FOURIER TRANSFORM ON BASIC AFFINE SPACE

1. Introduction

This is a study note for Gourvich-Kazhdan papers on the construction of Fourier transform on basic affine space for quasisplit groups, [GK23], [GK19]. Their main result is

Theorem 1.1. *Let G be a quasisplit group, $B = TU$ the Borel subgroup, then there exists unique family of unitary operators $\Phi_w,\ w \in W$ on $L^2(X, \omega_X)$ preserves $S_0(X)$ and satisfying*

- $\Phi_\omega \circ \theta(g, t^\omega) = \theta(g, t) \circ \Phi_\omega$, *for all* $\omega \in W,\ t \in T,\ g \in G$.
- $\Phi_{\omega_1}\Phi_{\omega_2} = \Phi_{\omega_1\omega_2}$.
- $\kappa_\Psi(\Phi_\omega)(\varphi) = \omega \cdot \varphi$ *for all* $\omega \in W$, $\varphi \in S_0(T)$.

The space $S_0(X)$ and the Fourier transform Φ_{ω_α} for ω_α a simple reflection are defined for the rank one groups and then for the general groups.

2. Minimal representation of $O(8)$

We review the minimal representation of $O(8)$ and its relation with the Fourier transform on the basic affine space for SU_3.

Let F be a non-archimedean local field, with $\mathcal{O}_F$ ring of integers. Let (V, q) be a non-degenerate quadratic space of dimension $2n+2$ $n \geq 3$ and the Witt index at least n over F. We fix the decomposition $V = \mathbb{H} \oplus V_1$ where $\mathbb{H}$ is a hyperbolic plane and V_1 is a non-degenerate quadratic space of dimension $2n$. The group $G = O(V, q)$ is quasisplit and it contains a maximal parabolic subgroup $Q = MN$, where M is canonically isomorphic to $GL_1 \times O(V_1)$, we denote $\langle \cdot, \cdot \rangle :\ N \times \overline{N} \to F$ the non-degenerate pairing given by the Killing form, fix a basis $\{e, e^*\}$ in $\mathbb{H}$ of isotropic vectors such that $\langle e, e^* \rangle = 1$ and N fixes the vector e^*. This gives an isomorphism $\varphi : \overline{N} \cong V_1,\ \varphi(\overline{n}) = \mathrm{proj}_{V_1}(\overline{n}e^*)$, where $\overline{N}$ is the radical of the opposite parabolic $\overline{Q}$.

Denote by $C \subset \overline{N}$ the cone of isotropic vectors in V_1, the group $G_1 = O(V_1)$ acts transitively on $C_0 = C \backslash \{0\}$, we define an action of Q on the space $S^\infty(C_0)$ of smooth functions on C_0 by

$$(a, h) \cdot f(\omega) = \chi_K(a)|a|^{n-1} f(a\omega h)\ a \in GL_1,\ h \in O(V_1)$$
$$n \cdot f(\omega) = \psi(-\langle n, \omega \rangle) f(\omega)\ n \in N$$

The normalized principal series $\mathrm{Ind}_Q^G \chi_K | \cdot |^{-1}$ contains a unique irreducible representation, we denote by Π, we can show Π is a minimal representation of G. The pairing

$$\langle \cdot, \cdot \rangle :\ \overline{I}_{2n+2}(\chi_K | \cdot |) \times \overline{I}_{2n+2}(\chi_K | \cdot |^{-1}) \to \mathbb{C}$$

reduces to a non-degenerate pairing $\Pi \times S_c$. This implies that Π is embedded into a $\mathbb{Q}$-smooth dual of S_c that can be described as a space S of smooth functions.

