GENERIC REPRESENTATIONS FOR UNITARY GROUPS

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1. Introduction

This is a study note for the [GJR01] paper where they proved for the quasisplit unitary group in three variables, every tempered packet of cuspidal automorphic representation contains a globally generic representation.

2. Global and local Bessel distributions

Suppose that G is a reductive group defined and quasi-split over a number field F, let N be the maximal unipotent subgroup in G and θ a character of $N(F_{\mathbb{A}})$ trivial on $N(F)$ and generic. Let π be a cuspidal automorphic representation of $G(F_{\mathbb{A}})$ and V the isotypic component of the space of cusp forms corresponding to π, if ϕ is a smooth vector in V, we set

$$\mathcal{W}(\phi) := \int_{N(F)\backslash N(F_{\mathbb{A}})} \phi(n)\overline{\theta}(n)\ dn$$

we say that π is **globally generic** with respect to θ if the linear form $\mathcal{W}$ is not identically zero. If $\mathcal{W}$ is non-zero, then for each place v, π_v is generic with respect to θ_v.

We may decompose $V = V_1 \oplus V_0$ with V_1 the closure of the kernel of the map $\phi \mapsto W_\phi$. The representation on V_0 is irreducible. It is generally conjectured that inside a tempered L-packet of automorphic representations, there is exactly one component which is generic with respect to the given generic character θ.

For π as above, we define a global Bessel distribution as

$$J_\pi(f) := \sum_{\phi} \mathcal{W}(\rho(f)\phi)\overline{\mathcal{W}(\phi)}$$

here the sum is over an orthogonal basis of V_0.

At a place v, we may define the local Bessel distribution as

$$\mathcal{B}_v(f_v) := \sum_{\phi} \mathcal{W}_v(\pi_v(f_v)\phi)\overline{\mathcal{W}_v(\phi)}$$

this distribution is defined up to a positive factor.

It follows from the local uniqueness that the global Bessel distribution decomposes as in infinite product of the local Bessel distributions

$$J_\pi(f) = C\prod_{v} \mathcal{B}_v(f_v) \text{ for } f = \otimes f_v$$

the question at hand is to compute the constant in terms of L-functions attached to π.

Let E/F be a quadratic extension of number fields with Galois group $\{1, \sigma\}$, we write $\sigma(z) = \overline{z}$, we let U_1 the unitary group in one variable, we denote by G the group $GL(3)$ regarded as an algebraic group over E and we denote Z its center. We let H be the group $GL_3(F)$ regarded as an algebraic group over F and we denote by Z_H its center. We say that a Π is distinguished by $H(F_{\mathbb{A}})$ if the central character ω of Π is trivial on $F_{\mathbb{A}}^\times$ and there is a form ϕ in the space of Π such that

$$\mathcal{I}(\phi) := \int_{Z_H(\mathbb{A})H(F)\backslash H(F_{\mathbb{A}})} \phi(h)\ dh \neq 0$$

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It is a result of Flicker that Π is distinguished if and only if the Asai L-function attached to Π has a pole at $s=1$. It is another result of Flicker that if Π is distinguished, then $\Pi^\sigma = \tilde{\Pi}$. The condition of symmetry is equivalent to Π being the standard base change of a tempered stable packet of automorphic representation of the unitary group. This fact is predicted by the analysis of the potential pole of the Asai L-function in terms of the L-group.

Our main tool is the relative trace formula, the relative trace formula used here was first discussed in [Y] in the context of GL_2.

3. Sketech of the relative trace formula

If f is a smooth function of compact support on $GL_3(E_{\mathbb{A}})$ we define a kernel

$$K_f(x,y) = \int_{Z(E_{\mathbb{A}})/Z(E)} \Big(\sum_{\xi \in GL_3(E)} f(x^{-1}\xi y z)\omega(z)\Big)\ dz$$

our main object of study is the distribution

$$J(f) := \int_{H(F)Z_H(F_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} \int_{N(E)\backslash N(E_{\mathbb{A}})} K_f(h,n)\ \theta(n)\ dndh$$

this distribution can be computed in terms of the symmetric space

$$\mathfrak{S} = \{ s \in GL_3(E)\ :\ s\overline{s} = 1 \}$$

There is a smooth function of compact support Φ on $\mathfrak{S}(F_{\mathbb{A}})$ such that

