FACTORIZATION OF PERIOD INTEGRALS

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1. Introduction

This is my study note for Jacquet's paper [Jac01] and Lapid-Rogawski paper [LR00] for the Galois pair (Res GL_3, U_3), since there is no local multiplicity one result, the factorization of the global period 2.2 is not *formal*.

2. Global result

2.1. **The relative trace formula.** We will recall the relative trace formula of [JY96], the main theorem on cuspidal automorphic representations will be a consequence of this.

Let E/F be a quadratic extension of number fields, we will denote σ the non-trivial element of the Galois group for E/F, let H_0 be the 3×3 unitary group. We let Π be a cuapidal automorphic representation of $GL(3, E_{\mathbb{A}})$.

We say Π is *distinguished* by H_0 if the period integral

$$\mathscr{P}(\phi) := \int_{H_0(F)\backslash H_0(F_{\mathbb{A}})} \phi(h)\ dh$$

doesn't vanish. If Π is distinguished, then we have Π^σ is equivalent to Π for $\Pi^\sigma(g) = \Pi(g^\sigma)$. Moreover we have the following characterization of the local place v_0 of F inert in E: let v be the corresponding place of E and $\mathscr{H}_{v_0}(\Pi_v, H_{0,v})$ be the space of linear forms on the space $\mathscr{V}(\Pi_v)$ of smooth vectors of Π_v which is invariant under H_{0,v_0}, then $\mathscr{H}_{v_0} \neq 0$. If v_0 is a place of F which splits into v_1 and v_2 in E and let $\mathscr{H}_{v_0} = \mathscr{H}(\Pi_{v_1} \otimes \Pi_{v_2}, H_{0,v_0})$ be the space of linear forms on the space $\mathscr{V}(\Pi_{v_1} \otimes \Pi_{v_2})$ of smooth vectors for the tensor product $\Pi_{v_1} \otimes \Pi_{v_2}$ which are invariant under $H_0 \cong (g_{v_1}, g_{v_2})$, $g_{v_1} = {}^t g_{v_2}{}^{-1}$, $g_{v_1} \in GL(3, F_{v_0})$, then $\mathscr{H}_{v_0}$ is of dimension one.

Let S_0 be a finite set of places of F contains all the places at infinity, the even places and the places ramify in E. Let S_i be the set of places in S_0 which are inert in E and let S_s be the split places. S the set of places of E over a place of S_0, we assume Π is distinguished and unramify outside S.

For ψ a nontrivial character of $F_{\mathbb{A}}/F$ and $\psi_E(z) = \psi(z + \overline{z})$, we denote N the group of upper triangular matrices and θ the character of $N(F_{\mathbb{A}})$ defined by $\theta(n) = \psi(n_{1,2} + n_{2,3})$, similarly a character θ with $\theta(n) \mapsto \psi_E(n_{1,2} + n_{2,3})$ on $N(E_{\mathbb{A}})$. For $\phi \in \mathscr{V}(\Pi)$, we define

$$\mathscr{W}(\phi) = \int_{N(E)\backslash N(E_{\mathbb{A}})} \phi(n)\theta^{-1}(n\overline{n})\ dn$$

set $W(g) = \mathscr{W}(\Pi(g)\phi)$.

We let F^+ be the set of elements of $F^\times$ which are norm of an element of $E^\times$. We let $\mathfrak{S}$ be the Hermitian matrices in $GL(3, E)$ and $\mathfrak{S}^+(F)$ the set of elements in $\mathfrak{S}(F)$ with determinant in F^+.

If f_v is a function of compact support on $GL(3, E_v)$ and Φ_{v_0} the function on $\mathfrak{S}^+_{v_0}$ defined by

$$\Phi_{v_0}(g_v^t \overline{g}_v) = \int f_v(g_v h_{v_0})\ dh_{v_0}$$

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To functions f and f' we can attach kernels

$$K_f(x,y) = \int_{E_{\mathbb{A}}^\times/E^\times} \sum_{\xi \in GL(3,E)} f(x^{-1}\xi z y)\Omega(z)\ dz$$

$$K_{f'}(x,y) = \int_{F_{\mathbb{A}}^+/F^+} \sum_{\xi \in G^+(F)} f'(x^{-1}\xi z y)\Omega(z)\ dz$$

Then we have

$$J(\Phi) = \int_{N(E)\backslash N(E_{\mathbb{A}})\times H(F)\backslash H(F_{\mathbb{A}})} K_f(n,h)\theta^{-1}(n\overline{n})\ dn\ dh$$

$$J'(f') = \int_{N(F)\backslash N(F_{\mathbb{A}})\times N(F)\backslash N(F_{\mathbb{A}})} K_{f'}(n_1,n_2^t)\theta^{-1}(n_1)\theta(n_2)\ dn_2$$

We have a notion of *matching orbital integrals*, and for Φ and f' with matching integral $J(\Phi) = J'(f')$. It follows that we have an identity of integral of kernels attached to π and its base change.

