: a parameter of a p-adic Weil–Deligne group rarely has dense image.

4. Blocks and the swap criterion

Most rows of Wan's table that are not covered by [N] are built from one construction.

Definition 4.1. The *block* of size $p \geq 1$ is $Y_p = \mathrm{Sp}_{2p}\backslash\mathrm{Sp}_{2p+2}$, viewed as a variety for $\mathrm{Sp}_{2p+2}\times\mathrm{SL}_2$, where $\mathrm{SL}_2 = \mathrm{Sp}_2$ is the factor of $\mathrm{Sp}_2\times\mathrm{Sp}_{2p} \subset \mathrm{Sp}_{2p+2}$ centralizing Sp_{2p}. If Z is a spherical variety for $\mathbf{G}_1 \times \mathrm{SL}_2$, *attaching a block to the leg* SL_2 gives the $\mathbf{G}_1 \times \mathrm{Sp}_{2p+2}$-variety $X = Z \times^{\mathrm{SL}_2} Y_p$.

For an irreducible V_λ of Sp_{2p+2}, the space $V_\lambda^{\mathrm{Sp}_{2p}}$ is non-zero only for $\lambda = (a, b, 0, \ldots, 0)$, and is then the irreducible SL_2-module of highest weight $a - b$. So the weights of X are obtained from those of Z by replacing the leg weight c with (a, b), $a - b = c$, with $b \geq 0$ free.

Proposition 4.2 (swap criterion)**.** *Let X be obtained from Z by attaching a block of size p to a leg with simple root α, and write $\alpha'_1 = e_1 - e_2$ for the first simple root of* Sp_{2p+2}.

1. *If α is itself a spherical root of Z, then α'_1 and $e_1 + e_2$ are spherical roots of X, and* $\mathrm{Lie}\, K_X = \mathfrak{so}_4 \oplus \mathfrak{k}''$ *with* $\mathfrak{so}_4 \subset \mathfrak{so}_{2p+3}$ *and $\mathfrak{k}''$ in the dual Lie algebra of $\mathbf{G}_1$. An element s of $\mathrm{O}_4 \smallsetminus \mathrm{SO}_4$ lies in C_X, normalizes K_X and exchanges the two factors of* $\mathfrak{so}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. *Hence (3.1) fails and $(\mathbf{G}, \mathbf{H})$ is not generically global Eulerian.*
2. *If α occurs in the spherical roots of Z only through $\alpha + \alpha''$, with α'' a simple root of $\mathbf{G}_1$ orthogonal to α (the group case), then the spherical roots of X are $\alpha'_1 + \alpha''$ and $e_1 + e_2$, and s does not normalize K_X.*

Sketch. Map X to the symmetric variety $X_{\mathrm{sym}} = (\mathrm{Sp}_2 \times \mathrm{Sp}_{2p})\backslash\mathrm{Sp}_{2p+2}$, of rank one with spherical root $e_1 + e_2$ and generic stabilizer $\mathrm{SL}_{2,\alpha'_1} \times \mathrm{Sp}_{2p-2}$. The fibre is Z, on which this stabilizer acts through $\mathrm{SL}_{2,\alpha'_1} \simeq \mathrm{SL}_2$. By localization, the spherical roots of X in the span of α'_1 and the roots of $\mathbf{G}_1$ are those of Z with α replaced by α'_1. The little Weyl group W_X is a product of reflection subgroups of the Weyl groups of the factors of $\mathbf{G}$, and contains an element acting by -1 on $e_1 + e_2$ because $W_{X_{\mathrm{sym}}}$ is a subquotient of W_X. In case (1) it also contains $s_{\alpha'_1}$, hence $s_{e_1+e_2}$. The Arthur SL_2 is principal in SO_{2p-1}, so the standard representation of SO_{2p+3} is $\mathbb{C}^4 \oplus S_{2p-1}$ and C_X meets SO_{2p+3} in O_4; as K_X is connected, its part in SO_{2p+3} is all of SO_4. Case (2) is rows 17 and 18 of [N]. □

For $p = 1$ the standard representation of SO_5 is $\mathbb{C}^4 \oplus S_1$ and C_X contains all of SO_5; the conclusion of (1) still holds if $2e_2$ is not a spherical root,

which we expect. Each row below has a block with $p \geq 2$, and one swap suffices for a negative verdict.

Case (1) reproduces the known spherical roots α_1, $\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$ of $(\mathrm{Sp}_{2n-2} \times \mathrm{GL}_1)\backslash \mathrm{Sp}_{2n}$, and case (2) reproduces rows 17 and 18.