$$\Phi(\mathcal{P}(g)) = \int_{H(F_{\mathbb{A}})} f(hg)\ dh$$

then

$$J(f) = \int_{N(E)\backslash N(E_{\mathbb{A}})} \int_{U_1(F)\backslash U_1(F_{\mathbb{A}})} \Big(\sum_{\xi \in \mathfrak{S}(F)} \Phi(\overline{n}^{-1}\xi n u)\Big)\theta(n) dn\ \zeta(u)\ du$$

then

$$J(f) = \sum_{\alpha \in E^\times} \int_{U_1(F_{\mathbb{A}})} J(ud_\alpha, \Phi)\zeta(u)\ du$$

Likewise, we can consider the unitary group U for the Hermitian matrix ω, the group is defined by $\overline{g}^t \omega g = \omega$. We denote A' the group of diagonal matrices, by N' the group of upper triangular matrices with unit diagonal in U and by Z_U the center of U. Let f' be a smooth function of compact support on $U(F_{\mathbb{A}})$, we construct a kernel

$$K'_{f'}(x,y) = \int_{Z_U(F_{\mathbb{A}})/Z_U(F)} \sum_{\xi \in U(F)} f'(x^{-1}\xi y z)\zeta(z)\ dz$$

and the distribution

$$J'(f') := \int K_{f'}(n_1,n_2)\theta'(n_1)\theta'(n_2) dn_1\ dn_2$$

then

$$J'(f') = \sum_{\alpha} J'(ud_\alpha, f')\zeta(u)\ du$$

After a matching of local orbital integrals, we have

$$J(f) = J'(f')$$

As usual, we may decompose the kernel K and K' with respect to the cuspidal data, for the cuspidal data χ for G and the cuspidal data χ' for U, we obtain a kernel K_χ and $K'_{\chi'}$, we define a corresponding distribution $J_\chi(f)$ (resp. $J_{\chi'}(f')$).

Let χ be a cuspidal representation with central character ω, then $J_\chi(f) = 0$ unless χ is distinguished by H. Now if χ is distinguished, it follows that we have an identity

$$J_\chi(f) = \sum_{\chi'} J_{\chi'}(f)$$

where the sum on the right is over all χ' such that the unramified representation $\otimes_{v\notin S}\chi_v$ is the image of the unramified representation $\otimes_{v_0\notin S_0}\chi'_{v_0}$ under the standard base change. From the formula, it follows that there is at least one χ' such that $J_{\chi'}\neq 0$. This implies χ' is globally generic.

A natural question is to ask whether the sum on the right has only one term, thanks to the local result, we can prove this and we have an identity

$$J_\chi(f)=J_{\chi'}(f')$$

4. Factorization of the period

If Π is distinguished we will need to express the linear form

$$\mathcal{J}(\phi):=\int_{Z_H(F_{\mathbb{A}})H(F)\backslash H(F_{\mathbb{A}})}\phi(h)\ dh$$

on the space of Π as a product of local linear forms.

Let Φ be a Schwartz-Bruhat function on $F_{\mathbb{A}}^n$ and ϕ a vector in Π, set

$$W(g):=\int\ \phi(ng)\overline{\theta}(n)\ dn$$

we set

$$\Psi(s,W,\Phi):=\int_{N(F_{\mathbb{A}})\backslash H(F_{\mathbb{A}})}W(\epsilon h)\phi[(0,0,\cdots,1)h]|\det\ h|^s\ dh$$

we assume that W and Φ are product of local functions. We choose S_0 sufficiently large so that outside S the function W_v is K_v-invariant and outside S_0, Φ_{v_0} is the characteristic function of $\mathcal{O}_{v_0}^n$.

