GEOMETRIC COCYCLE

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1. Introduction

This is my study note for Spencer Leslie's paper [Les24], where he introduced the notion of geometric cocycles which together with the Galois action on spherical systems determines the well-adapted homogeneous spherical varieties up to G-inner twist.

2. Notation

We will fix k a characteristic zero field, $\Gamma = \mathrm{Gal}(\overline{k}/k)$, $\overline{k}$ its algebraic closure. We will let G be a connected reductive group over k.

X a spherical variety for G, and we introduce the following invariants for X (both are over $\overline{k}$): A_X the maximal torus of A, $\chi = X^*(A_X)$ the weight lattice of X, $\Sigma_X \subseteq \chi$ the set of (primitive) spherical roots for X, Σ_X^n the set of n-spherical roots, Λ_X the root lattice, $\mathscr{D}(X)$ the set of B-stable prime divisors of X.

3. Uniqueness of homogeneous spherical varieties

We will introduce the following types of spherical roots, they will play an important role in the calculation of various automorphism groups and in fact also in the proof of 3.4.

Definition 3.1. We define the following four types of spherical roots, $\Sigma_X^1, \Sigma_X^2, \Sigma_X^3, \Sigma_X^4$

(1) $\gamma \in \Sigma_G$ and there exists $D \in \mathscr{D}$ with $\rho(D) = \frac{1}{2}\alpha^\vee|_{\mathfrak{a}_X}$.
(2) there is a subset $\Sigma \subset \Sigma_G$ of type $B_n,\ n \geq 2$ such that

$$\gamma = \alpha_1 + \alpha_2 + \cdots \alpha_n$$

and $\alpha_i \in \Sigma_X^p$ for $i > 1$.
(3) there is a subset $\{\alpha_1,\ \alpha_2\} \subset \Sigma_G$ of type G_2 with α_1 short root and $\gamma = 2\alpha_1 + \alpha_2$.
(4) $\gamma \in X^*(A_X)$ but $\gamma \notin \mathbb{Z}\Phi$.

Following the approach of Knop, Losev proves a uniqueness result 3.2 for the homogeneous spherical varieties with given combinatorial invariants, one of the advantage of his approach is that he avoided the case-by-case considerations, and also he clarifies the relation between the (primitive spherical roots) Σ_X and the n-spherical roots Σ_X^n generates the root lattice Λ_X, he introduced the definition of *distinguished roots*, and this provides us with a way to calculate the G-equivariant automorphism group of X.

Given X_1, X_2 two spherical varieties, we write $\mathscr{D}(X_1) = \mathscr{D}(X_2)$ if there is a bijection $\psi : \mathscr{D}(X_1) \to \mathscr{D}(X_2)$, such that $G_D = G_{\psi(D)}$, $\rho(D) = \rho(\psi(D))$, here $G_D = \{g \in D \mid\ gD = D\}$.

Theorem 3.2. *Let H_1, H_2 be two spherical subgroups, $X_1 = G/H_1,\ X_2 = G/H_2$, if $\chi(X_1) = \chi(X_2)$, $\mathscr{V}_{X_1} = \mathscr{V}_{X_2}$, $\mathscr{D}(X_1) = \mathscr{D}(X_2)$, then H_1, H_2 are G-conjugate.*

Furthermore, it can be shown that once the assumptions in the theorem are satisfied, any bijection $\psi : \mathscr{D}(X_1) \to \mathscr{D}(X_2)$ is induced by some element of $\mathrm{Aut}^G(X_1) \cong \mathrm{Aut}^G(X_2)$.

Losev clarifies the distinction between Σ_X and Σ_X^n, and he introduced the notion of *distinguished spherical roots.*

Definition 3.3. Following Losev, we will call roots of type $(1), (2), (3)$ the set of distinguished roots.

The following theorem studies the relation between Σ_X and Σ_X^n.

Theorem 3.4. *Assume X is a homogeneous spherical variety, then we have*

$$\Sigma_X^n = (\Sigma_X \cap \Lambda_G \backslash \Sigma_X^{dist}) \sqcup \{2\alpha \mid \alpha \in \Sigma_X^{dist} \cup (\Sigma_X \backslash \Lambda_G)\}$$

in other words, Σ_X^n is obtained from Σ_X by replacing spherical roots α of type $(1),(2),(3),(4)$ by 2α.

We have the following functorial properties between distinguished roots

Lemma 3.5. *Let X_1, X_2 be two spherical G-varieties and let $\varphi: X_1 \to X_2$ be a dominant G-equivariant morphism, then $\Sigma_{X_1}^i \cap \Sigma_{X_2} \subseteq \Sigma_{X_2}^i$ for $i = 1,2,3$. If φ is generically etale, then we have the equalities hold.*

The distinguished roots also behave well under parabolic induction

Proposition 3.6. *Let X be a spherical G-variety of the form $G \times_H V$, where H is a reductive subgroup of G and V is an H-module, then $\Sigma_X^i \subset \Sigma_{G/H}^i$ for $i = 1,2$.*

4. Forms of spherical varieties

Definition 4.1. A k-form of the spherical pair (G,X) is a k-form of the action $G \times X \to X$, that is a pair (G', X') where the reductive group G' over k acts on the variety X' over k, such that we have $\overline{k}$-isomorphisms $t_G : G_{\overline{k}} \longrightarrow G'_{\overline{k}}$ and $t_X : X_{\overline{k}} \longrightarrow X'_{\overline{k}}$, and the following diagram commutes

$$\begin{array}{ccc} G_{\overline{k}} \times X_{\overline{k}} & \longrightarrow & X_{\overline{k}} \\ \downarrow{\scriptstyle t_G \times t_X} & & \downarrow{\scriptstyle t_X} \\ G'_{\overline{k}} \times X'_{\overline{k}} & \longrightarrow & X'_{\overline{k}} \end{array}$$