Remark 4.3 (What the swap means for periods). Let Π be a representation of Sp_{2p+2} whose parameter is $(A \otimes B) \oplus S_{2p-1}$ with $[A, B] \in \mathrm{SO}_4$. The parameter does not record which of A, B sits on the factor $\mathrm{SL}_{2,\alpha_1'}$, and one expects the Sp_{2p}-coinvariants of Π, as a representation of SL_2, to involve both π_A and π_B. In case (1) the period on X is then the sum of the Z-periods with π_A and with π_B on the leg: two Euler products. In case (2) the other end of the leg selects one of them. This is a heuristic, consistent with Proposition 4.2.

5. Wan's Table 1

5.1. **Results.** Table 1 lists, for each row of [W, Table 1], whether the pair is global Eulerian (Definition 2.4) and generically global Eulerian (Definition 2.5). Groups are normalized by Convention 2.1; K_X and C_X are given up to isogeny. The number of lifts is that of a parameter of general type.

Rows	Dual data and reason	All ϕ_A	General type
3, 4, 8, 10–14	$C_X = K_X$	yes	yes
1, 2	$K_X = \mathrm{Sp}_{2a} \subset \mathrm{GL}_{2a}$; symplectic structures are unique up to the centralizer [GGP, Thm. 8.1]	yes	yes
9	$C_X = K_X \times \mathrm{SO}_2$	yes	yes
6, 7 ($m = 1$)	$C_X = K_X \times \{\pm 1\}$	yes	yes
15, 16	$K_X = \mathrm{GL}_4 \times \mathrm{GL}_2$, $C_X = K_X \cdot Z_{C_X}(K_X)$	yes	yes
17	$K_X \simeq \mathrm{SO}_4$, graph in $\mathrm{O}_4 \times \mathrm{SO}_3$; unique lift	yes	yes
18	$K_X = \mathrm{SL}_2^3/\mu_2$ in $\mathrm{O}_4 \times \mathrm{O}_4$; $N_{C_X}(K_X) = K_X$, but $[A, B, A]$ has two lifts	no	yes
5	$K_X = \mathrm{Spin}_4$, $C_X \supset K_X$ of index 2; swap; 2 lifts	no	no
19	$K_X = \mathrm{SO}_8 \times \mathrm{SO}_5$, $C_X = \mathrm{O}_8 \times \mathrm{SO}_5$; 2 lifts	no	no
20, 21, 22	core $\mathrm{GL}_{N-2}\backslash \mathrm{GL}_N$, one block on a leg of type T; 2 lifts	no	no
23, 24	Gross–Prasad core on Sp_4; 1, resp. 2 blocks; 2, resp. 4 lifts	no	no
25, 26, 27	triple product core; 1, 2, 3 blocks; 2, 4, 8 lifts	no	no
6, 7 ($m \geq 2$)	T^*X not hyperspherical (type N roots)	not defined	

Table 1. Global Eulerian pairs in Wan's Table 1.

5.2. **Rows covered by earlier computations.** For rows 1–4, 6–14, 17 and 18 the groups $K_X \subseteq C_X$ are those of [N], and unique lifting was checked in [LE, §5]; passing to $\mathbf{G}/\mathbf{N}$ preserves injectivity of the lifting map. For row 13 the normalized pair has $C_X = K_X$ for each of the central embeddings considered in [N]. Rows 6 and 7 with $m = 1$ have two fixed points, exchanged by a central element.

In row 17, $K_X = \{(d, j(d))\} \subset \mathrm{O}_4 \times \mathrm{SO}_3$ with $j : \mathrm{SO}_4 \to \mathrm{PGL}_2$ the projection to one adjoint factor, and a parameter is $([A, B], \bar{B})$. On the identity component of $C_X/K_X \simeq \mathrm{PGL}_2$ the fixed points are the centralizer of $\bar{B}$, a single orbit of $Z_{\mathrm{SO}_3}(\bar{B}) \subseteq S_{\phi_A}$. The other component contributes only if $\bar{A}$ and $\bar{B}$ are projectively equivalent. In that case the 4-dimensional orthogonal representation $A \otimes B$ contains an invariant line, so its centralizer in O_4 is not contained in SO_4, and the new lift is conjugate to the old one in K_X.

In row 18, $K_X = \{[x, y, z]\}$ embeds in the dual group by $[x, y, z] \mapsto ([x, y], [y, z])$. For $A \not\simeq B$ the parameters $[A, B, A]$ and $[B, A, B]$ have conjugate images, by the element exchanging the two factors of each SO_4, but are not conjugate in K_X. On the automorphic side this is the case where the two representations Π_1, Π_2 come from the same pair $\{A, B\}$, and the period has the two terms of Remark 4.3. The image of $[A, B, A]$ is not dense, in agreement with $N_{C_X}(K_X) = K_X$.

