PERIODS OF EISENSTEIN SERIES-GALOIS CASE

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1. Introduction

This is a study note on the paper [LR03] on regularized periods of Eisenstein series in the Galois case, I am mainly interested in generalizing their local functional equation to more general situation.

Let G be a connected reductive group over a number field F, and let H be the fixed subgroup of an involution, let ϕ be an automorphic form on $G(\mathbb{A})$, if ϕ is a cusp form, then the following integral converges and is called the *period integral* of ϕ relative to H

$$P^H(\phi) = \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \phi(h)\ dh$$

we say that ϕ is *distinguished* by H if $P^H(\phi) \neq 0$ and a cuspidal representation (π, V) is distinguished by H if there exists a $\phi \in V$ distinguished by H. It is of interest to define $P^H(\phi)$ via regularization for general automorphic forms for which the integral may not converge. For example, the regularized periods of Eisenstein series appear in the contribution of the continuous spectrum of Jacquet's relative trace formula and the relative trace formula is a key tool for characterizing distinguished cuspidal representations in terms of Langlands functoriality. The regularized periods are also related to L-functions in many case.

This paper continues the study of regularized period integrals initiated in [JLR99] in a more general context. We consider the Galois pairs (G, θ), that is $G = \mathrm{Res}_{E/F} H_1$, where H_1 is a connected reductive group over F, E/F is a quadratic extension and θ is induced by the Galois involution of E/F. The pair (G, θ) is said *relatively quasi-split* if θ stabilizes a minimal parabolic subgroup P_0 of G, we assume throughout the paper that (G, θ) is relatively quasi-split. The symmetric space attached to θ is the variety

$$\mathscr{C}' = \{\epsilon \in G :\ \theta(\epsilon)\epsilon = 1\}$$

Let $H_\epsilon = \mathrm{Stab}_G(\epsilon)$ be the stabilizer of ϵ, H_ϵ is a inner form of H_1, we consider periods relative to all of the subgroups H_ϵ.

In [JLR99], the regularization of the period integral was introduced for the Galois periods such that H is split, and an explicit formula for the periods of cuspidal Eisenstein series was obtained in the case $G = GL_{n,E}$ and $H = GL_{n,F}$. Our goal is to extend these two results to the groups G and H_ϵ, it is relatively straightforward to adapt the regularization procedure in this setting. However, the combinatorics and analysis involved in computing the regularized periods of Eisenstein series are more complicated because some key simplifying features of GL_n no longer apply. We rely here a detailed study of the double coset spaces $P(F)\backslash G(F)/H_\epsilon(F)$ for P contains P_0 and of twisted conjugation relative to θ in the Weyl group of G.

Let us fix $\epsilon_0 \in \mathscr{C}'$ and let us set $\tilde{H} = H_{\epsilon_0}$, the coset space $P\backslash G/\tilde{H}$ is identified with the set of P-orbits contained in the G-orbit $G * \epsilon_0$ in $\mathscr{C}'$. It is natural to consider the set of all P-orbits in $\mathscr{C}'$, and suppose that ϵ belongs to a Bruhat cell $P_0 n P_0$ where n lies above the element ξ in the Weyl group W_G, then ξ is necessarily a *twisted involution* that is, it satisfies $\theta(\xi) = \xi^{-1}$. Let $\mathcal{J}_0(\theta)$ be the set of twisted involutions in W_G. Springer introduced a map

$$\iota_0 :\ P_0\backslash\mathscr{C}' \to \mathcal{J}_0(\theta)$$

induced by the map sending ϵ to the twisted involution ξ indexing the Bruhat cell containing ϵ. More generally, there is a map

$$\iota_M :\ P\backslash\mathscr{C}' \longrightarrow W_M\backslash W_G/W_{\theta M}$$

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we identify $W_M\backslash W_G/W_{\theta M}$ with the set of reduced representatives and we say that a P-orbit $\mathcal{O}$ lies above ξ if $\iota_M(\mathcal{O}) = \xi$. We call ξ admissible if $\xi\theta(M)\xi^{-1} = M$ and $\mathcal{O}$ admissible if $\iota_M(\mathcal{O})$ is admissible. The set of admissible twisted involutions is denoted by $\mathcal{J}_M(\theta)$, any admissible P-orbit above ξ intersects $M\xi$ in at least one point x and this gives rise to a Galois pair (M, θ_x).