Theorem 4.2. *Given a spherical pair (G,X) defined over k, then the k-forms of a spherical pair (G,X) are classified by $H^1(k, Aut(G_{\overline{k}}, X_{\overline{k}}))$ up to k-isomorphisms.*

We will denote $\mu_{(G,X)}: \Gamma \to \mathrm{Aut}(G,X)$ the k-structure determined by the k-spherical pair (G,X). Given (G',X') a k-form of (G,X) the 1-cocycle $c : \Gamma \to \mathrm{Aut}(G,X)$ is given $\sigma \mapsto \frac{\mu_{(G',X')}(\sigma)}{\mu_{(G,X)}(\sigma)}$.

We are actually more interested in the k-forms (G', X') with $G' \cong G$ over k.

Corollary 4.3. *Given a spherical pair (G,X) defined over k, then the k-forms of a spherical pair (G,X) are classified by $H^1(k, Aut^{G_{\overline{k}}}(X_{\overline{k}}))$ up to k-isomorphisms.*

Given X a k-spherical variety for G, we will denote $\mu_X : \Gamma \to \mathrm{Aut}^G(X)$ the k-structure on $\mathrm{Aut}^G(X)$ determined by X. Let X' be a k-form of X, then the 1-cocycle $[c_{X',X}]: \Gamma \to \mathrm{Aut}^G(X)$ is given by

$$\sigma \mapsto \mu_{X'}(\sigma)/\mu_X(\sigma) \tag{4.1}$$

Now we restrict to homogeneous spherical varieties, we choose $x_0 \in X(k)$, hence a k-isomorphism $X \cong H\backslash G$, then we have $\mathrm{Aut}^G(X) \cong H \backslash N_G(H)$, we have a natural map $\mathrm{Aut}^G(H) \to \mathrm{Aut}(H)$ via conjugation, composing with $\mathrm{Aut}(H) \to \mathrm{Out}(H)$, we obtain a map

$$\mathrm{Aut}^G(X) \longrightarrow \mathrm{Out}(H)$$

we denote $\mathrm{Out}_X(H) \subset \mathrm{Out}(H)$ its image and $\mathcal{A}_X^b \subset \mathrm{Aut}^G(X)$ the kernel of this map, both $\mathcal{A}_X^b$ and $\mathrm{Out}_X(H)$ are defined over k.

Lemma 4.4. *The map $out_H^G: Aut^G(X) \longrightarrow Out(H)$ is Γ-equivariant for the natural action on $Out(H)$.*

Definition 4.5. Suppose we have (G,X) with $X = H\backslash G$ and (G,X') with $X' = H'\backslash G$ is a k-form of (G,X). We say X' is a G-inner form of X if the cocycle $c_{X,X'} \in Z^1(k, \mathrm{Aut}^G(X))$ takes value in $\mathcal{A}_X^b$.

From the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{A}_X^b & \longrightarrow & \mathrm{Aut}^G(X) & \longrightarrow & \mathrm{Out}_X(H) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H_{ad} & \longrightarrow & \mathrm{Aut}(H) & \longrightarrow & \mathrm{Out}(H) & \longrightarrow & 1 \end{array}$$

we see if $X' = H'\backslash G$ is a G-inner form of X, i.e. it corresponds to a class $[c] \in H^1(k, \mathrm{Aut}^G(X))$ which is the image of $H^1(k, \mathcal{A}_X^b)$, we have H' corresponds to a class $[\tilde{c}] \in H^1(k, \mathrm{Aut}(H))$ which comes from $H^1(k, \mathrm{Inn}(H))$, hence then H' is an inner form of H.

Lemma 4.6. *Suppose G is a reductive k-group and $X = H\backslash G$ is a homogeneous k-variety, let X' be a G-form of X, then the cocycle $c_{X,X'}$ induces the trivial cohomology class in $H^1(k, Out_X(H))$ if and only if X and X' are G-inner forms.*

Proof. This follows from the long exact sequence associated with the short exact sequence:

$$1 \longrightarrow \mathcal{A}_X^b \longrightarrow \mathrm{Aut}^G(X) \longrightarrow \mathrm{Out}_X(H) \longrightarrow 1$$

we get $H^1(k, \mathcal{A}_X^b) \longrightarrow H^1(k, \mathrm{Aut}^G(X)) \longrightarrow H^1(k, \mathrm{Out}_X(H))$. □

5. Existence of equivariant models

We summarize the result of Borovoi and Gagliardi [BG22] concerning the existence of equivariant models of spherical varieties in the following sections.

5.1. **Galois descent theory.** Let k be a field of characteristic zero, and $\Gamma = \mathrm{Gal}(\overline{k}/k)$, for $\gamma \in \Gamma$, we denote $\gamma^* : \mathrm{Spec}\ k \to \mathrm{Spec}\ k$ the morphism of schemes induced by γ, then we have $(\gamma_1\gamma_2)^* = \gamma_2^* \circ \gamma_1^*$.