Let S_i be the set of places in S_0 which are inert in E and let S_s be the set of places in S_0 which split in E, from the theory of Zeta integrals, we have

$$\begin{aligned}\int\phi(h)(\sum_{\xi\in F^n-0}\Phi(\xi h))\ |\det\ h|_F^s\ dh&=\Phi(s,W,\Phi)\\ &=L^{S_0}(s,\Pi,\mathrm{Asai})\times\prod_{v_0\in S_i}\Psi(s,W_v,\Phi_{v_0})\prod_{v_0\in S_s}\Psi(s,W_{v_1},W_{v_2},\Phi_{v_0})\end{aligned}$$

Taking residue at $s=1$, we get

$$\begin{aligned}c\mathcal{J}(\phi)\int\Phi(x)\ dx=&\ \mathrm{Res}_{s=1}L^{S_0}(s,\Pi,\mathrm{Asai})\\ &\times\prod_{v_0\in S_i}\Psi(1,W_v,\Phi_{v_0})\prod_{v_0\in S_s}\Psi(1,W_{v_1},W_{v_2},\Phi_{v_0})\end{aligned}$$

For a suitable constant c', we get

$$c'\mathcal{J}(\phi)=\mathrm{Res}_{s=1}\ L^{S_0}(s,\Pi,\mathrm{Asai})\prod_{v_0\in S_i}\mathcal{J}_{v_0}(W_v)\prod_{v_0\in S_s}\mathcal{J}_{v_0}(W_{v_1}\otimes W_{v_2})$$

Definition 4.1. We define the relative global Bessel distribution $\mathcal{J}_\Pi$ as follows: if f is a smooth function of compact support, K-finite on both sides, then we set

$$\mathcal{J}_\Pi(f)=\sum_i\ \mathcal{J}(\pi(f)v_i)\overline{\mathcal{W}(v_i)}$$

Definition 4.2. For $v_0\in S_i$, we introduce a local relative Bessel distribution $\mathcal{B}_{v_0}$, if f_{v_0} is K_{v_0}-finite, it is then given by

$$\mathcal{B}_{v_0}(f)=\sum_\phi\ \mathcal{J}_{v_0}(\Pi_{v_0}(f)\phi)\overline{\mathcal{W}_{v_0}(\phi)}$$

Assume $f=f_Sf^S$, we then find that the global distribution $\mathcal{J}_\Pi(f)$ can be written as

$$\mathcal{J}_\Pi(f)=c(\Pi)\prod_{v_0\in S_0}\ \mathcal{B}_{v_0}(f_{v_0})$$

5. The discrete part of the trace formula

For any smooth function of compact support, we have

$$J(f) = \sum_{\text{discrete}} J_\chi(f) + \sum_{\mu_1} J_{\mu_1}(f)$$

In the first sum, χ appears if it is represented by a pair (M, π) where π is a distinguished representation of M. Explicitly, the possibilities are $M = G$, M is of type $(2,1)$ and $M = A$.

6. Conclusion

We are now ready to prove the main result

Theorem 6.1. *Any stable tempered packet of cuspidal representations of U contains a globally generic representation.*

Proof. The packet has for base change a single irreducible cuspidal representation Π of $GL(3, E_{\mathbb{A}})$. By the results discussed in the second section the representation Π is distinguished. If f and f' have matching integrals, the absolute convergence of the relative trace formula allows us to write

$$J_\Pi(f) = \sum_\pi J'_\pi(f')$$

where the sum is over all the members of the packet. Since the distribution $J_\Pi(f)$ is a product of local relative Bessel distributions, we can choose a function f satisfying the simplifying assumption such that the left handside is nonzero, it follows that the right hand side is nonzero and there is at least one π such that $J'_\pi(f') \neq 0$, such a π is generic. □

Remark 6.2. The same proof applies to the case of an endoscopic packet by considering the discrete, non-cuspidal part of the trace formula.

We can also prove a local result

Theorem 6.3. *Let E/F be a local quadratic extension, any tempered L-packet of $U(F)$ contains exactly one generic component.*

We have the following corollary

Theorem 6.4. *Suppose that π is a cuspidal autormophic representation of $U(F_{\mathbb{A}})$ such that π_{v_0} is generic for every place v_0, then π is globally generic.*

As a consequence, we get

$$J_\Pi(f) = J'_\pi(f')$$

in the previous equality.

References

[GJR01] Stephen Gelbart, Hervé Jacquet, and Jonathan Rogawski. Generic representations for the unitary group in three variables. *Israel Journal of Mathematics*, 126(1):173–237, 2001.