Consider a cuspidal Eisenstein series $E(g, \varphi, \lambda)$ induced from a parabolic subgroup $P = MU$ of G, in the range of absolute convergence, this is given by

$$E(g, \varphi, \lambda) = \sum_{\delta \in P\backslash G} \varphi(\delta g) e^{\langle \lambda, H_M(\delta g)\rangle}$$

where φ is a suitable section. A formal computation of the period $P^{\tilde{H}}(E(\varphi, \lambda))$ gives

$$\int_{\tilde{H}\backslash \tilde{H}(\mathbb{A})^1} E(h, \varphi, \lambda)\, dh \text{ }'='\sum_{\eta} \int_{\tilde{H}^P_\eta\backslash \tilde{H}(\mathbb{A})^1} \varphi(\eta h) e^{\langle \lambda, H_M(\eta h)\rangle}\, dh \tag{1.1}$$

where $\{\eta\}$ is a set of representatives for the double cosets $P\backslash G/\tilde{H}$ and $\tilde{H}^P_\eta = \eta^{-1}P\eta \cap H$. Neither side of this equation converges, but the left hand side is defined by regularization, on the other hand, many of the terms on the right hand side are equal to zero. This is true for the nonadmissible cosets for cuspidality reasons. According to our main result, an admissible coset contributes only if it lies over an admissible twisted involution ξ such that $\xi\theta\alpha = -\alpha$ for all simple roots $\alpha \in \Delta_M$. In particular, if there does not exist any admissible ξ then $P^{\tilde{H}}(E(\varphi, \lambda))$ vanishes.

The terms on the right hand side of (1.1) are closely related to the *intertwining periods*. Fix an admissible twisted involution $\xi \in \mathcal{J}_M(\theta)$, let $\eta \in G$ and suppose that $x = \eta * \epsilon_0$ lies above ξ and satisfies $\theta_x(M) = M$, the intertwining period attached to η is defined as

$$j(\eta, \varphi, \lambda) = \int_{\tilde{H}^P_\eta(\mathbb{A})\backslash \tilde{H}(\mathbb{A})} \int_{M_x\backslash M_x(\mathbb{A}_F)^1} \varphi(m\eta h)\, dm\, e^{\langle \lambda, H_M(\eta h)\rangle}\, dh$$

We show that it converges for λ in a suitable conce and we set

$$J(\xi, \varphi, \lambda) = \sum_{\iota_M(\eta)=\xi} j(\eta, \varphi, \lambda)$$

the proof of convergence of j and J relies on a series of reduction steps. The convergence of $J(\xi, \varphi, \lambda)$ is easier to established when $G = GL_{n,E}$ and $\tilde{H} = GL_{n,F}$, in this case, the sum over η reduces to a single term and the integral has a majorant that can be transformed into a standard zeta integral whose convergence properties are well known. In fact $J(\xi, \varphi, \lambda)$ itself can be expressed as a ratio of Asai L-functions. In general, the sum over η is infinite and J is not a factorizable distribution. Therefore it cannot be expressed directly as a ratio of L-functions. It is likely, however J can be written as a finite sum of factorizable distributions whose local factors are related to ratios of L-functions. This was shown in the case $G = GL_{3,E}$ and $\tilde{H} = U(3)$ by applying a stabilization procedure to the sum over η and using the Jacquet-Ye comparison of the relative trace formula for G relative to $\tilde{H}$ with the Kunznetsov trace formula for $GL_{3,F}$.