Let $(Y, p_Y : Y \to \mathrm{Spec}\ k)$ be a k-scheme, a $\overline{k}/k$ semilinear automorphism of Y is a pair (γ, μ) where $\gamma \in \Gamma$ and $\mu : Y \to Y$ is an isomorphism of schemes such that the following diagrams commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\mu} & Y \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_Y} \\ \mathrm{Spec}\ k & \xrightarrow{(\gamma^*)^{-1}} & \mathrm{Spec}\ k \end{array}$$

we also say that μ is a γ-semilinear automorphism of Y. Note that if (γ, μ) is a semi-automorphism of Y, then μ determines γ uniquely.

Definition 5.1. We denote $\mathrm{SAut}(Y)$ the group of all γ-semilinear automorphisms μ of Y for γ runs over Γ. By a semilinear action of Γ on Y we mean a homomorphism of groups

$$\mu : \ \Gamma \to \mathrm{SAut}(Y), \ \gamma \mapsto \mu_\gamma$$

such that for each $\gamma \in \Gamma$ the automorphism μ_γ is γ-semilinear.

Example 5.2. Let G be a linear algebraic group over $\overline{k}$, we denote by $\mathrm{SAut}(G)$ the group of all γ-semilinear automorphisms τ of G, for γ runs over Γ. Any algebraic semilinear action σ of Γ on G comes from a k-model and conversely any algebraic semilinear action σ of Γ on G comes from a k-model of G as G is an affine variety.

Definition 5.3. Let Y be a $\overline{k}$-variety, we say that a semilinear action μ of Γ on Y is algebraic if there is a finite Galois extension ℓ/k of k and a ℓ-model Y_1 of Y inducing the action of μ on $\mathrm{Gal}(\overline{k}/\ell)$.

Lemma 5.4. *Let Y be a $\overline{k}$-variety, an algebraic semilinear action μ of Γ on Y comes from a k-model of Y if and only if Y admits a covering by Γ-stable open affine subvarieties. In particular, if Y is quasi-projective, then any algebraic semilinear Γ-action on Y comes from a k-model.*

Let $\mathcal{G}$ be a linear algebraic group over $\overline{k}$, and let $\mathcal{Y}$ be a $\mathcal{G}$-variety over $\overline{k}$, we let G be a k-model of $\mathcal{G}$.

Definition 5.5. Let $\mathcal{Y}$ be a $\mathcal{G}$-variety, and G a k-model of $\mathcal{G}$ with semilinear action $\sigma : \Gamma \to \mathrm{SAut}(G)$, let $\mu : \Gamma \to \mathrm{SAut}(Y)$ be a semilinear action, we say μ is σ-equivariant if

$$\mu_\gamma(gy) = \sigma_\gamma(g) \cdot \mu_\gamma(y)$$

for all $\gamma \in \Gamma, \ y \in \mathcal{Y}(\overline{k}), \ g \in \mathcal{G}(\overline{k})$.

If μ comes from a G-equivariant k-model Y of $\mathcal{Y}$, then it is σ-equivariant and conversely we have the following descent result

Lemma 5.6. *Let $\mathcal{G}, \mathcal{Y}, G$ and σ as in the previous definition, let μ be a σ-equivariant algebraic semilinear action of Γ on $\mathcal{Y}$, then the following two are equivalent*

(1) μ comes from a G-equivariant k-model Y of $\mathcal{Y}$.

(2) $\mathcal{Y}$ admits a covering by Γ-stable quasi-projective open subvarieties.

Example 5.7. If $\mathcal{Y}$ is a homogeneous $\mathcal{G}$-variety then it is quasi-projective and hence (2) is satisfied, so any σ-equivariant algebraic semilinear action of Γ on $\mathcal{Y}$ comes from a G-equivariant k-model Y of $\mathcal{Y}$.

Let $\mathcal{G}$ be a group scheme over $\overline{k}$, we an exact sequence

$$1 \to \mathrm{Aut}(\mathcal{G}) \to \mathrm{SAut}(\mathcal{G}) \to \Gamma$$

let G be a k-model of $\mathcal{G}$, then it defines a semilinear action σ, since $\mathrm{Aut}(\mathcal{G})$ is a normal subgroup of $\mathrm{SAut}(\mathcal{G})$, this action induces an action of Γ on the group $\mathrm{Aut}(\mathcal{G})$.

Recall that a 1-cocycle valued in $\mathrm{Aut}(\mathcal{G})$ is a map $c : \Gamma \to \mathrm{Aut}(\mathcal{G})$ such that $c_{\gamma\beta} = c_\gamma \cdot {}^\gamma c_\beta$ for $\gamma, \beta \in \Gamma$, we will denote the set of 1-cocycles by $Z^1(k, \mathrm{Aut}(\mathcal{G}))$. For every $c \in Z^1(k, \mathrm{Aut}(\mathcal{G}))$, we can consider the c-twisted semilinear action

$$\sigma' : \Gamma \to S\mathrm{Aut}(\mathcal{G}), \quad \gamma \mapsto c_\gamma \circ \sigma_\gamma$$

it follows from the cocycle condition $\sigma'_{\gamma\beta} = \sigma'_\gamma \circ \sigma'_\beta$ for all $\gamma, \beta \in \Gamma$.

The semilinear action comes from some k-model G' of $\mathcal{G}$, we will write $G' = G_c$ and say G' is a twisted form of G by the 1-cocycle c.

5.2. **Models of affine spherical varieties.** In the following, $k = \overline{k}$ will be an algebraically closed characteristic zero field, G will be a connected reductive group over k, B a Borel subgroup of G with unipotent radical U, T a maximal torus of B and X will be an affine G-variety.

Definition 5.8. For any affine G-variety X, weight monoid of X $\mathscr{V}_{wt} \subset X^*(T)$ will be the $\lambda \in X^*(T)$ such that

$$k[X] = \oplus_{\lambda \in X^*(T)} V(\lambda)$$

here $V(\lambda)$ is the irreducible representation of G with highest weight λ.