Theorem 2.1. *We have the followings hold*

- *There exists a Q equivariant embedding $\Pi \hookrightarrow S^\infty(C_0)$, we denote the image by S.*
- *The space S contains the space $S_c(C_0)$ of smooth functions of compact support and is contained in the space of smooth functions of bounded support.*

Let $Y \subset C_0 \subset V_1$, for each $\omega \in C_0$ we can define a map

$$\varphi_\omega :\ C_0 \longrightarrow F,\ \varphi_\omega(v) = \langle v, \omega \rangle$$

and let $Y_\omega(t)$ be the fiber over t. For any $\omega \in C_0$, we can consider a family of distributions $\mathcal{R}_\omega(t)$ on $S_c(C_\omega^{\mathrm{sm}})$, defined by

$$\mathcal{R}_\omega(t)(f) = \int_{Y_\omega \mathrm{sm}} f(t)\omega_{\omega,t} f(v),$$

The normalized Radon transform $\hat{\mathcal{R}}: S_c \to S^\infty$ is defined by

$$\hat{\mathcal{R}}(f)(\omega) = \mathcal{R}(t)(f)(\omega)\psi(t)$$

Definition 2.2. For any $f \in S_c$ define for $\omega \in C_0$

$$\Phi(f)(\omega) = \gamma(\chi_K, \psi) \int_{F^\times} \hat{\mathcal{R}}(f)(x\omega)\psi(x^{-1})\chi_K(-x)|x|^{n-2} \, d^\times x$$

We have the following operator $\Pi(r) \in \mathrm{Aut}(S)$, where $r \in O(V)$ is the involutive element

$$r(e) = e^*, \; r(e^*) = e, \; r|_{V_1} = \mathrm{Id}$$

the operator $\Pi(r)$ is a unitary involution on commuting with $O(V_1)$ and is called the Fourier transform on the cone as we have the following result

Theorem 2.3. *The restriction of* $\Pi(r)$ *to* S_c *is equal to* Φ.

The idea of the proof is as follows: Recall we have the decomposition $V = \mathbb{H} \oplus \mathbb{H} \oplus V_2$ and the basis $\{e_1, e_1^*\}$, $\{e_2, e_2^*\}$ for the first and second copy of $\mathbb{H}$, we define two involutions $r_1, r_2 \in \mathcal{O}(V)$ such that

$$r_1(e_1) = e_2, \; r_1(e_2) = e_1, \; r_1|_{V_2} = \mathrm{Id}$$
$$r_2(e_1) = e_1, \; r_2(e_1^*) = e_2^*, \; r_2|_{V_2} = \mathrm{Id}$$

then we can show that $r = r_1 \cdot r_2 \cdot r_1$, and note $r_2 \in M$, we have the formula for $\Pi(r)$.

We have a formula for the action of $\Pi(r_1)$ on a subspace of S_c, we will use a mixed model for the minimal representation realized on a space of functions on $F^\times \times V$, the map

$$F^\times \times V_2 \hookrightarrow C_0, \; (y, \omega) \mapsto (-y, \omega, q(\omega)/y)$$

we denote by C_0^1 its image, this gives rise to a G-equivariant isomorphism

$$L^2(C_0) \longrightarrow L^2(F \times V_2)$$

Proposition 2.4. *The operator* $\Pi(r_1)$ *preserves* $S_c(C_0^1)$ *and the action is given by*

$$\Pi(r_1)(f)(y, \omega, -q(\omega)/y) = \gamma(\chi_K, \psi) \int_{V_2} (-y) \cdot f(1, u)\psi(\langle u, v\rangle) \, du$$

Proof. Note that for $P = LU$ the Heisenberg parabolic subgroup of G, the unipotent radical is isomorphic to the symplectic space $W \otimes V_2$, and let P' be the derived subgroup of P, there is an explicit description of the action of P' on $\omega_{\psi,q}$.

Then this proposition follows from the isomorphism

$$\beta: \; \mathrm{ind}_{P'}^{P} \omega_{\psi,q} = \mathrm{ind}_{P'}^{P}(S_c(V_2)) \cong S_c(F^\times \times V_2)$$

and the fomula for r_1

$$r_1(f)(y, \omega) = \gamma(\chi_K, \psi) \int_{V_2} t^{-y} \cdot f(1, u)\psi(\langle u, v\rangle) \, du$$

□

3. $G = SL_2$

In this section, We will review the Fourier transfrom on basic affine space for $G = SL_2$.