We let π be a cuspidal automorphic representation of $GL(3, F_{\mathbb{A}})$ and let Π be its base change, for K_f^{Π} and K_f^{π} the kernels attached to the representations Π and π we have

$$K_f^{\Pi}(x,y) = \sum \Pi(f)\phi_i(x)\overline{\phi_i(y)}$$

$$K_{f'}^{\pi}(x,y) = \sum \pi(f')\phi_i'(x)\overline{\phi_i'(y)}$$

where the sum runs over the orthogonal basis of $\mathscr{V}(\Pi)$ and $\mathscr{V}(\pi)$.

Using the Whittaker functional $\mathscr{W}$ and the period integral $\mathscr{P}$, K_f^{Π} can be written as

$$\sum_{\phi_i} \mathscr{W}(\Pi(f)\phi_i)\overline{\mathscr{P}(\phi_i)}$$

similarly K_f^{π} can be written as

$$\sum_{\phi_i'} \mathscr{W}'(\pi(f')\phi_i')\overline{\mathscr{W}'(\pi(\omega)\phi_i')}$$

We can obtain

Proposition 2.1. *We have*

$$\sum_{\phi_i} \mathscr{W}(\Pi(f)\phi_i)\overline{\mathscr{P}(\phi_i)} = \sum_{\phi_i'} \mathscr{W}'(\pi(f')\phi_i')\overline{\mathscr{W}'(\pi(\omega)\phi_i')}$$

whenever f and f' have matching orbital integrals.

2.2. **Main result for cuspidal automorphic representations.**

Theorem 2.2. *There exist a constant $c \neq 0$ and for each $v_0 \in S_0$ a smooth vector $\mathscr{P}_{v_0} \in \mathscr{H}_{v_0}$ such that for every pure tensor $\phi \in \mathscr{V}^S$*

$$\mathscr{P}(\phi) = c \prod_{v_0 \in S_i} \mathscr{P}_{v_0}(W_0) \prod_{v_0 \in S_s} \mathscr{P}_{v_0}(W_{v_1} \otimes W_{v_2})$$

We note this factorization is not formal as we don't have local multiplicity one result also the proof of the theorem will provide us with a specific choice of local linear forms and also a specific value for the constant c in terms of L-functions. The main tool used in the proof is the relative trace formula introduced in [JY96].

We let Π be a distinguished cuspidal representation of $GL(3, E_{\mathbb{A}})$ with central character Ω, it is thus the base change of a unique cuspidal representation π of $GL(3, F_{\mathbb{A}})$ with central character ω. If f is a smooth function with compact support on G_S and K_S finite, we set

$$\mathscr{R}_{\Pi}(f) = \sum_{\phi_i} \mathscr{W}(\Pi(f)\phi_i)\overline{\mathscr{P}(\phi_i)}$$

the sum is over the basis (ϕ_i) of $\mathscr{V}^S$. This linear form can be thought as the *relative Bessel distribution* attached to Π.

On the group $G^+_{S_0}$, we can define

$$\mathscr{B}_\pi(f') = \sum_{\phi'_i} \mathscr{W}'(\pi(f')\phi'_i)\overline{\mathscr{W}'(\pi(\omega)\phi'_i)}$$

If f and f' have matching orbital integrals, from 2.1 we have

$$\mathscr{R}_\Pi(f) = \mathscr{B}_\pi(f')$$

For each $v_0 \in S_i$ we can define a distribution on $\mathscr{B}_v$ such that $\mathscr{R}_v(f_v) = \mathscr{B}_{\pi_{v_0}}(f'_{v_0})$ for f_v and f'_v with matching orbital integrals. At a place $v_0 \in S$, we can also define $\mathscr{R}_{v_0}$ such that

$$\mathscr{R}_{v_0}(f_{v_1} \otimes f_{v_2}) = \mathscr{B}_{v_0}(f'_{v_0})$$

where f'_{v_0} have matching orbital integrals with $f_{v_1} \otimes f_{v_2}$.