Up to G-equivariant isomorphism, there may be several different affine spherical varieties with the given weight monoid $\mathscr{V}_{wt}$ as there might be different multiplicative structures on the G-vector space $k[X]$. On the other hand, the ring of U-invariants $k[X]^U$ is always T-equivariantly isomorphic to the group algebra $k[\Gamma]$.

From now on k will be a characteristic zero field, $\mathcal{G}$ a connected reductive group over $\overline{k}$, G a k-model of $\mathcal{G}$. $\mathcal{X}$ a $\mathcal{G}$ spherical variety over $\overline{k}$, suppose X is a G-equivariant k-model of $\mathcal{X}$, then the Γ-action defined by G preserves $\mathscr{V}_{wt}$ and the valuation cone $\mathscr{V}$.

It turns out that the converse is also true when G is a quasisplit k-group, the converse is also true

Theorem 5.9. *Let $\mathcal{X}$ be an affine spherical $\mathcal{G}$-variety over $\overline{k}$ with wight monoid $\mathscr{V}_{wt}$ and valuation cone $\mathscr{V}$. Let G be a quasisplit k-model of $\mathcal{G}$, then $\mathcal{X}$ has a G-equivariant k-model if and only if the Γ-action defined by Γ preserves $\mathscr{V}_{wt}$ and $\mathscr{V}$.*

Their proof uses the moduli scheme $\mathcal{M}_\Lambda$, the $\overline{k}$-points of $\mathcal{M}_\Lambda$ corresponds to the equivalence classes $[X, \tau]$ with X an affine spherical $\mathcal{G}$-variety with weight monoid Λ and $\tau : k[X]^U \to k[\Lambda]$ a T-equivariant isomorphism of $\overline{k}$-algebras.

5.3. **Models of homogeneous spherical varieties for quasisplit groups.** In this section, X will be a G-homogeneous spherical variety over k.

Let $\mathscr{D}(X)$ be the set of B-stable prime divisors of X, we let G_D be the stabilizer of D in G. We have a valuation map $\rho : \mathscr{D}(X) \to \mathfrak{a}_X$. For $D \in \mathscr{D}(X)$, we let $\Sigma(D)$ be the set of $\alpha \in \Sigma_G$ such that $P_\alpha \nsubseteq G_D$. We obtain a map

$$\zeta : \ \mathscr{D}(X) \longrightarrow \mathscr{P}(\Sigma) \quad D \mapsto \Sigma(D)$$

for $\mathscr{P}(\Sigma)$ the power set of Σ.

We consider the map

$$\rho \times \zeta : \ \mathscr{D}(X) \longrightarrow \mathfrak{a}_X \times \mathscr{P}(\Sigma)$$

we denote Ω by its image. The fibers of this map is either 1 or 2 colors. Let $\Omega^{(1)}, \Omega^{(2)}$ denote the corresponding subsets determined by the size of the fiber.

Theorem 5.10. *We let G be a connected reductive group over $k = \overline{k}$. We denote $\Omega_X = (\chi, \Sigma_X, \Omega^{(1)}, \Omega^{(2)})$, suppose $X_1 = H_1\backslash G$, $X_2 = H_2\backslash G$ are two homogeneous spherical varieties over k and $\Omega_{X_1} = \Omega_{X_2}$, then X_1 and X_2 are G-isomorphic.*

We note that the spherical systems associated to X is $\mathcal{S}_X = (\chi, \Sigma_X, \mathscr{D}(X), \rho)$, so $\mathcal{S}_X$ is in fact part of the data of Ω_X.

We recall the following construction from Brion and Gagliardi, which allows us to write any homogeneous spherical varieties $\mathcal{H}\backslash\mathcal{G}$ as a quotient of a quasi-affine spherical homogeneous space by a torus.

Let $\mathcal{C}$ be a $\overline{k}$-torus with character group $\langle D\rangle$ for $D \in \mathscr{D}$, then $\mathcal{C}(\overline{k})$ is the group of maps $\mathscr{D} \to \overline{k}^\times$, we may define $\mathcal{G}' = \mathcal{G} \times \mathcal{C}$ and $\mathcal{B}' = \mathcal{B} \times \mathcal{C}$, so $X^*(\mathcal{B}') = X^*(\mathcal{B}) \oplus X^*(\mathcal{C})$, and χ can be regard as a sublattice of $X^*(\mathcal{B}')$, and $(\chi, \Sigma_\mathcal{X}, \mathscr{D})$ are the combinatorial invariants of the spherical subgroup $\mathcal{H} \times \mathcal{C} \subset \mathcal{G}'$.

Now we define some new invariants

$$\chi' = \chi + \langle \lambda_D \rangle,\ D \in \mathscr{D} \ \subset\ X^*(\mathcal{B}')$$

we also define ρ' via

$$\langle \rho'(D), \lambda\rangle \begin{cases} \langle \rho(D), \lambda\rangle & \text{for } \lambda \in \chi \\ q & \text{for } \lambda = \lambda_D \\ 0 & \text{for } \lambda = \lambda_{D'} \text{ with } D' \neq D \end{cases}$$

and we can set $\zeta'(D) = \zeta(D)$. So we obtain a new set of colors $\mathscr{D}'$ which is the same set as $\mathscr{D}$ but with different maps ρ' and ζ', we set $\Omega' = \mathrm{im}(\rho' \times \zeta')$, then we have $\Omega'^{(2)} = \emptyset$ and $\Omega'^{(1)} = \mathscr{D}' = \mathscr{D}$.