Let F be a non archimedean local field and let $(V, \langle\cdot,\cdot\rangle)$ be a symplectic vector space over F with basis vectors e_1, e_2 and $\langle e_1, e_2\rangle = 1$. G acts on right with $B = TU$ stabilizing Fe_2, $X = U\backslash G$ can be identified with $V - \{0\}$ via $[g] \mapsto e_2 \cdot g$. $\Phi \in \mathrm{Aut}(S_c(V))$ is defined as

$$\Phi(f)(\omega) = \int_V f(v)\psi(\langle \omega, v\rangle) \, dv$$

Proposition 3.1. *We have*

- Φ *extends to a unitary involution on* $L^2(X, \omega_X)$.
- $\theta(g, t) \circ \Phi = \Phi \circ \theta(g, t^{-1})$.

4. $G = SU_3$

Let K be a quadratic extension of F with Galois involution $x \mapsto \overline{x}$, $|x|_K = |\mathrm{Nm}(x)|_F$, $\tau \in \mathcal{O}_F$ such that $\mathcal{O}_K = \mathcal{O}_F + \sqrt{\tau}\mathcal{O}_F$.

K admits a quadratic form $x \mapsto \mathrm{Nm}(x)$ and the associated bilinear form on K is $(x, y) \mapsto \mathrm{Tr}(x\overline{y})$. $W = K^3$, (W, h) the Hermitian spaces with Hermitian form

$$h(v_1, v_2) = x_1\overline{z_2} + z_1\overline{x_2} + y_1\overline{y_2}$$

with $v_i = (x_i, y_i, z_i)$. $G = SU(W, h)$ acts on the right, $B = TU$ preserving the line $K(0, 0, 1)$, $X = U\backslash G$ is identified with W^0 of h-isotropic non-zero vectors in W. $\langle v_1, v_2\rangle = \mathrm{Tr}\ h(v_1, v_2)$ F-bilinear form. (V_K, q_K) is isomorphic to (W, q).

Recall that X can be identified with the W^0 of non-zero isotropic vectors in W, for any vector $\omega \in W^0 = X$ we consider

$$p_\omega :\ W^0 \longrightarrow F,\ p_\omega(v) = \langle v, \omega\rangle$$

the measure ω_X and the measure dx on F give rise to a measure $\omega_{\omega,a}$ on the fiber $p_\omega^{-1}(a)$ for any $a \in F$. For any $a \in F$ we define the Radon transform $\mathcal{R}(a) : S_c(X) \to S^\infty(X)$ by

$$\mathcal{R}(a)(f)(\omega) = \int_{p_\omega^{-1}(a)} f(v)\omega_{\omega,a}(v)$$

The normalized Radon transform on $S_c(X)$ is defined by

$$\hat{\mathcal{R}}(f)(\omega) = \int_F \mathcal{R}(a)(f)(\omega)\psi(a)\ da$$

Definition 4.1. We define the Fourier transform $\Phi : S_c(X) \to S^\infty(X)$

$$\Phi(f)(\cdot) = \int_{F^\times} \theta(t(x))\hat{R}(f)(\cdot)\psi(x^{-1})\chi_K(x)|x|^{-1}\ d^\times x$$

Proposition 4.2. *For $f \in S_0(X)$, we have*

$$\Phi(f)(\omega) = \int_X f(v)\mathscr{L}(\langle v, \omega\rangle)\ \omega_X(v)$$

where for $a \in F^\times$

$$\mathscr{L}(a) = \gamma(\chi_K, h)\int_{F^\times} \psi(ax + x^{-1})\chi_K(-x)|x|\ d^\times x$$

References

[GK19] Nadya Gurevich and David Kazhdan. Fourier transforms on the basic affine space of a quasi-split group. *arXiv preprint arXiv:1912.07071*, 2019.

[GK23] Nadya Gurevich and David Kazhdan. Fourier transform on a cone and the minimal representation of even orthogonal group. *arXiv preprint arXiv:2304.13993*, 2023.