From the decomposition

$$\mathscr{B}_\pi(f') = c(\pi) \prod_{v \in S_0} \mathscr{B}_{v_0}(f'_{v_0})$$

we can write

$$\mathscr{R}_\Pi(f) = c(\pi) \prod_{v_0 \in S_i} \mathscr{R}_{v_0}(f_v) \prod_{v_0 \in S_s} \mathscr{R}_{v_0}(f_{v_1} \otimes f_{v_2})$$

Let $\tilde{\mathscr{P}}$ be the linear form on $\mathscr{V}^S(\Pi)$ defined by

$$\tilde{\mathscr{P}}(\phi) = \prod_{v_0 \in S_i} \mathscr{P}_{v_0}(W_v) \prod_{v_0 \in S_s} \mathscr{P}_{v_0}(W_{v_1} \otimes W_{v_2})$$

We have

$$\begin{aligned}\sum_\phi \mathscr{W}(\Pi(f)\phi_i)\overline{\tilde{\mathscr{P}}(\phi_i)} &= c(\Pi) \prod_{v_0 \in S_i} \mathscr{R}_{v_0}(f_v) \prod_{v_0 \in S_s} \mathscr{R}_{v_0}(f_{v_1} \otimes f_{v_2}) \\ &= \frac{c(\Pi)}{c(\pi)} \mathscr{R}_\Pi(f) \\ &= \frac{c(\Pi)}{c(\pi)} \Sigma_\phi \mathscr{W}(\Pi(f)\phi_i)\overline{\mathscr{P}(\phi_i)}\end{aligned}$$

since Π is irreducible, we get $\mathscr{P} = \frac{c(\pi)}{c(\Pi)} \tilde{\mathscr{P}}$.

2.3. **Stabilization for Eisenstein series.** We summarize the main result of [LR00], they studied the regularized periods of Eisenstein series associated to the pair $(\mathrm{Res}_{E/F} GL_3, U_3)$.

For G a reductive group over a number field and H the fixed point set of an involution θ, regularized period integral $\mathscr{P}^H(\phi)$ of $\phi \in V_\pi$, for π an automorphic form of G has be defined in [JLR99].

When $\phi = E(\varphi, \lambda)$ is a cuspidal Eisenstein series, $\mathscr{P}^H(\phi)$ can be expressed in terms of certain linear functionals $J(\eta, \varphi, \lambda)$ called *intertwining periods* in [JLR99].

Assume now $G = \mathrm{Res}_{E/F} H$ with E/F a quadratic extension and θ is the involution induced by the Galois conjugation. Let $B = TN$ be a θ-stable Borel subgroup with T, N also θ-stable. Given a character χ of $[T](\mathbb{A}_E)$ trivial on $Z(\mathbb{A}_E)$ and λ in $\mathfrak{a}^*_{0,\mathbb{C}}$ the Eisenstein series

$$E(g, \varphi, \lambda) = \sum_{\gamma \in B(E)\backslash G(E)} \varphi(\gamma g) e^{\langle \lambda, H(\gamma g)\rangle}$$

converges for Re λ sufficiently large. Here $\varphi : G(\mathbb{A}) \to \mathbb{C}$ satisfies $\varphi(bg) = \delta(b)^{1/2}\chi(b)\varphi(g)$.

According to a result of Springer, the double coset in $B(E)\backslash G(E)/H(F)$ are parametrized by η such that $\eta\theta(\eta)^{-1} \in N_G(T)$, hence we obtain

$$\iota : \ B(E)\backslash G(E)/H(F) \longrightarrow W$$

for each η, we set $H_\eta = H \cap \eta^{-1} N \eta$. The intertwining period attached to η is

$$J(\eta, \varphi, \lambda) = \int_{H_\eta(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} e^{\langle \lambda, H(\eta h)\rangle} \ \varphi(\eta h) \ dh$$

For suitable φ and λ we have

$$\mathscr{P}^H(E(\varphi,\lambda)) = \delta_\theta \cdot c \cdot \sum_{\iota(\eta)=\omega} J(\eta,\varphi,\lambda)$$

here ω is the longest element of the Weyl group.

Now we special to $G = \mathrm{Res}_{E/F} GL(3,E)$, $H = U(3)$, $G' = GL(3,F)$. Let T and T' be the diagonal tori of G and G' and $Nm :\ T \to T'$ the norm mapping. We fix χ a unitary character of $T(\mathbb{A}_E)$ and $\mathscr{B}(\chi) = \{\nu\}$ the set of characters of $T'(F)Z'(\mathbb{A}_F)\backslash T'(\mathbb{A}_F)$ such that $\chi = \nu \circ Nm$.