5.3. **Rows 15 and 16.** Here $\mathbf{G} = (\mathrm{GL}_N \times \mathrm{GL}_2)/\mathbb{G}_m$ and $\mathbf{X} = \mathrm{GL}_{N-2}\backslash\mathrm{GL}_N$. For $\lambda = (\lambda_1, \lambda_2, 0, \ldots, 0, \lambda_{N-1}, \lambda_N)$ the GL_{N-2}-invariants of V_λ are, as a GL_2-module, a tensor product of two irreducibles of dimensions $\lambda_1 - \lambda_2 + 1$ and $\lambda_{N-1} - \lambda_N + 1$. The spherical roots are α_1, $e_2 - e_{N-1}$, α_{N-1} and the root of GL_2, of type $A_3 \times A_1$. So $K_X = \{(g, h) \in \mathrm{GL}_4 \times \mathrm{GL}_2 : \det g \det h = 1\}$, the dual group of the strongly tempered pair $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$ of [WZ], embedded through $\mathbb{C}^N = \mathbb{C}^4 \oplus S_{N-4}$. The centralizer C_X adds only the scalars on S_{N-4}, so $C_X = K_X \cdot Z_{C_X}(K_X)$ and Theorem 3.4(3) applies.

5.4. **Rows 5 and 20–27.** Each of these is obtained by attaching blocks to a core Z, and in each case the leg is of type T, so Proposition 4.2(1) applies to every block.

- Row 5: $Z = \mathbb{G}_m\backslash\mathrm{SL}_2$.
- Rows 20–22: $Z = \mathrm{GL}_{N-2}\backslash\mathrm{GL}_N$ as a variety for $\mathrm{GL}_N \times \mathrm{SL}_2$. By the computation for rows 15, 16 its weights on the three SL_2-directions α_1, α_{N-1} and the leg satisfy the Clebsch–Gordan condition, so the leg root is a spherical root. (For $N = 3$, Z is SL_3 acted on by $\mathrm{SL}_3 \times \mathrm{GL}_2$, the Rankin–Selberg variety, with the same conclusion.)
- Rows 23, 24: $Z = (\mathrm{Sp}_2 \times \mathrm{Sp}_2)\backslash(\mathrm{Sp}_4 \times \mathrm{SL}_2 \times \mathrm{SL}_2)$, the Gross–Prasad variety for $\mathrm{SO}_5 \times \mathrm{SO}_4$, which is strongly tempered. In row 23 one leg carries a block; in row 24 both do.
- Rows 25–27: $Z = \mathrm{SL}_2\backslash\mathrm{SL}_2^3$, the triple product variety, strongly tempered, with blocks on one, two or three legs.

In all these rows the generic stabilizer of T^*X computed in [N] is $\mu_2 \times$ (connected) with $\mu_2 = Z_{\mathbf{G}} \cap \mathbf{H}$, so T^*X is hyperspherical after Convention 2.1. A parameter of general type with r blocks has 2^r lifts, obtained by exchanging A_i and B_i in each block.

5.5. **Row 19.** Here $\mathbf{X} = \mathrm{Sp}_{2p}\backslash\mathrm{Sp}_{2p+4}$ for $\mathrm{Sp}_{2p+4} \times \mathrm{Sp}_4$, a block of rank two. The fibre over the symmetric variety $(\mathrm{Sp}_4 \times \mathrm{Sp}_{2p})\backslash\mathrm{Sp}_{2p+4}$ is the Gross–Prasad variety of rows 23, 24, so α_1, α_3 and the two simple roots of the external Sp_4 are spherical roots. The little Weyl group must also contain lifts of the Weyl group of type C_2 of the symmetric variety. A reflection subgroup of $W(C_4)$ with these properties contains $W(D_4)$. Since the standard representation of SO_{2p+5} is $\mathbb{C}^8 \oplus S_{2p-3}$, the group C_X is $\mathrm{O}_8 \times \mathrm{SO}_5$ and K_X, being connected of rank 6, is $\mathrm{SO}_8 \times \mathrm{SO}_5$. The outer automorphism of SO_8 coming from O_8 gives two lifts of an irreducible 8-dimensional orthogonal parameter.

5.6. **Caveats.**

(1) The reformulation assumes that $M^\vee$ is Whittaker-induced along ι_X.
(2) The entries for rows 1–4, 6–14, 17, 18 depend on the dual data of [N].
(3) Proposition 4.2 is proved here only in outline. It uses localization of spherical roots and the behaviour of the little Weyl group under an equivariant fibration, which should be checked against the literature on spherical systems.
(4) Row 19 is the least certain entry: the argument determines W_X only from below, and identifies K_X using its inclusion in the identity component of C_X.
(5) Rows 15 and 16 use that $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$ is strongly tempered, quoted from memory of [WZ].
(6) Parameters of general type are assumed to exist for each $G_X^\vee$.

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