Proposition 5.11. *The invariants $(\chi', \Sigma_\mathcal{X}, \mathscr{D}')$ comes from a quasi-affine spherical homogeneous space $\mathcal{H}'\backslash\mathcal{G}'$ such that $\mathcal{H}' \subset \mathcal{H} \times \mathcal{C}$.*

To study the existence of the k-form, we first note the following result

Proposition 5.12. *Suppose that G is a quasisplit connected reductive group over k and $X = H\backslash G$ is a spherical homogeneous k-variety, then the induced Γ-action on $X^*(T)$ and Σ_G preserves Ω_X.*

When G is quasisplit, Borovoi and Gagliardi proves the following result as an inverse to the previous result, when $\sigma\Omega = \Omega$, we may choose a continuous lift

$$\alpha_\mathscr{D} :\ \Gamma \times \mathscr{D}(X) \longrightarrow \mathscr{D}(X)$$

which is the same as choosing an action of Γ on $\Omega^{(2)}$.

Theorem 5.13. *Suppose G is quasisplit and $\mathcal{X} = \mathcal{H}\backslash G_{\overline{k}}$ is a spherical homogeneous space of $G_{\overline{k}}$. We assume the Γ-action induced from G preserves Ω_X, then for any lift $\alpha_\mathscr{D}$ of the Γ-action from Ω to $\mathscr{D}(X)$, there exists a G-equivariant k-model X of $\mathcal{X}$ inducing $\alpha_\mathscr{D}$. Moreover $X(k) \neq \emptyset$, so that $X = H\backslash G$ for H a k-subgroup of G.*

We note the proof is reduced to the affine case and an existence result of spherical subgroups.

Proof. Let $\mathcal{X}' = \mathcal{H}\backslash\mathcal{G}$ be the quasi-affine spherical homogeneous space from with combinatorial invariants $(\chi', \Sigma_\mathcal{X}, \Omega'^{(1)}, \Omega'^{(2)})$, and let $\mathfrak{X}$ be its affine closure. According to a result of Grosshans, the codimension of $\mathfrak{X}\backslash(\mathcal{H}'\backslash\mathcal{G}')$ in $\mathfrak{X}$ is at least two, which means there are no $\mathcal{G}'$-invariant prime divisors in $\mathfrak{X}$, hence the set of colors $\mathscr{D}'$ is the set of all B'-invariant prime divisors in $\mathfrak{X}$. A rational B'-semi-invariant function $f \in k(X)^{(B')}$ is regular if and only if $v_D(f) \geq 0$ for every $D \in \mathscr{D}'$. Since by definition $\langle \rho'(D), \lambda\rangle = v_D(f)$ where λ is the B'-weight of f, the weight monoid of $\mathfrak{X}$ is

$$\mathscr{V}_{wt} = \{\lambda \in \chi' \mid \langle \rho'(D), \lambda\rangle \geq 0 \text{ for every D} \in \mathscr{D}'\}$$

We set $G' = G \times C$, then from the construction, the Γ-action on $\mathcal{C}$ defined by C comes from $\alpha_\mathscr{D}$. Then the Γ-action defined by G' preserves $(\chi', \Sigma_\mathcal{X}, \Omega'^{(1)}, \Omega'^{(2)})$, now from the existence of equivariant model for affine spherical varieties 5.9, we see there exists a G'-equivariant k-model $\tilde{X}$ of $\mathfrak{X}$, which induces a G'-equivariant k-model X' of the open G'-orbit $\mathcal{H}'\backslash\mathcal{G}'$, its quotient by C is a G-equivariant k-model X of $\mathcal{H}\backslash\mathcal{G}$, by construction, the Γ-action on $\Omega'^{(1)} \cong \mathscr{D}$ is $\alpha_\mathscr{D}$, this is what we want. $\square$

Example 5.14. For $\mathcal{X} = \mathbb{G}_m \backslash PGL_2$ over $\overline{k}$, we have $\mathscr{D}(X) = \{D_1, D_2\}$, so there are two lifts of Γ-action from Ω to $\mathscr{D}(X)$. The trivial Γ-action corresponds to the k-form $\mathbb{G}_m \backslash PGL_2$ over k, and the non-trivial Γ-action corresponds to the k-form $\mathrm{Res}^1_{\ell/k}\mathbb{G}_m \backslash PGL_2$, for ℓ/k a quadratic extension.

6. Group of doubling automorphisms

In this section, X will be a G-spherical variety over k. Following Leisle, let's introduce the group of doubling automorphisms, which encodes the arithmetic information for a spherical variety X over k.

We begin with the connected case: $X = H\backslash G$, $\pi_0(H) = 1$, $\Sigma_X^{dist} \subset \Sigma_X$ the set of distinguished spherical roots, Σ^d is obtained from Σ_X^n by replacing 2α for all α with $\alpha \in \Sigma_X^{dist}$, so only the spherical roots $\alpha \in \Sigma_X^4$ are doubled, we let $\Lambda^d := \mathbb{Z}\Sigma^d$, this is a Γ-stable subset of Λ_X.

We obtain a short exact sequence of Γ-modules

$$0 \longrightarrow \Lambda_X^d/\Lambda_X \longrightarrow \chi/\Lambda_X \longrightarrow \chi/\Lambda_X^d \longrightarrow 0$$

and a dual sequence of diagonalizable k-groups

$$1 \longrightarrow \mathcal{A}_X^d \longrightarrow \mathcal{A}_X \longrightarrow \mathrm{Aut}_d(X) \longrightarrow 1$$

Definition 6.1. We call $\mathrm{Aut}_d(X)$ the group of doubling automorphisms.