Definition 2.3. For E/F a quadratic extension of local fields or $E = F \oplus F$, we define the stable local period as

$$J^{st}(\nu,\varphi,\lambda) = \sum_{\iota(\eta)=\omega} \Delta_{\nu,\lambda}(\eta)^{-1} \int_{H_\eta(F)\backslash H(F)} e^{\langle \lambda, H(\eta h)\rangle}\ \varphi(\eta h)\ dh$$

We define the global stable period as

$$J^{st}(\nu_0,\varphi,\lambda) = \prod_v J_v^{st}(\nu_{0v},\varphi_v,\lambda)$$

Theorem 2.4. *We have*

$$\mathscr{P}^H(E(\varphi,\lambda)) = \sum_{\nu\in\mathscr{B}(\chi)} J^{st}(\nu,\varphi,\lambda)$$

the right-hand side is a sum of factorizable distributions.

Furthermore, after some local unramified computations, we can show the local factors at unramified places are given by ratios of L-functions.

Proposition 2.5. *For E/F an unramified extension of p-adic fields and $p \neq 2$, χ unramified, for $\varphi_0 \in I(\chi,\lambda)$ with $\varphi_0(e) = 1$, we have the stable local period $J^{st}(\nu,\varphi_0,\lambda)$ is equal to*

$$\frac{L(\nu_1\nu_2^{-1}\omega, s_1)L(\nu_2\nu_3^{-1}\omega, s_2)L(\nu_1\nu_3^{-1}\omega, s_3)}{L(\nu_1\nu_2^{-1}, s_1+1)L(\nu_2\nu_3^{-1}, s_2+1)L(\nu_1\nu_3^{-1}, s_3+1)}$$

for $s_i = \langle \lambda, \alpha_i^\vee\rangle$.

Definition 2.6. We define the relative Bessel distribution in terms of regularized period as

$$J(f,\chi,\lambda) = \sum_{\phi_i} \mathscr{P}^H(E(I(f,\chi,\lambda)\varphi,\lambda))\ \overline{\mathscr{W}}(\varphi,\lambda)$$

here ϕ_i runs over an orthogonal basis of $I(\chi,\lambda)$.

We define the Bessel distribution for $\nu \in \mathscr{B}(\chi)$ as

$$J'(f',\nu,\lambda) = \sum_{\phi_i'} \mathscr{W}'(I(f',\nu,\lambda)\varphi,\lambda)\ \overline{\mathscr{W}'}(\varphi',\lambda)$$

We introduce

$$J^{st}(f,\nu,\lambda) = \sum_{\phi_i} J^{st}(\nu,\varphi,\lambda)\ \overline{\mathscr{W}}(\varphi,\lambda)$$

as a consequence of theorem 2.4, we have $J(f,\chi,\lambda) = \sum_{\nu\in\mathscr{B}(\chi)} J^{st}(f,\nu,\lambda)$.

The second main result of [LR00] is which is an analog of 2.1 for Eisenstein series

Theorem 2.7. *Assume that the global quadratic extension E/F is split at the real archimedean places for χ a unitary character and $\nu \in \mathscr{B}(\chi)$, we have*

$$J^{st}(f,\nu,\lambda) = J'(f',\nu,\lambda)$$

for functions f and f' with matching orbital integrals.

3. Local result

In this section E/F will be a quadratic extension of non-archimedean local fields, we say that Π is distinguished if the space $\mathscr{H}(\Pi, H)$ of H-invariant forms is non-zero, then the central character Ω of Π is trivial on U_1. We fix ω a character of $F^\times$ with $\Omega(z) = \omega(z\bar{z})$.

Theorem 3.1. *Suppose Π is supercuspidal then Π is distinguished if and only if $\Pi^\sigma = \Pi$. The dimension of $\mathscr{H}(\Pi, H)$ is then one. Then Π is the base change of a unique cuspidal representation π of $GL(3, F)$ with central character ω and there exists a unique $\mathscr{P}_\pi$ of $\mathscr{H}(\Pi, H)$ with*

$$\sum_{\phi_i} \mathscr{W}(\Pi(f)\phi_i)\overline{\mathscr{P}_\pi(\phi_i)} = \mathscr{B}_\pi(f')$$

for f and f' with matching orbital integrals.

This is proven by realizing Π as the local component of a cuspidal distinguished automorphic representation with central character Ω.

References

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