We now state the main result of the paper

Theorem 1.1. *Let $E(\varphi, \lambda)$ be a cuspidal Eisenstein series induced from a parabolic subgroup P with Levi subgroup M, let ξ be the longest element in $W(\theta(M))$, if $P^{\tilde{H}}(E(\varphi, \lambda))$ is nonzero, then the following conditions are satisifed*

- *$\xi\theta(M) = -M$ and $\xi\theta\alpha = -\alpha$ for all $\alpha \in \Delta_M$.*
- *there exists an element $x \in G * \epsilon_0$ with $\theta_x(M) = M$ lying above ξ such that for some $g \in G(\mathbb{A})$, the cusp form $m \to \varphi(mg)$ on $M(\mathbb{A})$ is distinguished by M_x.*

Under these conditions,

$$P^{\tilde{H}}(E(\varphi, \lambda)) = J(\xi, \varphi, \lambda)$$

There is a regularization process and it is based on a *mixed truncation operator* Λ^T_m defined on automorphic forms ϕ on $G(\mathbb{A})$. The mixed truncation is a variant of Arthur's truncation operator Λ^T that is well adapted to the problem of computing periods. For a sufficiently regular truncation parameter T, $\Lambda_m^T\phi$ is rapidly decreasing on $\tilde{H}(\mathbb{A})$ and the regularized period can be defined in terms of the convergent integral

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} \Lambda_m^T\phi(h)\ dh$$

this integral is a polynomial exponential as a function of T. This is equal to $\sum p_\lambda(T)e^{\langle\lambda,T\rangle}$. Let $\mathscr{A}_0(G)$ be the subspace of automorphic forms such that the polynomial $p_0(T)$ corresponding to the zero exponent has degree at most k. A cuspidal Eisenstein series at a generic parameter lies in $\mathscr{A}_0(G)$ and for them $p_0(T)$ is a constant whose value we take as the regularized period.

Our results play a role in analyzing the continuous contribution to the relative trace formula, as mentioned above, the relative trace formula is a tool for invesgating distinguished representations, one would like to know for example, that the distinguisehd represnetations arise via base change or some other functorial transfer from an auxiliary group G'.

2. Galois pairs

Let F be a characteristic zero field, let E/F be a quadratic extension, and let $G \cong \mathrm{Res}_{E/F}\ H_E$, more precisely, G is obtained by restriction of scalars of the E-points of H. Let θ be the involution of G defined by Galois conjugation. It is defined over F. We say that (G,θ) is a Galois pair with respect to E/F.

The symmetric space attached to (G,θ) is the variety

$$\mathscr{C}' = \{\epsilon \in G\ :\ \theta(\epsilon) = \epsilon^{-1}\}$$

the natural action of G on $\mathcal{C}'$ is denoted by $*$

$$g * x = gx\theta(g)^{-1}$$

Definition 2.1. The Galois pair (G,θ) is called *relatively quasisplit* if θ stabilizes a minimal parabolic P_0 of G.

We assume that (G,θ) is relatively quasisplit. In this case, P_0 admits a θ-stable Levi decomposition $P_0 = M_0U_0$.

3. Twisted involutions

In this section, we consider an involution σ acting on the vector space $\mathfrak{a}_0$ and preserving the set of simple roots Δ_0. Any involution of G fixing a minimal pair (P_0, M_0) induces such an action.

The Weyl group W acts on $J_0(\sigma)$ by twisted conjugation, we denote this action by $*$ so that

$$\omega * \xi = \omega\xi\sigma(\omega)^{-1}$$

Definition 3.1. Suppose that $D \in W_M\backslash W/W_{\sigma M}$ satisfies $\sigma(D) = D^{-1}$, and let ξ be the reduced representative for D, then D or ξ is called an *admissible twisted involution* if $\xi \in W(\sigma M, M)$, that if $\xi\sigma(M) = M$, the set of admissible twisted involutions is denoted by $\mathcal{J}_M(\sigma)$.

Definition 3.2. An admissible twisted involution $\xi \in \mathcal{J}_M(\sigma)$ is called *minimal* if there exists a σ-stable Levi subgroup L with $M \subset L$ such that $\xi = \omega_{\sigma M}^L$ and $\xi\sigma\alpha = -\alpha$ for all $\alpha \in \Delta_M^L$. In this case, L is uniquely determined by ξ and is denoted by $L_{\xi,\sigma}$. Let $\chi_M(\sigma)$ denote the set of minimal twisted involutions in $\mathcal{J}_M(\sigma)$.