For each $\alpha \in \Sigma_X^{dist}$, it determines a canonical character $\mu_\alpha = \langle \alpha, - \rangle : \mathcal{A}_X(\overline{k}) \to \mu_2(\overline{k})$

$$\mathrm{Aut}_d(X)_{\overline{k}} \xrightarrow{\prod_\alpha \mu_\alpha} \prod_\alpha \mu_2 \tag{6.1}$$

this is a $\overline{k}$-isomorphism.

Next we determines the k-structure on $\mathrm{Aut}_d(X)$: we will see that the canonical k-structure on $\mathcal{A}_X$ induces a k-structure on $\mathrm{Aut}_d(X)$, let I_X be the index set of Γ-orbits on Σ_X^{dist}, then

$$\Sigma_X = \sqcup_{i \in I_X} \mathcal{O}_i = \sqcup_{\gamma_i \in \mathcal{O}_i} (\Gamma/\Gamma_i)\gamma_i$$

for $\Gamma_i = \mathrm{Gal}(\overline{k}/k_i)$, stabilizer of γ_i under Γ-action. We get

$$X^*(\mathrm{Aut}_d(X)) \cong \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\mathbb{Z}/2\mathbb{Z}\gamma_i) \cong (\mathbb{Z}/2\mathbb{Z})^{\Sigma_X^{dist}}$$

and hence $\mathrm{Aut}_d(X) \cong \prod_{i \in I_X} \mathrm{Res}_{k_i/k}(\mu_2)$.

For $X_{\overline{k}}$ a spherical variety for $G_{\overline{k}}$, the k-structure on the group $\mathrm{Aut}_d(X)$ is independent of the G-equivariant k-model of $X_{\overline{k}}$ as the Γ-action preserves Ω_X and the Γ-action on Σ_X^n is independent of the choice of forms, but as we saw before, the canonical k-structure of $\mathcal{A}_X$ depends on the k-form X.

Lemma 6.2. *Suppose $X = H\backslash G$ is spherical and H is geometrically connected, in the long exact sequence on cohomology, the map*

$$H^1(k, \mathcal{A}_X) \longrightarrow H^1(k, Aut_d(X))$$

is surjective.

Now assume $\pi_0(H)$ is not trivial, H^0 the connected component of the identity, $X^\circ = H^\circ\backslash G$. The first naive attempt is to define $\mathrm{Aut}_d^{naive}(X)$ as $\mathcal{A}_X/\mathcal{A}_X^d$, however the following example shows that this group maybe too small.

Example 6.3. For $G = SL_2 \times SL_2$, $H = (T_0 \times T_0) \sqcup (T_0\omega \times T_0\omega)$, $\theta(g) = {}^tg^{-1}$, $\omega \in N_{SL_2}(T_0)\backslash T_0$, $H^\circ = T_0 \times T_0$, then we have $\mathcal{A}_{X^\circ} \to \mathcal{A}_X$. We have $\mathcal{A}_{X^\circ} \cong \mu_2 \times \mu_2$, $\mathcal{A}_X \cong \mu_2 \times \mu_2/\Delta_{\mu_2} \cong \mu_2$. Both spherical roots of X° are distinguished, for X both spherical roots are of type N, and so $\mathrm{Aut}_d^{naive}(X) = 1$. Moreover, we can calculate $\mathrm{Out}_{X^\circ}(H^\circ) \cong \mathrm{Aut}_d(X^\circ)$, and $\mathrm{Out}_X(H) \cong \mu_2 \cong \mathcal{A}_X$, if we take $\mathrm{Aut}_d(X^\circ)/\mathrm{Aut}_d(\pi_0(H))$ as the definition, then we have $\mathrm{Out}_X(H) \cong \mathrm{Aut}_d(X)$.

Definition 6.4. We define the group of doubling automorphism of $X = H\backslash G$ as $\mathrm{Aut}_d(X) := \mathrm{Aut}_d(X^\circ)/\pi_0(H)$, here $X^\circ = H^\circ\backslash G$.

With this definition, we have the following commutative diagram for $\mathrm{Aut}_d(X)$ with exact rows

$$\begin{array}{ccccc} \pi_0(H) & \longrightarrow & \mathcal{A}_{X^\circ} & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Aut}_d(\pi_0(H)) & \longrightarrow & \mathrm{Aut}_d(X^\circ) & \longrightarrow & \mathrm{Aut}_d(X) \end{array}$$

from this we deduce that $\mathrm{Aut}_d(X^\circ) \cong \mathrm{Aut}_d(X)$ if $H^\circ \subset H \subset H^\circ \cdot Z(G)$.

Leisle introduced the following conjecture motivated by the study of G-inner forms

Conjecture 6.5. *Suppose G is a quasisplit reductive k-group and $X = H\backslash G$ is a homogeneous spherical variety then the map*

$$H^1(k, \mathcal{A}_X) \longrightarrow H^1(k, Aut_d(X))$$

is surjective.

7. Geometric cocycle

Let $X = H\backslash G$ be a homogeneous spherical G-variety, we first assume $H^\circ \subset H \subset H^\circ Z(G)$, so we have the k-isomorphism

$$\mathrm{Aut}_d(X) = \prod_{i \in I_X} \mathrm{Res}_{k_i/k}(\mu_2)$$

Recall that we have the short exact sequence

$$1 \longrightarrow \mathcal{A}_X(\bar{k}) \longrightarrow S\mathcal{A}_X \longrightarrow \Gamma$$

here the multiplication on $S\mathcal{A}_X$ is given by $(a \rtimes \sigma)(b \rtimes \tau) = a\sigma(b) \rtimes \sigma\tau$.

We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{A}_X(\bar{k}) & \longrightarrow & S\mathcal{A}_X & \longrightarrow & \Gamma \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{Aut}_d(X)(\bar{k}) & \longrightarrow & S\mathrm{Aut}_d(X) & \longrightarrow & \Gamma \end{array}$$

here $S\mathrm{Aut}_d(X) = S\mathcal{A}_X/\mathcal{A}_X^d(\bar{k})$.

The k-structure of X gives a splitting $\mu_X : \Gamma \to S\mathcal{A}_X$, which descends to $\Gamma \to S\mathrm{Aut}_d(X)$, under the isomorphism (6.1), we can obtain a canonical extension

$$1 \longrightarrow \mu_2(\bar{k})^{\Sigma_X^{dist}} \longrightarrow \mu_2(\bar{k})^{\Sigma_X^{dist}} \rtimes \Gamma \longrightarrow \Gamma$$

we have the following group law on $\mu_2(\bar{k})^{\Sigma_X^{dist}} \rtimes \Gamma$

$$[(\epsilon_\alpha^\sigma)_\alpha, \sigma][(\epsilon_\alpha^\tau)_\alpha, \tau] = [(\epsilon_\alpha^\sigma \epsilon_{\sigma^{-1}(\alpha)}^\tau)_\alpha, \sigma\tau]$$

so we obtain a morphism $\epsilon : S\mathcal{A}_X \to \mu_2(\bar{k})^{\Sigma_X^{dist}} \rtimes \Gamma$ via $\epsilon(s) = [(\mu_\alpha(s))_\alpha, \sigma]$, indeed we have the map $a \rtimes \sigma \mapsto [(\mu_\alpha(a))_\alpha, \sigma]$, and it can be checked that $\epsilon((a \rtimes \sigma)(b \rtimes \tau)) = \epsilon(a\sigma(b) \rtimes \sigma\tau) = [(\mu_\alpha(a\sigma(b)))_\alpha, \sigma\tau] = [\mu_\alpha(a), \sigma][\mu_\alpha(b), \tau] = \epsilon(a \rtimes \sigma)\epsilon(b \rtimes \tau)$. Here we use the identity that

$$\mu_\alpha(a\sigma(b)) = \mu_\alpha(a)\mu_\alpha(\sigma(b)) = \mu_\alpha(a)\mu_{\sigma^{-1}(\alpha)}(b)$$

and the group law on $S\mathcal{A}_X$.

Definition 7.1. Let $\mathcal{X}$ be a $\mathcal{G}$-spherical variety. Given X a k-form of $\mathcal{X}$, this induces a section $\mu_X : \Gamma \to S\mathcal{A}_X$, composing with ϵ, we obtain a section

$$\tilde{\mu}_X^{dist} : \Gamma \longrightarrow S\mathcal{A}_X \longrightarrow \mu_2(\bar{k})^{\Sigma_X^{dist}} \rtimes \Gamma$$

composing with the trivial section $\mu_2(\bar{k})^{\Sigma_X^{dist}} \rtimes \Gamma \to \mu_2(\bar{k})^{\Sigma_X^{dist}}$, we get

$$\mu_X^d : \Gamma \longrightarrow \mu_2(\bar{k})^{\Sigma_X^{dist}} = \prod_{\alpha \in \Sigma_X^{dist}} \{\pm 1\}, \quad \sigma \mapsto \prod_{\alpha \in \Sigma_X^{dist}} \mu_\alpha(\mu_X(\sigma))$$

we will call μ_X^d the geometric cocycle for X.

Here we check that $\mu_X^d(\sigma\tau) = \prod \mu_\alpha(\mu_X(\sigma\tau)) = \prod \mu_\alpha((a \rtimes \sigma)(b \rtimes \tau)) = \prod \mu_\alpha(a\sigma(b)) = \prod \mu_\alpha(a \rtimes \sigma)(\prod \mu_\alpha(b \rtimes \tau))^\sigma = \mu_X^d(\sigma)\mu_X^d(\tau)^\sigma$, hence μ_X^d is indeed a 1-cocycle valued in $\mathrm{Aut}_d(X)$.

The following proposition explains why geometric cocycle is useful in the study of G-inner forms.

Proposition 7.2. *Let G be a quasisplit group over k, X a homogeneous G-spherical variety with $X \cong H\backslash G$ and H geometrically connected, X' a k-form of X, then for the cohomology class $[c_{X',X}] \in H^1(k, Aut^G(X))$ under the map*

$$\psi : \quad H^1(k, Aut^G(X)) \longrightarrow H^1(k, Aut_d(X))$$

we have $\psi([c_{X',X}]) = [\mu_X^d] - [\mu_{X'}^d]$.