We can define a weighted directed graph $\mathfrak{G}$ to an associate class $\mathcal{M}$ of Levi subgroups. The vertices of $\mathfrak{G}$ are the pairs (M,ξ) with $M \in \mathcal{M}$ and $\xi \in \mathcal{J}_M(\sigma)$. The set of edges connecting $(M_1,\xi_1),\ (M_2,\xi_2)$ is the set of $\alpha \in \Delta_{M_1}$ such that $s_\alpha M_1 s_\alpha^{-1} = M_2,\ \xi_2 = s_\alpha * \xi_1$ and $\xi_1(\sigma\alpha) \neq \alpha$. Let $\mathfrak{G}^0$ be the subgraph with the same set of vertices but where we retain only those edges for which $\xi_1\sigma(\alpha) \neq \pm\alpha$.

The following is a useful characterization of the sets $W(\xi,\xi')$ and $W^0(\xi,\xi')$. To each path

$$\xi_1 \xrightarrow{\alpha_1} \xi_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \xi_n$$

Let $W(\xi_1,\xi_2)$ be the set of words defined by paths from ξ_1 to ξ_2. Let $W^0(\xi_1,\xi_2)$ be the set of words arising from paths contained in $\mathfrak{G}^0$.

Proposition 3.3. *Every $\xi \in \mathcal{J}_M(\sigma)$ is twisted conjugate to a minimal twisted involution. In fact, there exists $\xi' \in \chi_{M'}(\sigma)$ such that $\omega * \xi = \xi'$ with $\omega \in W^0(\xi, \xi')$.*

Proposition 3.4. *With the above notation and assumptions, we have*

- *$\omega \in W(M')$ and $M_1' = \omega M' \omega^{-1}$ is a Levi subgroup of L.*
- *$\xi_1' = \omega * \xi'$ belongs to $\mathcal{J}_{M_1'}(\sigma)$.*
- *$s' \in W_L(M_1)$.*
- *ω belongs to $W^0(\xi', \xi_1')$.*

thus we have a commutative diagram

$$\begin{array}{ccc} \xi & \xrightarrow{\omega} & \xi_1 \\ \downarrow{\scriptstyle s_\alpha} & & \downarrow{\scriptstyle s'} \\ \xi' & \xrightarrow{\omega} & \xi_1' \end{array}$$

4. Double cosets

We assume that (G, θ) is relatively quasisplit. Our goal in this section is to analyze the P_0-orbits in $\mathscr{C}'$.

If $\epsilon \in \mathscr{C}'$, then the P_0-orbit $\mathcal{O} = P_0 * \epsilon$ is contained in the double coset $P_0 \epsilon P_0$, we define a map

$$\iota_0 : \ P_0 \backslash \mathscr{C}' \to W$$

by sending $\mathcal{O} = P_0 * \epsilon$ to the element $\xi \in W$ corresponding to $P_0 \epsilon P_0$ in the Bruhat decomposition. If $\iota_0(\mathcal{O}) = \xi$, then we say $\mathcal{O}$ lies above ξ.

Lemma 4.1. *Each P_0-orbit in $\mathscr{C}'$ intersects $N_G(T_0)$.*

Proposition 4.2. *The map*

$$\mathcal{O} \to \mathcal{O} \cap N_G(T_0)$$

defines a bijection between the set of P_0-orbits in $\mathscr{C}'$ and the set of M_0-orbits in $\mathscr{C}' \cap N_G(T_0)$.

Proposition 4.3. *The image of ι_0 is all of $\mathcal{J}_0(\theta)$.*

5. Intertwining periods

5.1. **Definition of intertwining periods.** These functionals play a role in the relative trace forula analogous to that of the intertwining operators in the ordinary trace formula. Let $\varphi \in \mathscr{A}_P^1(G)$, recall $\tilde{H}$ is the stabilizer of a point $\epsilon_0 \in \mathscr{C}'$ in G.

Recall that the admissible P-orbits of $\mathscr{C}'$ above ξ are in one-to-one correspondence with the M-orbits in $\mathscr{C}' \cap M\xi$. Let $\mathcal{O}$ be an M-orbit in $\mathscr{C}' \cap M\xi$, choose any $x \in \mathcal{O}$ and a Haar measure on $M_x(\mathbb{A})^1$, then the period integral

$$P^{M_x}(\varphi)(g) = \int_{M_x \backslash M_x(\mathbb{A})^1} \varphi(mg)\ dm$$

is well defined. Let η be chosen so that $x = \eta * \epsilon_0$, the intertwining period is defined by

$$j(\mathcal{O}, \varphi, \lambda) = \int_{\tilde{H}_\eta^P(\mathbb{A}) \backslash \tilde{H}(\mathbb{A})} P^{M_x}(\varphi)(\eta h) e^{\langle \lambda, H_M(\eta h) \rangle}\ dh$$

We set

$$J(\xi, \varphi, \lambda) = \sum_{\mathcal{O} \subset \mathscr{C} \cap M\xi} j(\mathcal{O}, \varphi, \lambda)$$

6. Functional equations

We have the simple functional equations

Proposition 6.1. *Let $\xi \in \mathcal{J}_M(\theta)$, let $\mathcal{O} \subset \mathscr{C} \cap M\xi$, and let $\alpha \in \Delta_M$ with $-\alpha \neq \xi(\theta\alpha) < 0$, then for $\varphi \in \mathscr{A}_P^1(G)$ and $\lambda \in \mathcal{D}_\xi$, we have*

$$J(s_\alpha * \xi, M(s_\alpha, \lambda)\varphi, s_\alpha\lambda) = J(\xi, \varphi, \lambda)$$

this is because the intertwining operators for $s_\alpha * \mathcal{O}$ and $\mathcal{O}$ differ by a unipotent integral which is given by a standard intertwining operator.

Theorem 6.2. *Let $\xi \in \mathcal{J}_M(\theta)$ and let $\varphi \in \mathscr{A}_P^1(G)_c$, then*

- *$J(\xi, \varphi, \lambda)$ extends to a meromorphic function on $((\mathfrak{a}_{M,\mathbb{C}}^G)^*)_{\xi\theta}^-$.*
- *If $\xi' \in \mathcal{J}_{M'}(\theta)$ and $s \in W(\xi, \xi')$, we have*

$$J(\xi', M(s, \lambda)\varphi, s\lambda) = J(\xi, \varphi, \lambda)$$

Proof. From the previous proposition 6.1, it remains to prove the functional equations in the case $\xi(\theta\alpha) = -\alpha$. Let the notation as in proposition 3.4.3, we have the commutative diagram

$$\begin{array}{ccc} \xi & \xrightarrow{\omega} & \xi_1 \\ \downarrow{\scriptstyle s_\alpha} & & \downarrow{\scriptstyle s'} \\ \xi' & \xrightarrow{\omega} & \xi_1' \end{array}$$

Since $s' \in W_{L_{\xi_1,\theta}}(M_1)$, we get

$$J(\xi_1, \varphi', \lambda') = J(s' * \xi_1, M(s', \lambda')\varphi', s'\lambda')$$

together with the functional equation for $\omega \in W^0(\xi, \xi_1)$, we get

$$J(\xi, \varphi, \lambda) = J(\xi_1, M(\omega, \lambda)\varphi, \omega\lambda) = J(s' * \xi_1, M(s'\omega, \lambda)\varphi, s'\omega\lambda)$$

Also by proposition 3.4.3, $\omega \in W^0(\xi', \xi_1')$, so that

$$J(\xi', M(s_\alpha, \lambda)\varphi, s_\alpha\lambda) = J(\xi_1', M(\omega s_\alpha, \lambda)\varphi, \omega s_\alpha\lambda)$$

all together, we get

$$J(\xi', M(s, \lambda)\varphi, s\lambda) = J(\xi, \varphi, \lambda)$$

□

References

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