Proof. We note that the 1-cocycle $[c_{X',X}]$ valued in $\mathrm{Aut}^G(X)$ is given by $\mu_{X'}/\mu_X : \Gamma \to \mathrm{Aut}^G(X)$ as in (4.1). Hence under the k-map

$$\mathrm{Aut}^G(X) \to \mathrm{Aut}_d(X) \tag{7.1}$$

$$a \mapsto \prod \mu_\alpha(a) \tag{7.2}$$

it gives us a 1-cocycle valued in $\mathrm{Aut}_d(X)$, so we get $\psi([c_{X',X}]) : \Gamma \to \mathrm{Aut}_d(X)$ is given by $\sigma \mapsto \prod \mu_\alpha(\frac{\mu_{X'}(\sigma)}{\mu_X(\sigma)}) = \frac{\mu_{X'}^d(\sigma)}{\mu_X^d(\sigma)} = [\mu_{X'}^d] - [\mu_X^d]$ by definition. ∎

In the general case, we can use the commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_d(X^\circ)(\overline{k}) & \longrightarrow & \mathrm{Aut}_d(X)(\overline{k}) \\ \downarrow & & \downarrow \\ \mu_2(\overline{k})^{\Sigma_{X^\circ}^{dist}} & \longrightarrow & \mu_2(\overline{k})^{\Sigma_{X^\circ}^{dist}}/\mathrm{Aut}_d(\pi_0(H)) \end{array}$$

Definition 7.3. Given two G-forms X and X' of $X_{\overline{k}}$, we say that they are in the same geometric class if their geometric cocycles satisfy $[\mu_X^d] = [\mu_{X'}^d]$ in $H^1(k, \mathrm{Aut}_d(X))$.

We can ask about the existence of the geometric class accociated to each geometric cocycles

Proposition 7.4. *Let $\mathcal{X}$ be a $\mathcal{G}$-spherical variety over $\overline{k}$ and G a quasisplit form of $\mathcal{G}$ over k, when $\pi_0(H)$ is trivial, there exists G-forms X associated to each geometric class as soon as the Γ-action induced from G on $\mathcal{X}$ preserves $\Omega_{\mathcal{X}}$.*

Proof. Indeed, we have a surjecture map $H^1(k, \mathcal{A}_X) \to H^1(k, \mathrm{Aut}_d(X))$, since the Γ-action on $\mathcal{X}$ preserves $\Omega_{\mathcal{X}}$, we have a G-spherical variety X' which is a k-model of $\mathcal{X}$, to any geometric cocycle $[\mu^d] \in H^1(k, \mathrm{Aut}_d(X'))$, we can find $[c] \in H^1(k, \mathrm{Aut}^G(X'))$, and this corresponds to X a k-form of $\mathcal{X}$ which has $[\mu^d]$ as the given geometric cocycle. ∎

Proposition 7.5. *Suppose conjecture 6.5 is true, then there exists G-forms X associated to each geometric class as soon as the Γ-action induced from G on $\mathcal{X}$ preserves $\Omega_{\mathcal{X}}$.*

Proof. The proof is similar to the previous proposition, again we use that $H^1(k, \mathrm{Aut}^G(X)) \to H^1(k, \mathrm{Aut}_d(X))$ is surjective. ∎

Recall that, we have defined the canonical map out : $\mathcal{A}_X \longrightarrow \mathrm{Out}(X)$, and various automorphism groups $\mathcal{A}_X^b$, $\mathcal{A}_X^d$, $\mathcal{A}_X^\#$.

Lemma 7.6. *We have natural inclusions $\mathcal{A}_X^b \subseteq \mathcal{A}_X^d \subseteq \mathcal{A}_X^\#$. In particular, there are surjective morphisms:*

$$Out_X(H) \longrightarrow Aut_d(X) \longrightarrow Aut_\Omega(\mathscr{D}(X))$$

The G-inner forms X' are characterized by that $c_{X,X'}$ being trivial in $H^1(k, \mathrm{Out}(X))$, and by definition X and X' are in the geometric class if $[\mu_X^d] = [\mu_{X'}^d]$ in $H^1(k, \mathrm{Aut}_d(X))$, the following class of spherical varieties is a natural class to study

Definition 7.7. Suppose G is a quasisplit group and X a homogeneous spherical G-variety, we say X is *well-adapted* if the natural morphism $\mathrm{Out}(X) \longrightarrow \mathrm{Aut}_d(X)$ is an isomorphism.

The following result is proven in Leisle's paper which characterizes the classes of G-inner form of X using Ω_X and $[\mu_X^d]$ the geometric cocycle.

Proposition 7.8. *If $X_1 = G/H_1$ and $X_2 = G/H_2$ are two well-adapted homogeneous spherical varieties for G with G quasisplit, then we have*

$$(\Omega_{X_1}, [\mu^d_{X_1}]) = (\Omega_{X_2}, [\mu^d_{X_2}])$$

if and only if X_2 is a G-inner form of X_1.

Proof. If X_2 is a G-inner form of X_1, then $(X_2)_{\overline{k}}$ is $G_{\overline{k}}$-isomorphic to $(X_1)_{\overline{k}}$, hence we have $\Omega_{X_1} = \Omega_{X_2}$, also from definition $0 = [c_{X_1,X_2}] = [\mu^d_{X_1}] - [\mu^d_{X_2}]$ in $H^1(k, \mathrm{Out}_{X_1}(H_1)) \cong H^1(k, \mathrm{Aut}_d(X_1))$.

If $(\Omega_{X_1}, [\mu^d_{X_1}]) = (\Omega_{X_2}, [\mu^d_{X_2}])$, from $\Omega_{X_1} = \Omega_{X_2}$, we know that X_1 and X_2 are $G_{\overline{k}}$ isomorphic, and from proposition 7.2 $[c_{X_1,X_2}] = [\mu^d_{X_1}] - [\mu^d_{X_2}] = 0$ in $H^1(k, \mathrm{Out}_{X_1}(H_1)) \cong H^1(k, \mathrm{Aut}_d(X_1))$, hence X_2 is a G-inner form of X_1. □

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